studia z HISTORII FILOZOFII KWARTALNIK 4(16)/2025

ISSN 2083-1978
e-ISSN 2391-775X

http://dx.doi.org/10.12775/szhf.2025.031

Krystyna Mruczek-Nasieniewska
Nicolaus Copernicus University in Toruń
mruczek@umk.pl
ORCID: 0000-0001-9125-2446

On certain Płonka's constructions

Abstract: We would like to present and discuss certain ideas developed by Jerzy Płonka. Jerzy Płonka was a Polish algebraist whose scientific interests included equational logics considered in terms of the structure of the identities that define them. In this paper, we present two constructions by Jerzy Płonka, which served as tools in his work on semantics for classes determined by the so-called *P*-compatible and regular identities. We are referring to the *P*-dispersion and the Płonka sum.

Keywords: Jerzy Płonka, regular identity, *P*-compatible identity, Płonka sum, *P*-dispersion

Introduction

There are various concepts important for logic and algebraic logic in particular, for example, notions such as variety or quasi-variety. In the 1930s, Garret Birkhoff introduced the concept of variety of algebras, and since then, many works related to this topic have been developed. Research on varieties of algebras has been conducted by, among others: A. Tarski, B. Jónsson, R. Dedekind, J. von Neuman, G. Gräter, R. McKenzie. One of the key results concerning this matter is a theorem (called the Tarski-Birkhoff theorem) which states that a class of algebras of a fixed type is a variety iff it is an equationally definable class. Thus, this result naturally connected algebra with mathematical logic, contributing to the rapid development of research on classes of algebras.

Having a notion of the variety of algebras, the question arises about subclasses of the initial class that are closed under the same operators as the initial class of algebras. In this way – very natural in algebra – the notion of a subvariety of a given variety has been considered. It is also known that all subvarieties of a given variety with an inclusion relation form a lattice. This lattice is dually isomorphic to the lattice of equational theories, which extend the theory defining the initial variety. In this way, what can be said about the lattice of equational theories can be expressed in the language of varieties.

The difficulty, however, is that relatively not too much can be said about this matter in the general case (Jónson (1968), Ježek (1981a,1981b,1982); McNulty (1981); Lampe (1986)). The lattices of varieties – or equational theories – for various classes of algebras have been and are still widely studied (see Jónson and Rival (1979); Lee (1970); McKenzie (1972), Nelson, (1971a, 1971b) for some classic results).

1. A background

Issues concerning the structure of equality have been the subject of research since the 1960s. At that time, notions such as regular, biregular, normal, externally compatible identities, and generalisation of the last two notions – P-compatible identities, but also other types of identities were introduced (see, for example, Płonka (1967, 1969); Graczyńska (1983); Płonka (1990a); Mel'nik (1971, 1973a); Płonka (1974); Chromik (1990); Płonka (1990b) and Płonka (2001)). It turns out that the structure of the terms occurring in an identity is important in certain fields, such as the theory of automata, unification theory, and more broadly – in those fields of research where the goal is to automatize reasoning, which seems to be essential in the era of artificial intelligence development. In particular, it is interesting to consider those subsets of the set of identities defining a given variety which form equational theories. These theories determine – in the general case – varieties which are larger in the sense of inclusion than the initial class of algebras. Such an approach to the subject allows a "broader view" on the variety and its subvarieties. This kind of investigation has been undertaken for many varieties of algebras (see, for example Chromik and Hałkowska (1991); Hałkowska, Cholewińska and Wiora (1997); Hałkowska (1998); Mruczek (2000); Płonka (1990b, 1988)).

2. Jerzy Płonka and his investigations on P-compatibility

Before going to presentation of Płonka's investigations, let us say few words about him as a person. He was born on June 7, 1930 in the village of Kończyce Małe in

Cieszyn County. During the occupation, as a boy, he lived in Kraków, combining Catholic attitudes with patriotism, belonged to a rose group led by Karol Wojtyła, and was a participant in religious-patriotic meetings (see Szczepański 1999).

While attending elementary school in Kraków, after suffering from the flu, he lost his sight. After graduating in 1949 from high school in Bytom, he began working at the State Educational Institution for Blind Children in Wrocław, but also enrolled at the State Institute of Special Education in the vocal faculty of the State Higher School of Music in Wrocław. From 1956 to 1961, Płonka studied mathematics at the University of Wrocław. For fifteen years, he worked as a teacher at the Wrocław center, where he applied the qualifications he had acquired during his studies. Despite the loss of his eyesight, as we read in Ogonowski (2020), he was an admirer of the Polish Tatry Mountains and an active participant in tourist and sightseeing events. He was also a teacher of singing and, of course, mathematics. Despite his own blindness, he was able to teach his blind pupils geometry and stereometry.

He defended his doctoral thesis in 1964, and three years later, in 1967, he obtained his habilitation degree. He received his full professorship in 1980. He was an employee of the Department of Algebra and Algebraic Geometry of the Institute of Mathematics of the Polish Academy of Sciences from 1964 to 2007. Additionally, in 1973, he began working at the Higher Pedagogical School of Opole.

The theory of equational logics defined by identities of a special form by Prof. Jerzy Płonka is one of the key achievements of 20th-century Polish algebraic logic. It was developed in the context of research on non-classical logics and their algebraic semantics, and its origins date back to the late 1960s. Below, we will attempt to outline some of the basic elements of this theory.

2.1. Logical context

In the 20th century, research was conducted on logical systems with the aim of assigning appropriate algebraic structures to logic. For example, in the works of Boole, Post, and Tarski, the relationships between logic and algebra had been investigated. In general, it is not a trivial task to identify the appropriate classes of algebras that determine the adequate semantics of a given system in the case of many non-classical logics (to mention various classes of modal logics, different paraconsistent logics, or substructural logics). In addition, there are also negative results on this issue (consider, for example, intermediate predicate logics), but also general results on the algebraizability of logics, obtained by Blok and Pigozzi, and later by Czelakowski and Font.

Although, to my knowledge, Jerzy Płonka's research was not focused on the use of algebraic structures for semantic tasks, it turns out that his own discoveries have recently found such applications.

2.2. Birkhoff's rules

Let $Id(\tau)$ denote the set of all identities of a given type, and let Σ be a subset of this set. We can assign a set $\mathsf{Cn}(\Sigma)$ to Σ in the following way:

- Every identity of the form $p \approx p$ belongs to this set.
- If an identity $p \approx q$ belongs to this set, then so does the identity $q \approx p$.
- If the identities $p \approx q$ and $q \approx r$ belong to the set $\mathsf{Cn}(\Sigma)$, then so does the identity $p \approx r$.
- Additionally, this set is closed under replacement (of an 'equal' by 'equal') and substitution.

The result of replacing any number of occurrences of term p with term q in term r is denoted below by $r(p/\!/q)$.

More formally, we can write it down as follows. For a set $\Sigma \subseteq Id(\tau)$, we denote by $\mathsf{Cn}(\Sigma)$ – the so-called closure of the set Σ under the consequence Cn – where:

Definition 2.1. $\mathsf{Cn}(\Sigma)$ is the smallest subset of the set $Id(\tau)$ containing Σ, such that:

(21) $p \approx p \in \mathsf{Cn}(\Sigma)$ *for any term* p *of type* τ*;*

(22) *if* $p \approx q \in \mathsf{Cn}(\Sigma)$*, then* $q \approx p \in \mathsf{Cn}(\Sigma)$ *for any terms* p, q *of type* τ*;*

(23) *if* $p \approx q \in \mathsf{Cn}(\Sigma)$ *and* $q \approx r \in \mathsf{Cn}(\Sigma)$*, then* $p \approx r \in \mathsf{Cn}(\Sigma)$ *for any terms* p, q, r *of type* τ*;*

(24) $\mathsf{Cn}(\Sigma)$ *is closed under replacement, that is, for any identity* $p \approx q$ *belonging to the set* $\mathsf{Cn}(\Sigma)$ *and for any term* r *of type* τ*, if* p *is a subterm of* r*, then for any* $r(p/\!/q)$ *– a result of replacing of term* p *by term* q *in term* r *we have* $r \approx r(p/\!/q) \in \mathsf{Cn}(\Sigma)$*;*

(25) $\mathsf{Cn}(\Sigma)$ *is closed under substitution, that is, for every identity* $p \approx q$ *belonging to the set* $\mathsf{Cn}(\Sigma)$ *and every term* r *of type* τ*, if we replace every occurrence of the variable* x *in the identity* $p \approx q$ *with* r*, then the resulting identity belongs to* $\mathsf{Cn}(\Sigma)$*.*

The above five rules are called *Birkhoff's rules*.

Jerzy Płonka observed that if we consider the set of all identities satisfied in a given equational theory Σ, then there are some subsets of this set, that are closed under the same rules as the original theory (in the same way as it was formulated for closure under replacement and substitution). Of course, this is not true for every subset of the set $Id(\tau)$. Only 'closed' subsets are interesting since they constitute subtheories of the initial theory Σ. As an example, let us consider the set of all identities satisfied in a fixed theory Σ that have the property that the variables appearing on both sides of the identity are different. An example of such an identity is absorption in theories of Boolean algebra (e.g., $x + (x \cdot y) \approx x$). However, if we substitute a single, fixed variable for each variable appearing in this identity, we obtain an identity that has the same variables on both sides. Therefore, the set of such identities is not closed under substitution, i.e., it is not an equational theory.

2.3. P-compatible identities and P-dispersion

For a term ϕ of type τ that is not a variable, we denote by $ex(\phi)$ the outermost operation symbol in the term ϕ. It is easy to see that for a term ϕ that is a zero-argument operation symbol, $ex(\phi)$ is precisely that symbol. Let Π_F denote the set of all partitions (divisions) of the set F and let $P \in \Pi_F$. We will denote the block of the partition P containing $f \in F$ by $[f]_P$.

Definition 2.2. An identity $\phi \approx \varphi$ of type τ is called *P-compatible* iff it is of the form $x \approx x$ or none of the terms ϕ and φ is a variable, and $ex(\phi) \in [ex(\varphi)]_P$.

This definition comes from J. Płonka (Płonka 1988) and is a generalization of the definitions of externally compatible identity and normal identity. The first was given by W. Chromik (Chromik 1990), the second – independently – by J. Płonka (Płonka 1974) and I. J. Melnik (Mel'nik 1973b).[1] Let us recall these definitions.

Definition 2.3. An identity $\phi \approx \varphi$ of type τ is called *externally compatible* iff it is of the form $x \approx x$ or none of the terms ϕ and φ are variables, and $ex(\phi)$ and $ex(\varphi)$ are the same.

Definition 2.4. An identity $\phi \approx \varphi$ of type τ is called normal iff it is of the form $x \approx x$ or none of the terms ϕ and φ are variables.

[1] For some further investigations concerning varieties determined by externally compatible identity, see, for example, Gajewska-Kurdziel and Mruczek-Nasieniewska (2007).

Since the above definitions use the notion of a term, we will now write down them again in a slightly more formal way, showing more precisely how equalities are formed from terms. Let V be a variety of algebras of the type $\tau : F \longrightarrow N$, where F is the set of function symbols (where τ gives information how many arguments a given symbol requires), N is the set of natural numbers and P some partition of the set F. An identity of type τ is called P-compatible if it is of the form $x \approx x$ or of the form $f(p_0,\ldots,p_{\tau(f)-1}) \approx g(q_0,\ldots,q_{\tau(g)-1})$, where f and g belong to the same block of partition P and $p_0,\ldots,p_{\tau(f)-1},q_0,\ldots,q_{\tau(g)-1}$ are terms of type τ. If a partition P contains only one-element blocks, then the identity P-consistent will be called an externally consistent identity.

Before giving a formal definition of P-dispersion, we will describe what this construction is all about. Let us imagine that we have two algebras $\mathfrak{A}$ and $\mathfrak{B}$, which are of the same type, and we have a certain partition of the set of basic operations F. We say that the algebra $\mathfrak{A}$ is a P-dispersion of the algebra $\mathfrak{B}$ if it is possible to divide the universe of the algebra $\mathfrak{A}$ indexed by elements of the universe of the algebra $\mathfrak{B}$ and indicate functions from $\mathfrak{B}$ to $\mathfrak{A}$ indexed by blocks of the considered partition in such a way that operations in the algebra $\mathfrak{A}$ are performed in such a way that in order to calculate the result of a certain operation for a given set of arguments in the algebra $\mathfrak{A}$, we first “return” to the algebra $\mathfrak{B}$, and then for the element that is the result of this operation, we transform it by functions indexed by blocks of the given partition.

The formal definition is as follows:

Definition 2.5. Let $\mathfrak{A} = \langle A, F^{\mathfrak{A}}\rangle$ and $\mathfrak{B} = \langle B, F^{\mathfrak{B}}\rangle$ be algebras of type τ and let $P \in \Pi_F$. We call the algebra $\mathfrak{A}$ a *P-dispersion* of the algebra $\mathfrak{B}$ if there exists a partition $\{A_i\}_{i\in B}$ of the set A and a family of functions $\{c_{[f]_P}\}_{f\in F}$ mapping B to A that satisfies the following conditions:

(26) $c_{[f]_P}(i) \in A_i$ *for each* $i \in B$,

(27) *for every* $f \in F$ *and* $a_i \in A_{k_i}, i = 0,\ldots,\tau(f)-1$, $f^{\mathfrak{A}}(a_0,\ldots,a_{\tau(f)-1}) = c_{[f]_P}(f^{\mathfrak{B}}(k_0,\ldots,k_{\tau(f)-1}))$,

(28) *if* $f \in [g]_P$, *then* $c_{[f]_P}(i) = c_{[g]_P}(i)$, *for every* $i \in B$.

Using the above definition, it is easy to verify that the following properties hold.

Fact 1. *1. The equivalence relation* $\sim$ *induced on* A *by the family* $\{A_i\}_{i\in I}$ *is a congruence of the algebra* $\mathfrak{I}_D$, *and the algebra* $\mathfrak{I}_D/\sim$ *is isomorphic to* $\mathfrak{I}$.

2. *If the algebras $\mathfrak{J}$ and $\mathfrak{I}$ are isomorphic and $\phi{:}J \longrightarrow I$ is the corresponding isomorphism, then there exists a P-dispersing system D such that $\mathfrak{I}_D$ is a P-dispersion of the algebra $\mathfrak{J}$.*

Let us note, following Płonka, that if for a given algebra $\mathfrak{A} = \langle A, F_{\mathfrak{A}}\rangle$ we consider the following P-dispersing system $D = \langle P, \mathfrak{A}, \{\{a\}\}_{a\in A}, \{c_{[f]_P}\}_{f\in F}\rangle$, where for every $f \in F$, $c_{[f]_P}$ is the identity function on the set A, then the algebra obtained in this way is isomorphic to the algebra $\mathfrak{A}$.

For a variety V of type τ, let $D_P(V)$ denote the class of all P-dispersions of algebras with V. If the partition P contains only single-element blocks, then instead of 'P-dispersion' we say simply 'dispersion' and write c_f instead of $c_{[f]_{Ex}}$ for $f \in F$.

As already mentioned, one of the basic properties of the P-dispersion operator is the preservation of P-compatible identities. The following holds:

Lemma 1 (Płonka 1990b)**.** *Let $\mathfrak{A}$ be P-dispersion of an algebra $\mathfrak{B}$ of the type τ. Then the algebra $\mathfrak{A}$ satisfies all P-compatible identities satisfied in the algebra $\mathfrak{B}$.*

Let us consider an identity $\phi \approx \psi$ and let it be a P-compatible identity satisfied in $\mathfrak{B}$ and ϕ, ψ be n-ary terms. The following cases may happen:

1) $\phi \approx \psi$ is an identity of the form $x \approx x$. Then, of course, it is satisfied in algebra $\mathfrak{A}$.

2) $\phi \approx \psi$ is an identity of the form

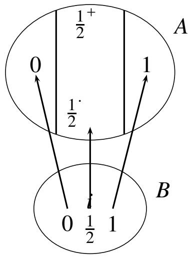
$$f(\phi_0, \ldots \phi_{\tau(f)-1}) \approx g(\psi_0, \ldots \psi_{\tau(g)-1}).$$

Let us take $a_k \in A_{i_k}(k = 0, \ldots, n-1)$. We know that the identity $\phi \approx \psi$ is satisfied in the algebra $\mathfrak{B}$ and the outermost symbols of operations in terms ϕ and ψ belong to the same block of the partition P. But from that we obtain: $c_{[ex(\phi)]_P}(\phi^{\mathfrak{B}}(i_0, \ldots, i_{n-1})) = c_{[ex(\psi)]_P}(\psi^{\mathfrak{B}}(i_0, \ldots, i_{n-1})) = \psi^{\mathfrak{A}}(a_0, \ldots, a_{n-1})$.

It turns out that not only an algebra that is a dispersion of a certain algebra from a given class satisfies all P-compatible identities satisfied in that algebra. A stronger statement is also true. Namely:

Theorem 2 (Płonka 1990b)**.** *The variety of algebras V is satisfied only by P-compatible identities iff it is closed under P-dispersions of algebras from the class V.*

Consider the following example.



Example 2.1. Let algebra $\mathfrak{A} = \langle\{0, \frac{1}{2}^{+}, \frac{1}{2}^{\cdot}, 1\}; +, \cdot, ^{-}\rangle$ is the following dispersion of the algebra $\mathfrak{B} = (\{0, \frac{1}{2}, 1\}; +, \cdot, ^{-})$:

$c_{+}(k) = c_{\cdot}(k) = c_{-}(k) = k$, for $k \in \{0, 1\}$,
$c_{+}(\frac{1}{2}) = c_{-}(\frac{1}{2}) = \frac{1}{2}^{+}$,
$c_{\cdot}(\frac{1}{2}) = \frac{1}{2}^{\cdot}$.

As one can see $\overline{\overline{\frac{1}{2}^{\cdot}}} = \frac{1}{2}^{+}$. Thus, the identity $\overline{\overline{x}} \approx x$ is not satisfied in the algebra $\mathfrak{A}$. It can be shown that this algebra satisfies all externally compatible identities true in the class determined by externally compatible the so-called **MV** algebras (constituting semantics for many-valued logics).

3. Płonka sums and regular identities

Jerzy Płonka proposed the structure of his sums in Płonka (1967). The main aim of his construction was to analyze classes of equations defined by the so-called *regular identities*, i.e., identities in which variables appearing on both sides are the same. This notion has many connections. So, in universal algebra it is the main aim mentioned above which consists in the analysis of equational classes determined by regular identities (see Płonka 1967; Graczyńska 1990). Regular identities have recently been used to conduct semantic research related to the so-called variable inclusion logic (see, for example, Bonzio, Paoli and Baldi 2022).

Let us recall the concept of a 'direct system' (see, for example, Płonka 1967). We will assume that the set F of basic operations has a form $\{f_t\}_{t \in T}$ with some set of indices T.

Definition 3.1. A *direct system* is a triple $\mathbf{A} = \langle\langle I, \leqslant\rangle, \{\mathfrak{A}_i\}_{i \in I}, \{\varphi_{ij} : i, j \in I \,\&\, i \leqslant j\}\rangle$, where:

1. $\langle I, \leqslant\rangle$ is a partially ordered set with the ordering relation $\leqslant$,
2. For every $i \in I$, $\mathfrak{A}_i = \langle A_i, \{f_t^{\mathfrak{A}_i}\}_{t \in T}\rangle$ is an algebra and all elements of the family $\{\mathfrak{A}_i\}_{i \in I}$ consists of algebras of the same type.

3. For every $i, j \in I$ where $i \leqslant j$, a homomorphism $\varphi_{ij} : \mathfrak{A}_i \longrightarrow \mathfrak{A}_j$ is given and the resulting set $\{\varphi_{ij} : i, j \in I \,\&\, i \leqslant j\}$ of homomorphisms fulfils the following conditions:
   (a) for every $i, j, k \in I$, whenever $i \leqslant j \leqslant k$, $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$,
   (b) for every $i \in I$, φ_{ii} is the identity homomorphism on $\mathfrak{A}_i$.

If we have the so-called simple system, we can introduce the notion of *Płonka sum*. In the context of Płonka sum only finitary operations are considered but without nullary basic operations. Additionally, the set I fulfils the requirement that every its two elements have a least common upper bound while carriers of the algebras from $\{\mathfrak{A}_i\}_{i \in I}$ are mutually disjoint (otherwise one should consider some isomorphic copies of them).

Definition 3.2 (Płonka 1967)**.** Let $\mathbf{A} = \langle \langle I, \leqslant \rangle, \{\mathfrak{A}_i\}_{i \in I}, \{\varphi_{ij} : i, j \in I \,\&\, i \leqslant j\} \rangle$ be a direct system. A *Płonka sum of the direct system* $\boldsymbol{A}$, denoted as $\mathbf{S}(\mathbf{A})$, is an algebra $\mathfrak{A} = \langle \bigcup_{i \in I} A_i ; \{f_t^{\mathfrak{A}}\}_{t \in T} \rangle$ similar to algebras from $\{\mathfrak{A}_i\}_{i \in I}$, where:

- for every $t \in T$, for arbitrary $a_1, \ldots, a_{\tau(f_t)} \in \bigcup_{i \in I} A_i$:

$$f_t^{\mathfrak{A}}(a_1, \ldots, a_{\tau(f_t)}) := f_t^{\mathfrak{A}_{i_0}}(\varphi_{i_1 i_0}(a_1), \ldots, \varphi_{i_{\tau(f_t)} i_0}(a_{\tau(f_t)})),$$

  where for any $1 \leq j \leq \tau(f_t)$, $a_j \in A_{i_j}$, where $i_j \in I$ and i_0 is the least upper bound in $\langle I, \leqslant \rangle$ of the set $\{i_1, \ldots, i_{\tau(f_t)}\}$.

We can see, therefore, that if we want to 'calculate' the result of an operation in the Płonka sum, we have to use homomorphisms leading from the algebras from which the arguments originate to their upper bound, and calculate this result in the algebra indexed by this upper bound. Thus, if we consider two terms that have different sets of variables in them, their realization in the Płonka sum will give different elements. Therefore, no equality that is not regular can be satisfied in the Płonka sum. Furthermore, it can be seen that the Płonka sum of algebras from a fixed class will satisfy all regular equalities satisfied in every algebra of that class.

Let us now give a simple example of Płonka's sum. Consider three two-element Boolean algebras $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3$ and a partially ordered set $I = \{i_1, i_2, i_3\}$, in which the elements i_1, i_2 are incomparable, and i_3 is their upper bound. In this case, homomorphisms have to be functions that transform the zeros of one algebra into the zeros of another algebra, analogously with ones. It can be verified that the mentioned absorption is not satisfied (e.g., $(\mathsf{0}_1 + (\mathsf{0}_1 \cdot \mathsf{0}_2)$ is not equal to $\mathsf{0}_1$).

The following statement is true.

Theorem 3 (Płonka 1967)**.** *If* $\boldsymbol{A} = \langle\langle I, \leqslant\rangle, \{\mathfrak{A}_i\}_{i\in I}, \{\varphi_{ij} : i, j \in I \,\&\, i \leqslant j\}\rangle$ *is a direct system of algebras, where for every two elements of I the least upper bound exists in I with respect to* $\leqslant$*, containing at least two algebras, then in the algebra* $\boldsymbol{S}(\boldsymbol{A})$ *all regular equations satisfied in all algebras from* $\{\mathfrak{A}_i\}_{i\in I}$ *are satisfied, whereas every other equation is false in* $\boldsymbol{S}(\boldsymbol{A})$.

4. Conclusion

As we can see, both constructions satisfy the predefined identities while they are 're-jecting' those identities that are not to be satisfied. And although these are different constructions, certain similarities can be observed. In each construction, in order to refute certain identities, it is necessary to calculate the value of terms in another 'world', another algebra, in which, thanks to certain functions, we are able to differentiate the values of the terms that compose a given equality. It is crucial not only to indicate another world that is 'sensitive' to the aspects that interest us, but also to select the correct functions. In the case of P-dispersion, these functions 'remember' the block from which the outermost symbol of the operation originates; in the case of Płonka's sums, these are homomorphisms of algebras.

For a few years, I participated in a seminar led by Prof. J. Płonka. During lectures and discussions, it was clear that Professor could "see" the models under consideration, and his lack of sight did not interfere with this "vision". This brings to mind the analogous ability of Beethoven, who, despite almost complete deafness, continued to compose, using his inner musical sense and composed his outstanding works.

References

Bonzio Stefano, Francesco Paoli, Michele Pra Baldi. 2022. *Logics of variable inclusion*. Vol. 59 of *Trends in Logic* Springer. https://doi.org/10.1007/978-3-031-04297-3

Chromik Wiesława. 1990. "On externally compatible identities of algebras". *Demonstratio Mathematica* 23(2): 345–355.

Chromik Wiesława, Katarzyna Hałkowska. 1991. "Subvarieties of the variety defined by externally compatible identities of distributive lattices". *Demonstratio Mathematica* 24(1–2): 345–350.

Gajewska-Kurdziel Katarzyna, Krystyna Mruczek-Nasieniewska. 2007. "The Lattice of Subvarieties of the Variety Defined by Externally Compatible Identities of Abelian Groups of Exponent *n*". *Studia Logica* 85(3): 361–379. https://doi.org/10.1007/s11225-007-9038-9

Graczyńska E. 1990. "On normal and regular identities". *Algebra Universalis* 27(3): 387–397. https://doi.org/10.1007/BF01190718

Graczyńska Ewa. 1983. "On regular identities". *Algebra Universalis* 17(1): 369–375. https://doi.org/10.1007/BF01194544

Hałkowska Katarzyna, Barbara Cholewińska, Ramona Wiora. 1997. "Externally compatible identities of Abelian Groups". Acta Universitatis Wratislaviensis 1890. *Logika* 17: 163–170. 1997.

Hałkowska Katarzyna. 1998. "Lattice of equational theories of *P*-compatible varieties". In: *Logic at Work. Essays dedicated to the memory of Helena Rasiowa*, ed. Ewa Orlowska. Title Studies in Fuzziness and Soft Computing. Heidelberg–New York: Springer. pp. 587–595.

Ježek Jaroslav. 1981a. "The lattice of equational theories. Part I". *Czechoslovak Mathematical Journal* 31: 127–157.

Ježek Jaroslav. 1981b. "The lattice of equational theories. Part II". *Czechoslovak Mathematical Journal* 31: 573–603.

Ježek Jaroslav. 1982. "The lattice of equational theories. Part III". *Czechoslovak Mathematical Journal* 32: 129–164.

Jónson Bjarni. 1968. "Equational classes of lattices". *Mathematica Scandinavica* 22: 187–196.

Jónson Bjarni, Ivan Rival. 1979. "Lattice varieties covering the smallest non-modular lattice variety". *Pacific Journal of Mathematics* 82(2): 463–478. https://doi.org/10.2140/pjm.1979.82.463

Lampe William A. 1986. "A property of the lattice of equational theories". *Algebra Universalis* 23: 61–69. https://doi.org/10.1007/BF01190912

Lee Kee-Beng. 1970. "Equational classes of distributive pseudo-complemented lattices". *Canadian Journal of Mathematics* 22(4): 881–891. https://doi.org/10.4153/CJM-1970-101-4

McKenzie Ralph N. 1972. "Equational bases and non-modular lattice varieties". *Transactions of the American Mathematical Society* 174: 1–43. https://doi.org/10.2307/1996095

McNulty George F. 1981. "Structural diversity in the lattice of equational theories". *Algebra Universalis* 13: 271–292. https://doi.org/10.1007/BF02483841

Mel'nik I. I. 1971. "Normal closures of perfect varieties of universal algebras". *Ordered sets of lattices* 14: 56–65. [In Russian].

Mel'nik I. I. 1973a. "Nilpotent shifts of varieties". *Mathematical Notes* 14(5): 962–966. Eng. translation of Mel'nik (1973b). https://doi.org/10.1007/BF01462258

Mel'nik I. I. 1973b. "Nilpotent shifts of varieties". *Matematicheskie Zametki* 14(5): 703–712. [In Russian].

Mruczek Krystyna. 2000. "On some lattice of varieties related to changes of the type". In: *General Algebra and Applications. Proceedings of the 59th Workshop on General Algebra*, ed. Klaus Denecke, Hans-Júrgen Vogel. Aachen: Shaker Verlag. pp. 147–153.

Nelson Evelyn. 1971a. "The lattices of equational classes of commutative semigroups". *Canadian Journal of Mathematics* 23(5): 875–895. https://doi.org/10.4153/CJM-1971-098-0

Nelson Evelyn. 1971b. "The lattices of equational classes of semigroups with zero". *Canadian Mathematical Bulletin* 14(4): 531–535.

Ogonowski Jerzy. 2020. "Zmarł Jerzy Płonka [Jerzy Płonka has died]" https://pzn.org.pl/zmarl-jerzy-plonka/. Accessed: 2025-07-09. [In Polish].

Płonka Jerzy. 1967. "On a method of construction of abstract algebras". *Fundamenta Mathematicae* 61(2): 183–187.

Płonka Jerzy. 1969. "On equational classes of abstract algebras defined by regular equations". *Fundamenta Mathematicae* 64: 241–147.

Płonka Jerzy. 1974. "On the subdirect product of some equational classes of algebras". *Mathematische Nachrichten* 63: 303–305.

Płonka Jerzy. 1988. "On varieties of algebras defined by identities of some special forms". *Houston Journal of Mathematics* 14: 253–263.

Płonka Jerzy. 1990a. "Biregular and uniform identities of algebras". *Czechoslovak Mathematical Journal* 40(3): 367–387.

Płonka Jerzy. 1990b. "*P*-compatible identities and their applications to classical algebras". *Mathematica Slovaca* 40(1): 21–30.

Płonka Jerzy. 2001. "The Lattice of Subvarieties of the Biregularization of the Variety of Boolean Algebras". *Discussiones Mathematicae General Algebra and Applications* 21: 255–268.

Szczepański Henryk. 1999. "Z modlitwą, muzyką i algebrą [With prayer, music, and algebra]". *Gość Niedzielny* 76(49): 19. [In Polish].