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On some concepts constituting discussive logic*

Abstract: We recall some basics of a system of discussive logic proposed by Polish logician Stanisław Jaśkowski and present and analyze some key concepts, mainly from the field of modal logic, important for the presentation of the discussive logic $\mathsf{D_2}$ and related systems. In particular, we focus on a presentation of M-counterpart, a notion central in the context of the project of discussive logic. We also give a summary of results on a generalization of this notion.

Keywords: Jaśkowski, discussive logic, **S5**, M-counterpart, M-fragment, translations of discussive connectives

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Introduction

Possibilities limit our reasoning, but also our decisions, behavior, choices, and so on. It is common that some options that are possible for a given person are not necessarily possible for another. So, one can assume that every person has their own individual and unique range of possibilities. From a more theoretical point of view, to describe such a situation, one should consider a kind of Cartesian product of an indexed family of sets of possibilities for particular beings. On the other hand, one can consider a respective subset of the Cartesian power of the set-theoretical sum of all possibilities of particular individuals obtaining the very same universe.[1] Moreover, thinking in terms of some history of a given agent, it might be described as a sequence of passing from one possibility to another, or to put it in another way, it can be seen as iterations of possibilities. Thus, it seems to be not only a formal toy to consider expressions that, after being preceded by a sequence of possibility operators, become a valid expression from the point of view of a certain logical system. It appears that such a perspective is also important from the point of view of the development of discussive logic $\mathsf{D_2}$ by the Polish logician Stanisław Jaśkowski, since in $\mathsf{D_2}$ the possibility operator, and in fact its iterations, plays the crucial role. Jaśkowski´s development of discussive logic was, up to our knowledge, the first consciously planned and formally constructed paraconsistent logic. [2]

In this paper, we present the historical overview of a central notion of discussive logic, the notion of an M-*counterpart* of a modal logic. The importance of this study is to contribute to the understanding of the logical basis of discussive logic, a logic aimed at representing discussions that can contain contradictions. In the paper, we try to spell out formal notions in more non-formal terms. Our aim is to broaden the investigation of discussive logic. When pertinent, we also try to give insights of formal notions in terms of discussive concepts.

This work is divided as follows. In the first part, we begin by introducing a few basic technical preliminaries for the rest of the paper. We then proceed systematically in the following way: after introducing the discussive logic $\mathsf{D_2}$ in the second section, we study the notion of M-counterpart and its relation to discussive logic. In the third section, we study the historical development of variants and extensions of

[1] Indeed, the first case can be expressed by the Cartesian product $\prod_{i\in I} A_i$ (where I is a set of parameters that can be identified with the set of all people and A_i, for a given i, represents the set of all possibilities of i), while the second by $(\bigcup\limits_{i\in I} A_i)^I$, and it is a set-theoretical definitional fact that, leaving aside trivial cases, the first set is a proper subset of the second one.

[2] Although the term "paraconsistent" was introduced in 1976 by the Peruvian logician Miró Quesada.

the discussive logic. An extensive summary and also interesting results and generalisations on the notion of M-counterparts can be found in Kotas and da Costa (1977).

In the paper we do not investigate the historical events that led to the creation of discussive logic, nor the relation of Polish paraconsistent logic to other paraconsistent traditions. For the former study, one can consult Nicolás-Francisco (2022), and for the latter, one can see Rodrigues, Nasieniewski and Nicolás-Francisco (2025), where, in particular, a comparison of the discussive logic with da Costa's hierarchy of paraconsistent systems C_n is presented.

1. Preliminaries

As it is common in logic, logical researchers use formal languages with several aims:

1. to express complex ideas in a synthetic way,
2. to be precise,
3. to analyze the logical structure of a given phenomenon.

The use of formal languages is not exclusive to pure logical investigations but is suggested by any topic that one wants to analyze in a certain recurring structure. In the present case of study, we refer to Stanisław Jaśkowski's logical system developed as a formalization of a model of discussion where participants have different opinions but where not everything that can be said is true. The use of formal language is not only to analyze the logical structure of a given 'static' phenomenon but also to analyze 'interactions' that are taking place during a discussion. For Jaśkowski, such use would be on the validity of arguments that are debated. From this point of view, the logic developed by Jaśkowski has a natural motivation. To present the discussive logic, we will use some technical notions, and we will explicate them in some detail when necessary.

We will work with two languages. This is not very surprising, as it is a fairly common situation where we have two different domains. Consider, for example, the specific terminology used in music, which may be unfamiliar to those not educated in the field. In this case, some concepts have to be translated into the language that a given person knows. Thus, let L_1 and L_2 denote two propositional languages built from a set of propositional variables indexed with natural numbers, that is $\mathrm{Prop} = \{p_i : i \in \mathbb{N}\}$, and the set of connectives $\mathrm{Con} = \{\wedge, \vee, \rightarrow, \neg, \Box, \Diamond, \wedge_{\mathrm{d}}, \rightarrow_{\mathrm{d}}, \leftrightarrow_{\mathrm{d}}\}$. The set Prop is a collection of indexed letters that aim to denote propositions, for instance:

'p_1' can denote "The radius of the moon is 1740 kilometers.", and similarly for other letters. The connectives help to build other propositions that can denote some relations. For instance, the connective $\wedge$ is generally used to conjoin two propositions (if we consider two propositions p_1 and p_2, to conjoin them we write $p_1 \wedge p_2$). As an example: if 'p_1' denotes "The radius of the moon is 1740 kilometers" and 'p_2' denotes "The radius of the earth at the poles is 6356 kilometers", we can conjoin both propositions to form the new proposition "The radius of the moon is 1740 kilometers and the radius of the earth at the poles is 6356 kilometers", and we can denote such proposition by $p_1 \wedge q_2$. In the case of classical conjunction, $\wedge$ only gathers information about the truthfulness of the two propositions combined by using it, but '$\wedge$' can have a different meaning, being a variant of the classical stipulation. The other connectives do a similar job but with its specific interpretation:

- $\vee$ denotes 'or'
- $\rightarrow$ denotes 'if... then' (the 'if-part' appears at the left of the arrow and the 'then-part' appears at the right of the arrow)
- $\neg$ is interpreted as 'not'
- $\Box$ denotes 'it is necessary that...'
- $\Diamond$ denotes 'it is possible that...'.

As we will see, for the case of $\wedge_{\mathrm{d}}$, $\rightarrow_{\mathrm{d}}$, $\leftrightarrow_{\mathrm{d}}$, we have a particular interpretation that involves reference to participants in a discussion. Note also that the symbols above denote the expressions on the right in some of their uses (see, e.g., Orayen 1989, p.174), but not in all of them, since one can use 'and' in ways that are not denoted by $\wedge$, as, for example, when two conjucts must be related and not only be true, like in relating semantics (see, e.g., Jarmużek (2021)).

We will consider in this paper the following subsets of the set of connectives: $\mathrm{Con_m} = \{\wedge, \vee, \rightarrow, \neg, \Box, \Diamond\}$, $\mathrm{Con_d^-} = \{\wedge, \rightarrow_{\mathrm{d}}, \leftrightarrow_{\mathrm{d}}, \vee, \neg\}$ and $\mathrm{Con_d} = \{\wedge_{\mathrm{d}}, \rightarrow_{\mathrm{d}}, \leftrightarrow_{\mathrm{d}}, \vee, \neg\}$. That is, we will consider $\mathrm{Con_m}$ as the usual set of connectives characteristic for modal approaches and the sets $\mathrm{Con_d^-}$ and $\mathrm{Con_d}$ as connectives that will have interpretation in terms of participants of discussions: the first with the classical conjunction and the second with a discussive one, yet to be explained.

The sets of expressions called formulas and denoted by $\mathrm{For_m}$, $\mathrm{For_d^-}$ and $\mathrm{For_d}$ are defined in the usual way from the set of propositional variables and the sets $\mathrm{Con_m}$, $\mathrm{Con_d^-}$ and $\mathrm{Con_d}$, respectively. In other words, $\mathrm{For_m}$ is meant as the smallest set of strings/expressions that meets the following requirements:

- propositional variables are formulas (elements of the set Prop are formulas)

- if A and B are formulas, then, as it is common, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, $\neg A$, $\Box A$ and $\Diamond A$ are formulas.

Thus, every propositional variable and any result of applying connectives from $\mathrm{Con_m}$ to the expressions obtained earlier, including propositional variables is a formula in $\mathrm{For_m}$, but nothing else. Similarly, the sets $\mathrm{For_d^-}$ and $\mathrm{For_d}$ can be defined, taking, instead of the set $\mathrm{Con_m}$, the sets $\mathrm{Con_d^-}$ and $\mathrm{Con_d}$, respectively. We will use Greek upper letters Γ, Δ, Σ to denote arbitrary sets of formulas (representing collections of sentences), where the context will make it clear whether they are formulas from the set $\mathrm{For_m}$, $\mathrm{For_d^-}$, or $\mathrm{For_d}$.

Let us recall here some formulations of the definitions of *regular* and *normal modal logic*. Let **PC** denote the set of modal formulas that are instances of classical tautologies. A *regular modal logic* $\boldsymbol{L}$ is a subset of $\mathrm{For_m}$ which satisfies the following conditions:

- $\mathbf{PC} \subseteq \boldsymbol{L}$,
- $\Diamond p \leftrightarrow \neg\Box\neg p \in \boldsymbol{L}$, $\quad(\mathtt{df}_\Box\Diamond)$
- $\Box(p \to q) \to (\Box p \to \Box q) \in \boldsymbol{L}$, $\quad(\mathtt{K})$
- $\boldsymbol{L}$ is closed under detachment for $\to$, i.e., if $A \in \boldsymbol{L}$ and $A \to B \in \boldsymbol{L}$ then $B \in \boldsymbol{L}$,
- $\boldsymbol{L}$ is closed under the monotonicity rule: if $A \to B \in \boldsymbol{L}$ then $\Box A \to \Box B \in \boldsymbol{L}$,
- $\boldsymbol{L}$ is closed under uniform substitution: if $A \in \boldsymbol{L}$ then $sA \in \boldsymbol{L}$, where sA is the result of uniform substitution of formulas for propositional variables in A.

A regular modal logic $\boldsymbol{L}$ is *normal* iff $\boldsymbol{L}$ satisfies the following condition:

- $\boldsymbol{L}$ is closed on the necessitation rule, i.e., if $A \in \boldsymbol{L}$, then $\Box A \in \boldsymbol{L}$.

It is a well-known fact that, on the basis of any normal logic, monotonicity rule is derivable. **C2** and **K** are, respectively, the smallest regular and normal logics, while **D2**[3] and **D** are, respectively, the smallest regular and normal modal logics that contain the axiom $(\mathtt{D})$: $\Box p \to \Diamond p$, or equivalently (by the dual version of $(\mathtt{df}_\Box\Diamond)$, **PC** and distributivity of $\Diamond$ with respect to $\vee$) $\Diamond(p \to p)$. **D2** and **D** are called *deontic*[4] logics.

[3] Notice that **D2** and D_2 are different logics.
[4] Von Wright (1951) expresses his indebtedness for this term to C. D. Broad.

2. Discussive logic $\mathsf{D_2}$

It is a common human practice that, given two languages (formal or not), we can translate the sentences of one language into the sentences of the other. The translation process can be complex in the case of non-formal languages that need to take into consideration cultural, semantic, and grammatical aspects. However, for formal languages, the case can be more easy as the conditions for the translation can be artificially imposed. To present the discussive logic, we will use some translations between the formulas of formal languages. The reason for doing so is to provide the language for discussive logic with an intuitive interpretation.

We will translate the sentences of $\mathrm{For}_\mathrm{d}^-$ to For_m according to the following conditions:

- any propositional sentence stays unchanged,
- the sentences expressed by the formulas $A \wedge B$, $A \vee B$, $A \to B$, $\neg A$ are translated recursively with the preservation of the given connective unchanged,
- the sentence expressed by the formulas $A \to_\mathrm{d} B$ is translated to $\Diamond \mathrm{i}(A) \to \mathrm{i}(B)$ (i.e., 'if it is possible that $\mathrm{i}(A)$, then $\mathrm{i}(B)$'), where $\mathrm{i}(A)$ and $\mathrm{i}(B)$ are results of a further recursive translation of A and B ending on atomic sentences. The intuitive interpretation is that the possibility operator indicates a statement by a participant in the discussion to which the given participant is responding, so $A \to_\mathrm{d} B$ is translated as 'if someone stated A in the discussion, then B',
- similarly to the previous case but slightly more complicated, the sentence expressed by the formula $A \leftrightarrow_\mathrm{d} B$ is translated to $((\Diamond \mathrm{i}(A) \to \mathrm{i}(B)) \wedge (\Diamond \mathrm{i}(B) \to \Diamond \mathrm{i}(A)))$ (i.e. 'if someone has asserted that $\mathrm{i}(A)$, then $\mathrm{i}(B)$, and if someone has asserted that $\mathrm{i}(B)$, then someone has asserted that $\mathrm{i}(A)$'). This asymmetrical form of the translation of equivalence has to do with Jaskowski's consideration of an external observer whose judgment is expressed by additional copy of possibility operator.

The provided translation is intended to interpret expressions but also whole arguments using modal language. In the case of discussive logic, the normal modal logic **S5** is used. It can be expressed semantically by means of relational structures (frames) built of a nonempty set (of the so-called worlds) and the universal relation on it, where given a valuation of propositional variables (that can be understood as collection of independent valuations for every world), a formula is interpreted in a given world by using standard conditions of classical logic for $\wedge$, $\vee$, $\to$, $\neg$, while in the case of $\Box$, the truth of $\Box A$ in a given world holds, if A is true in every world, while the truth of $\Diamond A$ is equivalent to the truth of A in some world. A formula is a tautology of **S5** iff it is true in every world, for every frame with universal relation and arbitrary valuation in it. A set of premises entails a formula A on the basis of **S5** iff for every frame, valuation and a world, whenever all premises are true in this world, so is A. The discussive logic D_2 is defined as the set of formulas that become theses of the modal logic **S5** (i.e,. that are valid formulas of **S5**, or, what is the same, valid arguments with no premises) after being preceded by the symbol $\Diamond$ and translated to the modal language as it was recalled above. The idea behind the use of $\Diamond$ is that assertions of participants in a discussion are seen as possible from an external point of view. Jaśkowski undoubtedly had also a consequence relation in mind, but it can be expressed by the theses of discussive logic (see Footnote 7).

Discussive logic was introduced by Stanisław Jaśkowski in (Jaśkowski 1948, 1999b, 1949, 1999a) in an attempt to provide such a propositional logical system that would solve the following three tasks:

1. do not always entail that every formula would be a thesis, when applied to inconsistent system (inconsistent theories),
2. be rich enough to allow for practical inferences, and
3. have intuitive justification.

As points two and three admitted several solutions, Jaśkowski considered different options to solve these tasks. The resolution of Jaśkowski with the use of the modal logic **S5** was based on the fact that **S5** is quite rich as an extension of classical propositional logic and can be embed into the first-order classical logic. Thus, the solution by means of a translation into **S5** using application of possibilities, corresponding to the point of view of an external observer:

- fulfills the first stipulation since the translation: $\Diamond A, \Diamond\neg A \vDash \Diamond B$ of the inference: $A, \neg A \vDash B$, for some A and B is not valid on the basis of **S5**; moreover
- it would allow for – what were considered at the time – practical inferences, since it could recover the whole classical logic (in the language with $\vee$ and $\neg$),
- finally, the discussive logic would have intuitive justification as it is based on discussive models where each participant can express some views, and his position is considered in the whole discussion as relevant. (For more on this observation see Rodrigues, Nasieniewski and Nicolás-Francisco 2025).

After the introduction of discussive logic, Jaśkowski published in 1949 another paper expanding the discussive logic with discussive conjunction $\wedge_{\mathrm{d}}$. So, this time the translation was from $\mathrm{For_d}$ to $\mathrm{For_m}$ with the above conditions adding the following condition:

- the sentence expressed by the formula $A \wedge_{\mathrm{d}} B$ is translated to ‘$\mathrm{i}(A)$ and it is possible that $\mathrm{i}(B)$, or ‘$\mathrm{i}(A)$ and someone has asserted $\mathrm{i}(B)$’.[5]

Jaśkowski also noted that one could simplify the definition of $A \leftrightarrow_d B$ using only connectives from discussive logic in the following terms: $(A \leftrightarrow_d B) =_{\mathrm{def}} (A \rightarrow_{\mathrm{d}} B) \wedge_{\mathrm{d}} (B \rightarrow_{\mathrm{d}} A)$. This is logically equivalent to the interpretation given above for $\leftrightarrow_d$.[6] Thus, the discussive logic $\mathsf{D_2}$ could be redefined as the set of formulas that become theses of the modal logic **S5** preceded by the symbol $\Diamond$ after being translated to the modal language including the new meaning of conjunction.

2.1. The concept of M-counterpart and discussive logic

For the purposes of this paper, we will identify a logic with the set of its theses or tautologies.[7] As it is clear from the definition of discussive logic, the crucial role for $\mathsf{D_2}$ is played by the set of modal formulas that are theses of **S5** and have $\Diamond$ at the beginning, that is, as the main connective. We call the set of all modal formulas of a given modal logic ***S*** having ‘$\Diamond$’ at the beginning, the M*-fragment of* ***S***.

Similarly, we call the set of all modal formulas of a given modal logic ***S*** having $\Box$ at the beginning, the L*-fragment* (also known as the ‘*L*-fragment’) of ***S***. Perzanowski (1975) proposed a general and systematic study of the L-fragments and M-fragments of normal modal logics, including the M-counterpart of **S5** and of the logic $\mathbf{S5}^{\mathsf{M}}$–‘the discussive companion of **S5**’. This modal logic was investigated in (Błaszczuk and

[5] One has to admit that in the case of discussive conjunction, Jaśkowski did not provide any intuitive reading. It seems that Max Urchs was the first who proposed it. The first author of the present paper remembers a talk on this matter. As for a published version, see Urchs (1995).

[6] Similarly, we may observe that discussive implication can be expressed without the explicit use of possibility, but with the help of conjunction, under which the possibility is hidden. In particular, one can define $A \rightarrow_{\mathrm{d}} B$ using only connectives from discussive logic in the following terms: $\neg((A \vee \neg A) \wedge_{\mathrm{d}} A) \vee B$.

[7] We note that this limitation is not restrictive. Jaśkowski observed that the inference from $\Diamond p$ and $\Diamond(p \rightarrow q)$ to $\Diamond q$ is not valid in the modal logic **S5**, and so not valid in discussive logic if classical implication would be used. With the introduction of the discussive conditional, it was possible to prove $\Diamond q$ from $\Diamond p$ and $\Diamond(\Diamond p \rightarrow q)$ since the last formula is equivalent to $(\Diamond p \rightarrow \Diamond q)$ in the system **S5**. By using such conditional, one can consider a propositional variable p_{n+1} as the consequent of the final conditional in a series: $p_1 \rightarrow_{\mathrm{d}} (p_2 \rightarrow_{\mathrm{d}} (\ldots(p_n \rightarrow_{\mathrm{d}} p_{n+1})\ldots))$. But this formula after translation becomes $\Diamond(\Diamond p_1 \rightarrow (\Diamond p_2 \rightarrow (\ldots(\Diamond p_n \rightarrow p_{n+1})\ldots)))$, which is equivalent on the basis of **S5** to $(\Diamond p_1 \rightarrow (\Diamond p_2 \rightarrow (\ldots(\Diamond p_n \rightarrow \Diamond p_{n+1})\ldots)))$. So, $(\Diamond A_1 \rightarrow (\Diamond A_2 \rightarrow (\ldots(\Diamond A_n \rightarrow \Diamond A_{n+1})\ldots))) \in \mathbf{S5}$ iff $\Diamond A_1, \ldots, \Diamond A_n \vDash_{\mathbf{S5}} \Diamond A_{n+1}$. Thus, one can treat $\mathsf{D_2}$ also as a consequence relation. For further reading see Nasieniewski and Pietruszczak (2013).

Dziobiak 1975c).

More formally, for the rest of the paper, we will consider the following definitions. For any $\boldsymbol{S} \subseteq \mathrm{For_m}$ and $A \in \mathrm{For_m}$:

- $\Diamond\text{-}\boldsymbol{S} = \{A : \Diamond A \in \boldsymbol{S}\}$
- $\Box\text{-}\boldsymbol{S} = \{A : \Box A \in \boldsymbol{S}\}$
- $\boldsymbol{S}^{\Diamond} = \{A : \exists B \in \boldsymbol{S}, A = \Diamond B\}$
- $\boldsymbol{S}^{\Box} = \{A : \exists B \in \boldsymbol{S}, A = \Box B\}$.

In other words, $\Diamond\text{-}\boldsymbol{S}$ denotes the set of all modal formulas that result from eliminating possibility being the main connective of elements of $\boldsymbol{S}$, whereas $\Box\text{-}\boldsymbol{S}$ denotes the set of all modal formulas that result from eliminating necessity being the main connective of elements of $\boldsymbol{S}$. We call the set of formulas $\Diamond\text{-}\boldsymbol{S}$, the M-*counterpart of* $\boldsymbol{S}$, while the set of formulas $\Box\text{-}\boldsymbol{S}$, the L-*counterpart of* $\boldsymbol{S}$.[8] By contrast, $\boldsymbol{S}^{\Diamond}$ denotes the set of all modal formulas of $\boldsymbol{S}$ that have $\Diamond$ at the beginning, and $\boldsymbol{S}^{\Box}$ denotes the set of all modal formulas of $\boldsymbol{S}$ that have $\Box$ at the beginning. More generally, one can define $\Box^n\text{-}\boldsymbol{S}$ and $\Diamond^n\text{-}\boldsymbol{S}$, the M^n-*counterpart* and L^n-*counterpart of* $\boldsymbol{S}$, respectively, by considering iterations of $\Box$'s and $\Diamond$'s. To this aim, one can use $\Box^0\text{-}\boldsymbol{S}$ (and $\Diamond^0\text{-}\boldsymbol{S}$) to denote $\boldsymbol{S}$, and $\Box^{n+1}\text{-}\boldsymbol{S}$ to denote $\Box\text{-}(\Box^n\text{-}\boldsymbol{S})$ (respectively, $\Diamond^{n+1}\text{-}\boldsymbol{S}$ denotes $\Diamond\text{-}(\Diamond^n\text{-}\boldsymbol{S})$). The core of these definitions consists of expressions $\Box^n A$ (and $\Diamond^n A$) that can be defined inductively and denoting formulas of the form $\underbrace{\Box\cdots\Box}_{n} A$ (resp. $\underbrace{\Diamond\ldots\Diamond}_{n} A$). Using such notations we can define $\boldsymbol{S}^{\Diamond^n}$ and $\boldsymbol{S}^{\Box^n}$ in the similar way as $\boldsymbol{S}^{\Diamond}$ and $\boldsymbol{S}^{\Box}$, as sets composed of elements of $\boldsymbol{S}$ of the form $\Diamond^n B$ and $\Box^n B$, respectively. Thus, $\boldsymbol{S}^{\Diamond^n}$ (the M^n-*fragment of* $\boldsymbol{S}$) denotes the set of all modal formulas of $\boldsymbol{S}$ that begin with n copies of $\Diamond$, and $\boldsymbol{S}^{\Box^n}$ (the L^n-*fragment of* $\boldsymbol{S}$) denotes the set of all modal formulas of $\boldsymbol{S}$ that begin with n copies of $\Box$. For any modal system $\boldsymbol{S}$, any modal formula, and any modality $Q \in \{\Diamond, \Box\}$, it is true that (see Perzanowski 1975, Fact 1):

$$A \in Q^n\text{-}\boldsymbol{S} \text{ iff } Q^n A \in \boldsymbol{S} \text{ iff } Q^n A \in \boldsymbol{S}^{Q^n}.$$

The importance of M-counterparts and L-fragments is coming from the fact that there is a correspondence between the M-counterpart and L-fragment of **S5** and $\mathsf{D_2}$.[9]

Kotas showed (see Lemma 6 in Kotas 1974) that the system $\Diamond$-**S5** is equivalent *via* respective translations to the discussive logic $\mathsf{D_2}$. After that, Furmanowski proved in (Furmanowski 1975) that the system $\Diamond$-**S4** can also be used to define the discussive system $\mathsf{D_2}$. So, in the same way as $\Diamond$-**S5**, $\Diamond$-**S4** is also equivalent *via* respective translations to the discussive logic. Moreover, Furmanowski proved that for any modal

[8] Notice that some authors (see, e.g., da Costa (1975)) are treating M-counterpart as discussive logic for which the symbol **J** is used.

[9] See, for example, Kotas (1975) and da Costa (1975).

system $\boldsymbol{S}$ between **S4** and **S5**, discussive system based on $\boldsymbol{S}$ equals the discussive system $\mathsf{D}_\mathbf{2}$. So, Furmanowski used sufficient notions to ask whether a given modal logic, when treated with Jaśkowski's translations, leads to the very same logic $\mathsf{D}_\mathbf{2}$ or to some other systems. More precisely, let us consider a fixed modal system $\boldsymbol{S}$. If one defines a discussive system $\mathsf{D}_{\boldsymbol{S}}$ as the set of formulas fulfilling the following condition:

$$\mathsf{D}_{\boldsymbol{S}} = \{A \in \mathrm{For_d} : \Diamond(\mathrm{i}(A)) \in \boldsymbol{S}\},$$

(where, again, $\mathrm{i}(A)$ is the result of the recursive translation of A ending on atomic sentences according to the conditions given above) one may obtain different discussive systems, although not all of them will result in a new logic. Taking into account the above formulation, $\mathsf{D}_{(-)}$ can be seen as an operator, that for any set of modal formulas X, gives a discussive logic $\mathsf{D}_{(X)}$. Thus, while searching for other discussive systems one has surely consider modal logics that 'escape' from those that are intermediate between **S4** and **S5**. One example of a not very useful discussive logic is the logic $\mathsf{D}_\mathbf{K}$; given the well-known fact that the modal logic **K** has no theorems beginning with possibility, $\mathsf{D}_\mathbf{K}$ is empty. On the other hand, it is also possible to study independently the systems $\Box^n$-$\boldsymbol{S}$ and $\Diamond^n$-$\boldsymbol{S}$ for any modal system $\boldsymbol{S}$.

Coming back to the matter of M-counterpart issues, Perzanowski in (Perzanowski 1975) axiomatized the M^n-counterparts of certain modal logics containing the modal logic **D**. One can easily see that the modal logic **D** is the smallest normal modal logic that has among its theses formulas beginning with possibility, and thus apt to define some discussive logic. Perzanowski also considered modal logics that have the same M-counterpart as **S5**, including the smallest such normal modal logic, which Perzanowski termed $\mathbf{S5}^\mathsf{M}$ (which also contains the modal logic **D**).

Studying other modal logics than the ones that can be used to define discussive logic $\mathsf{D}_\mathbf{2}$, in (Błaszczuk and Dziobiak 1976) some modal systems and their M-counterparts were investigated. In particular, due the fact that the method of axiomatization given in (Perzanowski 1975) does not cover the cases of **T**, **B** and $\mathbf{T}_n^+$, the respective investigations were undertaken by Błaszczuk and Dziobiak. It is natural to consider the system **T** since next to (T): $\Box p \to p$ it contains its dual version $p \to \Diamond p$, which, together with transitivity of implication allows to infer (D). The systems **B** and $\mathbf{T}_n^+$ are extensions of the system **T**: the system **B** extends the system **T** with the Brouwerian axiom (B): $p \to \Box\Diamond p$ (which says that every sentence is necessarily possible; equivalently, $\Diamond\Box p \to p$), and the system $\mathbf{T}_n^+$ extends **T** with the axiom $\Box^n A \to \Box\Box^n A$, for any natural number n. The systems $\mathbf{T}_n^+$ are called Sobociński systems and denoted as $\mathbf{S4_n}$. The class of systems $\{\mathbf{S4_n}\}_{n\geqslant 1}$ was considered in (Thomas 1964), where $\mathbf{S4_n}$ is denoted as $\mathbf{T_n}$, while the fact that $\mathbf{S4_{n+1}} \subsetneq \mathbf{S4_n}$, for $n \geqslant 1$ was attributed to an unpublished result by Sobociński. A similar reference to a forthcoming paper of Sobociński can be found in (Feys 1965), where again, '$\mathbf{S4^n}$', a slightly changed symbol for $\mathbf{S4_n}$ is used.

Błaszczuk and Dziobiak (1975b) showed how the M^n-counterparts of these logics can be axiomatized. Additionally, for every $n \geqslant 1$, $\mathbf{T}^*_\mathbf{n}$ – an extension of $\mathbf{T}$ by the rule $\Diamond^{n+1} A \vdash \Diamond^n A$ was considered. For such systems, the following holds:

- $\mathbf{T} \subsetneq \mathbf{T}^*_\mathbf{n} \subsetneq \mathbf{S4_n}$ ($n = 1, 2, \ldots$), i.e., the system $\mathbf{T}^*_\mathbf{n}$ strictly extends $\mathbf{T}$, while in turn, $\mathbf{T}^*_\mathbf{n}$ is strictly contained in $\mathbf{S4_n}$,
- $\Diamond^n$-$\mathbf{T}^*_\mathbf{n} = \Diamond^n$-$\mathbf{S4_n}$ ($n = 1, 2, \ldots$), i.e., that the M^n-counterpart of $\mathbf{T}^*_\mathbf{n}$ is the same as the M^n-counterpart of $\mathbf{S4_n}$.

Moreover, besides showing that the M^n-counterpart of $\mathbf{T}^*_\mathbf{n}$ equals the M^n-counterpart of the system $\mathbf{S4_n}$, Błaszczuk and Dziobiak proved that $\mathbf{T}^*_\mathbf{n}$ is the least system in the class of all normal modal systems with such property. They also indicated the greatest system in the class of all normal modal systems whose M^n-counterpart is the same as the M^n-counterpart of $\mathbf{S4_n}$, for the given n. The respective systems are denoted as $\mathbf{S5}^n$. $\mathbf{S5}^n$ are, as usual, determined by the rule of substitution, detachment for material implication, the rule of necessitation, and, as regards axioms, they are extensions of $\mathbf{S4_n}$ by the formula $\Diamond^n \Box^n p \to \Box^n p$. As suggested in (Błaszczuk and Dziobiak 1976), these investigations were not directly connected with discussive logic, but were generalizations that arose from it. The proofs of the above results are given mainly in (Błaszczuk and Dziobiak 1977)). In (Błaszczuk and Dziobiak 1975a) it was showed that there exist intermediate systems between $\mathbf{T}^*_\mathbf{n}$ and $\mathbf{S4_n}$, which, of course, have the same M^n-counterpart.

On more contemporary approaches to $\mathsf{D_2}$, Nasieniewski and Pietruszczak (2008) investigated the minimal regular modal logic that can define $\mathsf{D_2}$, corresponding to the logic $\mathbf{S5}^\mathsf{M}$, that they called $\mathbf{rS5}^\mathsf{M}$. Besides, other modal logics defining $\mathsf{D_2}$, smallest in a given class, are given; among the considered classes are: rte-logics, that is, modal logics closed under the rule of replacement of tautological equivalents (rte): if $(A \leftrightarrow B) \in \mathbf{PC}$ and $C \in \mathbf{PC}$ then $C[A/B] \in \mathbf{PC}$ (where $C[A/B]$ denotes the formula obtained by replacing any number of occurrences of A with B); classical modal logics, i.e., rte-logics having additionally, as axioms, formulas ($\mathtt{K}$) and $\Box(p \to p)$; congruential logics closed under the rule of congruence: $A \leftrightarrow B / \Box A \leftrightarrow \Box B$. Of course, each of these logics belongs to the set $\mathbf{S5}^\Diamond$ (for details, see Nasieniewski and Pietruszczak 2011). There are also investigations on a similar topic but concerning discussive consequence relation (Nasieniewski and Pietruszczak 2012, 2013).

The consideration of generalized M^n-counterparts of modal logics is an important contribution to the development of generalized versions of debatable logic that still wait to be undertaken.

3. Discussive logics and its variants

In the literature, there are known discussive logics alternative to the one obtained by means of the normal modal logic **S5**. Let us recall that the logic **D** has among its theses formulas beginning with $\Diamond$, so it can be used as an argument of the operator $\mathsf{D}_{(-)}$. The resulting discussive logic has been studied in (Mruczek-Nasieniewska and Nasieniewski 2019) under the name $\mathbf{D_0}$.

Besides to consider the operator $\mathsf{D}_{(-)}$, other strategy is to consider alternative formulations of the translations of the discussive connectives $\wedge_{\mathrm{d}}, \rightarrow_{\mathrm{d}}, \leftrightarrow_{\mathrm{d}}$. In particular, below we just mention three other translation functions i^{1}, i^{0}, i^{b} that resulted in logics that were investigated among others in (Nasieniewski and Pietruszczak 2014b).

Let i^{1} be a function from $\mathrm{For}_{\mathrm{d}}$ to $\mathrm{For}_{\mathrm{m}}$ with the following properties:

- for any $A, B \in \mathrm{For}_{\mathrm{d}}$
  - $\mathrm{i}^{1}(A \wedge_{\mathrm{d}} B) = \Diamond \mathrm{i}^{1}(A) \wedge \mathrm{i}^{1}(B)$
  - and all other cases are as in the function i.

Such interpretation of discussive language was undertaken in (da Costa and Dubikajtis 1977) and, for example, in (Achtelik et al. 1981).

Let i^{0} be a function from $\mathrm{For}_{\mathrm{d}}$ to $\mathrm{For}_{\mathrm{m}}$ with the following properties:

- for any $A, B \in \mathrm{For}_{\mathrm{d}}$
  - $\mathrm{i}^{0}(A \wedge_{\mathrm{d}} B) = \mathrm{i}^{0}(A) \wedge \mathrm{i}^{0}(B)$
  - and all other cases are as in the function i.

This is the interpretation of the original Jaśkowski's formulation from 1948.

Let i^{b} be a function from $\mathrm{For}_{\mathrm{d}}$ to $\mathrm{For}_{\mathrm{m}}$ with the following properties:

- for any $A, B \in \mathrm{For}_{\mathrm{d}}$
  - $\mathrm{i}^{\mathrm{b}}(A \wedge_{\mathrm{d}} B) = \Diamond \mathrm{i}^{\mathrm{b}}(A) \wedge \Diamond \mathrm{i}^{\mathrm{b}}(B)$
  - and all other cases are as in the function i.

Such a meaning of discussive language was considered, for example, in (Nasieniewski and Pietruszczak 2014a).

The discussive logics obtained by the above translations, D_2^*, D_2^-, and D_2^{**}, were defined using the very same method, as in the case of D_2. Strictly speaking, we have:

$$\mathsf{D}_2^* = \{A \in \mathrm{For_d} : \Diamond \mathrm{i}^1(A) \in \mathbf{S5}\}.$$
$$\mathsf{D}_2^- = \{A \in \mathrm{For_d} : \Diamond \mathrm{i}^0(A) \in \mathbf{S5}\}.$$
$$\mathsf{D}_2^{**} = \{A \in \mathrm{For_d} : \Diamond \mathrm{i}^b(A) \in \mathbf{S5}\}.$$

Let us finally mention a system suggested by Perzanowski in (Jaśkowski 1999a), in the "Comments to the translator", and investigated in (Ciuciura 2006), where a discussive negation has been introduced:

1. $\mathrm{i}^n(\neg_d A) = \ulcorner \Diamond \neg \mathrm{i}^n(A) \urcorner$.

Also, in this variant, other cases of connectives, that is of conjunction, implication and disjunction are meant in the same way as in the definition of i.

In summary, there are certainly interesting discussive logics that are determined by other translations that could be used to define new variants of discussive connectives.

4. Conclusion

There are of course other important problems and results that were inspired by Jaśkowski's discussive logic, to mention only a few examples (Béziau 2006; Mruczek-Nasieniewska and Nasieniewski 2005; Kovač 2008; Akama, Minoro Abe and Nakamatsu 2011; Dunin-Kęplicz, Powała and Szałas 2018). So, one has to admit that the above overview is very limited to a specific pattern connected basically to two translations used by Jaśkowski to define his discussive logic. Our goal was to show some spaces where further development could still be done using the original patterns proposed by Jaśkowski.

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