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A History of Positional Logic

Abstract: This paper offers a comprehensive historical study of positional logic, a branch of logic that is closely intertwined with temporal and epistemic logic. Given its significance in the realm of philosophical logic, there is a notable absence of a thorough historical analysis of this subject. To address this gap, our work aims to provide a detailed examination of the most important systems of positional logic, highlighting key results in the field, and offering a comprehensive list of publications that have contributed to the development of positional logic.

Keywords: positional logic, topological logic, temporal logic, epistemic logic

Introduction

Positional logic is a field with a rich history spanning almost eighty years, connected to the emergence of temporal epistemic logics. At its core, positional logic relies on the concept of a realization operator, which links the name of a proposition with a specific context or position. This operator enables the evaluation of the truth value of a proposition within some context, providing a framework to analyze propositions in relation to the context in which they are asserted.

The foundations of positional logic were laid by Polish logician Jerzy Łoś in 1947, with his pioneering work presenting the first system of positional logic. Remarkably, this system also served as the earliest instance of a temporal logic. In a subsequent work, Łoś extended the scope of positional logic by employing it to interpret systems of belief, effectively establishing it as the first formal system of epistemic logic.

The groundbreaking results achieved by Łoś in the field of positional logic remained largely unknown to non-Polish speaking logicians until the publication of short reviews by Roman Suszko and Henryk Hiż. Thanks to their efforts, positional logic gained recognition and began to be disseminated by influential figures such as Arthur Prior and Nicholas Rescher. However, after the last works on positional logics in the 1970s, the field experienced a dormant period lasting nearly forty years. This period of inactivity ended with a resurgence of interest, driven by the efforts of logicians associated with Toruń Logic Group.

Recognizing the significance of positional logic and the lack of comprehensive historical studies on the subject, our publication aims to address this gap and provide a thorough exploration of its history. Our work is organized into five sections. The first section serves as an introductory chapter that elucidates the fundamental concepts and principles underlying positional logic. It analyzes the philosophical ideas that motivated the development of positional logic and showcases its diverse applications in various branches of philosophy.

The second section of our work is dedicated to exploring the pioneering contributions of Jerzy Łoś in the early history of positional logic. In this section, we analyze Łoś's seminal works on the subject, highlighting his original systems of logic, his motivations for their creation, and their profound impact on the field. We aim to underscore the fact that Łoś's groundbreaking work not only established the foundations of positional logic but also introduced the first systems of temporal and epistemic logics.

The third section of our work focuses on the significant contributions of Arthur Prior to positional logic. Given Prior's dual role as both a continuator and a critic of Łoś's work, his involvement in the development of positional logic is particularly intriguing. We sought to capture both aspects of Prior's contribution, comparing the axiom systems of his positional logics while also providing an accurate presentation of his arguments against the use of positional logic as a framework for temporal logic. Despite Prior's criticisms of Łoś's positional logic, he himself developed two systems funded on Łoś's logic and outlined potential applications. In doing so, Prior expanded the scope of positional logic beyond the epistemology and philosophy of time, exploring new possibilities for interpretation and application.

In the subsequent section, we analyze the contributions of Nicholas Rescher, who drew direct inspiration from Prior's work on positional logic. Rescher's significant impact on the field cannot be overstated. He extensively investigated Łoś's assertion logic and formulated several systems of assertion logic based on Łoś's work. Furthermore, Rescher conducted extensive research on the role of positional logic in temporal logic. Recognizing the expressive power of the realization operator, he developed an abstract system of topological logic, which can be viewed as a direct precursor to the logic developed by Jarmużek and Pietruszczak. Logicians associated with Rescher's work were among the last to contribute to the study of positional logic in the 1970s until its resurgence.

In the final section of this paper, we inspect the latest advancements in positional logic, particularly those closely associated with the logic hub in Toruń. Firstly, we highlight the resurgence of research on positional logic pioneered by Jarmużek. We then explore the subsequent developments in positional logics, with a particular focus on their metalogical aspects. Next, we explore the historical research conducted by contemporary logicians, shedding light on the evolving understanding of positional logic within its historical context. Lastly, we conclude this section by showcasing the wide-ranging applications of positional logic in recent works, demonstrating its relevance across various fields of study.

In this paper, we present multiple formal systems, preserving their original notation and symbols to reflect their historical context. This includes the realization operator, sentence variables, metavariables, and other elements as they originally appeared. As a result, there may be inconsistencies between these systems. Furthermore, certain components, such as the rule of substitution, might appear to be absent, for example, where propositional variables are used. These inconsistencies are intentional and should be understood as serving historical purposes rather than logical ones.

1. The logic of context

Modern logic focuses primarily on three types of expression: propositions, predicates, and names. The most basic truth-bearing structure is the truth value of a proposition, which falls within the scope of propositional logic. Among the various propositional logics, classical propositional logic is the most well-known. There are also other types of propositional system, such as propositional systems of modal logic and propositional systems of tense logic, that deal with truth values in specific aspects.

Through more in-depth linguistic analysis, it has become possible to identify a more fundamental structure that consists of a predicate and one or more names. First-order logic, along with its nonclassical variants, provides us with the necessary tools to study these structures in greater detail. By using these tools, we can gain a deeper understanding of the relationships between predicates and names and explore the various ways in which they can be combined to form meaningful statements.

The two classes of logic, propositional logic and first-order logic, address distinct syntactic categories. Propositional logic primarily focuses on sentences as its category, whereas first-order logic encompasses names combined with sentence-forming functors (predicates) and operators (e.g. quantifiers). By integrating these two approaches, we arrive at positional logic, which deals with expressions constructed from a sentence, one or more names, and a sentence-forming operator. These expressions embody the essence of positional logic, which aim to situate sentences within specific contexts or positions (hence the name). Through an examination of the relationships between sentences and names, positional logic offers a nuanced and sophisticated framework for analyzing the intricate structure of language.

To understand this design, it is essential to analyze the difference between propositions and propositional functions (or indefinite propositions). Propositions, or sentences in the Fregean sense, express complete thoughts and are therefore capable of bearing truth values[1]. In contrast, propositional functions have an incomplete sense and cannot bear truth values without closure or transformation into a full-fledged sentence. This means that propositional functions must be instantiated with specific arguments to become propositions. Through this process, propositional functions become complete expressions that can bear truth values and be analyzed using propositional logic.

One of the first definitions of the mentioned term appeared in *Principia Mathematica*[2]:

> By a "propositional function" we mean something which contains a variable x, and expresses a proposition as soon as a value is assigned to x. That is to say, it differs from a proposition solely by the fact that it is ambiguous: it contains a variable of which the value is unassigned.

From a syntactic perspective, a propositional function is an expression that contains at least one variable. To transform a propositional function into a proposition, all occurrences of each variable within the expression must be substituted with spe-

[1] Matthew McGrath, Frank Devin, "Propositions", *The Stanford Encyclopedia of Philosophy*, Winter 2020 Edition, access 9.07.2023, https://plato.stanford.edu/archives/win2020/entries/propositions.

[2] Bertrand Russell, Arthur N. Whitehead, 1910, *Principia Mathematica*, (Cambridge: Cambridge University Press, 1910), 38.

cific values. These variables represent the incomplete portion of a propositional function, which must be instantiated with specific values to become a proposition.

The difference between a proposition and a propositional function can also be described using semantics. In a possible world semantics, a proposition generates a set of possible worlds in which it is true. In contrast, for a propositional function, we cannot point to a specific set of possible worlds. This is because the referent of a propositional function is not fixed, but rather it is a function that ranges over a power set of possible worlds[3]. The referent of a propositional function is incomplete without specific arguments, and thus the set of possible worlds in which it is true depends on the arguments with which it is instantiated.

Consider the expression "x is green". This expression is a propositional function because it contains at least one variable, in this case, the variable x. Additionally, the variable x is free in all occurrences within the specified scope. Therefore, without specifying what object the variable refers to, the expression cannot bear a truth value. Not all propositional functions contain apparent variables, as is the case with the example of "x is green". In natural language, many propositional functions contain a disguised variables. This means that while any variable does not occur within the expression explicitly, the truth value of the proposition could still depend on some unspecified element. For example, consider the proposition "The cat is green". Although there is no explicit variable within the expression, it is possible that the cat being referred to is indeed green at a certain time or place. The cat may have been dyed green, or it may have changed color due to the onset of mating season. Therefore, even expressions that do not contain explicit variables can still be a propositional function that require some element to be specified in order to become a complete proposition capable of bearing a truth value.

The object of positional logic can be understood as the set of propositions expanded to include indefinite statements, which are often presented in the guise of standard propositions. Such propositions may appear to be correctly formed sentences, but upon further analysis, it becomes evident that they lack some key information required to determine their truth value. If we do not restrict ourselves to a specific type of indefiniteness, we can say that every proposition is essentially a propositional function. Treating each proposition as a propositional function aligns well with the principles of positional logic, as it allows us to evaluate both definite and indefinite propositions, within any possible context. In some cases, this approach may not introduce new information; for instance, the sentence "It is raining in Toruń

[3] Edwin Mares, "Propositional Function". *The Stanford Encyclopedia of Philosophy*, Winter 2019 Edition, access 9.07.2023, https://plato.stanford.edu/archives/win2019/entries/propositional-function/.

on the 13th of November 2024" does not gain additional meaning when considered in the context of the date "13th of November 2024", as this information is already embedded in the sentence. However, even for this example, we can identify an infinitely large set of contexts not explicitly accounted for, such as the specific time of the observation.

To account for this, the syntactic structure of positional logic includes a way to bind a proposition to a context or position. This takes the form of $\mathcal{R}_\alpha A$, where $\mathcal{R}$ is a sentence-forming operator, α is a name, and A is a proposition. The realization operator effectively 'binds' the sentence A to the position indicated by α, forming a complete proposition from the components. In other words, the expression $\mathcal{R}_\alpha p$ indicates that the proposition p is realized in the context or position represented by α[4]. From the fact that the context or a position represented by α is fundamental for truth value of an expression, came the name of the family of logics.

The flexibility and generality of these systems come from the fact that the range of α is not fixed[5], allowing a wide range of applications. Rescher and Urquhart provided a list of possible applications to illustrate this point[6]:

> Here α may be any element of a range of positions. These may be spatial positions indicated by Cartesian coordinates, or by any positional scheme such as seat-numbers in a lecture hall. Or again, the positions at issue may be temporal, with α ranging over the integers (for days or years) or over the real numbers (for a more refined scheme of dating).

From a logical perspective, the structure $\mathcal{R}_\alpha A$ entails treating the sentence denoted by A as a propositional function with respect to the context α. This is true even if the proposition A appears to be complete.

Due to the fact that A is considered an incomplete proposition or a propositional function, it cannot be assigned a logical value. In order to determine its truth value, it needs to be complemented with a context. This context, along with the propositional function, is transformed into a complete proposition capable of bearing a truth value. However, there are some exceptions to this rule. For instance, if A is an expression that is invariably true regardless of the context, such as a logical law, then we can simplify the formula as follows: $\mathcal{R}_\alpha A \leftrightarrow A$. Since the truth value of such

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[4] Nicholas Rescher, Alasdair Urquhart, *Temporal Logic*, (New York: Springer Verlag, 1971), 13.

[5] At least not anymore. The first systems of positional logic were created for some specific purposes, like formalizing logic of time or modelling some epistemic notions. In those cases, range of α was fixed. The generality of positional logic was developed after years of studies.

[6] Rescher, Urquhart, *Temporal Logic*, 13.

an expression remains the same regardless of the context, we should not treat it as a propositional function.

The first systems of positional logic were developed in the 1940s by the Polish logician Jerzy Łoś. Although he used a different symbol for the realization operator than the commonly used $\mathcal{R}$ today, his work laid the foundation for the development of this branch of logic. Over the years, different authors used various symbols to denote the realization operator. The symbol used by Nicholas Rescher in his later work, $\mathcal{R}$, has become the standard notation for the operator. Rescher's choice of symbol reflects the meaning and function of the operator, as it stands for 'realization'. Therefore, the operator is commonly known as the realization operator. This notation has endured over the decades and is still in use today, particularly in the Toruń Logic Group where Łoś's ideas continue to be influential.

Rescher coined two names for this branch of logic: positional logic and topological logic. The former name emphasizes the most fundamental syntactic structure and semantic relations of the logic, while the latter focuses on the semantics and applications for modeling reasoning about topological structures. As Rescher demonstrated, topological logics can be used to model reasoning about Euclidean and non-Euclidean geometries[7]. Although both names capture different aspects of the logic, positional logic has persisted throughout the years and is the name used in this publication.

Considering established systems of positional logic, we can define positional logic in a broader sense as any logical system in which the realization operator is included in the language and positional expressions are part of the well-formed formulas. However, we can further narrow this definition to identify positional systems in a stricter sense - those that not only fall within the broader category but also meet the following specific conditions. Firstly, among the various positional logics, we can identify those in which the realization operator adheres to the foundational intuitions established by its original proponents. Secondly, from a historical perspective, we can further narrow the scope to systems that are directly linked to or influenced by the works of Łoś, Prior, and Rescher.

The realization operator is a central feature of positional logic. In order to be considered a realization operator in the standard sense, the symbol must have been used in the same syntactic category as those found in other positional logics. Specifically, it must be a sentence-forming operator that takes one or more names and a sentence as arguments.

[7] Ibidem, 22.

The second condition takes into account the axiom schemes. There has been a consensus regarding the basic properties of positional logic since the first works of Łoś, Nicholas Rescher on topological logics, and Jarmużek on **MR**[8]. The simplest systems are generated by the following axiom schemes.

Axiom 1. *$e(A)$, if A is a tautology of* CPL *(Classical Propositional Logic) and e is a mapping from language of* CPL *to language of* **MR**.

Axiom 2. $\mathcal{R}_\alpha \neg A \leftrightarrow \neg \mathcal{R}_\alpha A$

Axiom 3. $\mathcal{R}_\alpha (A \wedge B) \rightarrow \mathcal{R}_\alpha A \wedge \mathcal{R}_\alpha B$

Axiom 4. *$\mathcal{R}_\alpha A$, if A is a tautology of* CPL.

The last axiom was present in the early systems of positional logics as a rule of inference:

Rule (RG). *If $A \in$* CPL

$$\frac{A}{\mathcal{R}_\alpha A}$$

In the above axioms schemes and rules, A is a metavariable that stands for any formula in the language.

The historical condition we impose on positional logics in the narrower sense serves to distinguish "standard" positional logics from systems that may have developed independently within different logical traditions. This distinction is particularly relevant in the case of Prior, where positional logic emerged somewhat as a byproduct of his reflections on time and modality, though still in dialogue with the philosophical views of Łoś. When a work does not explicitly engage with the tradition of Łoś, Prior, and Rescher, it will not be considered a positional logic in the narrower sense and, therefore, will be excluded from this discussion. However, it should be noted that this study could be expanded in the future to encompass such works.

To understand the motivation behind the development of positional logic, we need to ask: why was the realization operator introduced in the first place? As we have previously mentioned, the main idea behind the operator is to highlight the crucial role of context in determining the truth value of a sentence. Specifically, the operator combines a sentence with a context to form a new proposition that incorporates both components. Therefore, the truth value of the sentence is not evaluated

[8] It should be precised here, that this is true for normal positional logics. There have been some developments, especially by Tkaczyk and Karczewska, in creating such logics that do not satisfy this condition, but they are still being considered positional logics.

in isolation but rather in the context in which it is being considered. Until a specific interpretation of the context α is given, it can be seen as an arbitrary context.

The initial purpose of creating positional logic was to leverage the possibilities offered by the realization operator, which binds the truth value of a sentence to a specific context, to solve particular problems. Those problems were mainly connected to formalization of modal notions such as time and knowledge. And solutions to that problems, were a beginning of non-classical logics, especially temporal and epistemic logics.

2. Jerzy Łoś and the beginning of positional logic

Jerzy Łoś was a renowned Polish logician who made significant contributions to the field of model theory and algebra. One of his most notable achievements is the fundamental theorem on ultraproducts, which is now known as Łoś theorem. Along with it, he co-authored the Tarski-Łoś preservation theorem. However, in addition to these accomplishments, Łoś also pioneered the first systems of positional logic. More importantly, he created the first systems of temporal logic and epistemic logic using this specific formal framework. The current section aims to present the origin of this branch of logic in the works of Jerzy Łoś, his motivations, and the impact he had on the history of logic.

Jerzy Łoś, born on March 22, 1920, in Lvov, had a notable philosophical career that was heavily influenced by representatives of the Lvov-Warsaw School of Philosophy. However, World War II disrupted his scientific career. After the war, he resumed his studies and began working at various universities, including Lublin, Warsaw, and Wrocław[9]. It is worth highlighting that Łoś made his most significant discoveries in logic during the late 1940s, just after the war. It was during this period that he wrote two papers which established his influence on the field of philosophical logic:

- "Podstawy analizy metodologicznej kanonów Milla" ("Foundations of the methodological analysis of Mill's canons") that was his master thesis from 1947;
- "Logiki wielowartościowe a formalizacja funkcji intensjonalnych" ("Multi-valued logics and the formalism of intensional functions") from 1948.

Both of Łoś's works utilize the framework of positional logic. However, they differ in several respects, such as philosophical motivation, applications of the formal

[9] Tomasz Jarmużek, Tomasz Kupś, "The Heritage of Jerzy Łoś's Philosophical Logic and the Polish Question. An Introduction to the Volume", *Studia z Historii Filozofii* 11(3) (2020): 8.

system, and formal properties of the two positional logics. What is interesting is that, in addition to those two works, Łoś did not write any other paper on the subject of positional logic. As both works were initially printed in Polish exclusively, the only way for international scientists to acknowledge those results were by the reviews by Suszko[10] and Hiż[11].

The first of the papers written by Łoś dealt with the formalization of Mill's canons. He believed that the formalization should embed the temporal aspect of these methodological tools. To do so, he developed a logic of physical time that is regarded as the first temporal logic[12]. In the second paper, Łoś explored a logic of assertions that considered the relationship between a cognitive agent and a proposition.

The two papers mentioned are significant contributions to the history of logic. Due to the fact that both papers were originally written in Polish, their reception in the international community was very limited. As a result, the only way for English-speaking readers to become familiar with the key ideas presented in these works was through short reviews. The review of "Podstawy Analizy Metodologicznej Kanonów Milla" was published in the *Journal of Symbolic Logic* in 1951 by Henryk Hiż, while the review of the latter paper was written by Roman Suszko and published in the same journal in 1949.

Considering the dates and the context of rapid development in formal logic, the time that elapsed between the publication of Łoś's articles and the publication of the reviews was long enough for the topics to lose their importance in current discussions. However, this was not the case, as a few years later, temporal logic and epistemic logic became mainstream concerns. Despite Łoś's pioneering work on developing the first formal system of temporal logic, his place in the history of temporal logic remains the subject of ongoing debate according to modern standards, as in "Jerzy Łoś Positional Calculus and the Origin of Temporal Logic", "The Significance of the Contributions of A. N. Prior and Jerzy Łoś in the Early History of Modern Temporal Logic", "The Influence of Jerzy Łoś on Early Developments in Temporal Logic". Before this discussion appeared, there was almost no reference to Łoś in works devoted to temporal logics[13].

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[10] Roman Suszko, "Review: Jerzy Łoś, Many-Valued Logics and Formalization of Intensional Functions", *Journal of Symbolic Logic* 14(1) (1949): 64–65.

[11] Henryk Hiż, "Review of: J. Łoś, Podstawy Analizy Metodologicznej Kanonów Milla", *Journal of Symbolic Logic* 16(1) (1951): 58–59.

[12] Tomasz Jarmużek, Marcin Tkaczyk, "Jerzy Łoś Positional Calculus and the Origin of Temporal Logic", *Logic and Logical Philosophy* 28 (2019): 259.

[13] Jarmużek, Kupś, "The Heritage of Jerzy Łoś's Philosophical Logic and the Polish Question. An Introduction to the Volume": 11.

2.1. The first temporal logic

In “Podstawy Analizy Metodologicznej Kanonów Milla”, Łoś stated at the outset that the purpose of his work was to provide a formal system as a tool for the methodological analysis of Mill’s canons. Why did he choose this particular methodological tool? It appears that his research was influenced by the mindset of the Lvov-Warsaw School. This mindset was characterized by studying classical philosophical texts, identifying problems described therein, and attempting to solve them by clarifying the language used by the author.

In this specific work, we can see the same pattern. Łoś was studying J.S. Mill’s text and discovered that the author postulated that the canons are dependent on causality. However, Łoś noticed that Mill did not justify his claim, nor was this issue resolved by later authors. Additionally, Łoś assessed that there was no formal framework available at the time that could assist in investigating such a connection[14]. This is aptly described by the following quotation:

> Mill formulated a hypothesis (or rather a thesis) about the dependence of his canons on the principle of causality. This hypothesis has not been properly resolved so far, and we do not even have a formally correct language to consider this relationship[15].

The problem described here led Łoś to the idea of constructing a system of philosophical logic that could handle the formalization of Mill’s canons. He began his investigation by recognizing that propositions describing physical facts are in fact disguised indexical expressions or propositional functions, in which the temporal factor is not specified. He identified undefined temporal indexical expressions, such as “It is raining in Toruń”, which have a logical value that varies depending on the time they are asserted. However, classical logic does not allow us to express the connection between a sentence and the time it was asserted. To address this limitation, Łoś created his own formal system, which included three types of variables: propositional variables $(p, q, r, ...)$, time variables $(t, t_1, t_2, ...)$, and interval variables $(\eta, \eta_1, \eta_2, ...)$. In addition, he incorporated connectives from first-order logic and two specific symbols into his language: δ and $\mathcal{U}$. The former is a term-forming operator

[14] Jerzy Łoś, “Podstawy Analizy Metodologicznej Kanonów Milla”, *Annales Universitatis Mariae Curie-Skłodowska* 2(5) (1947): 269.

[15] Translated by the author from Polish: Mill postawił hipotezę (właściwie twierdzenie) o zależności swoich kanonów od zasady przyczynowości. Hipoteza ta nie jest dotychczas należycie rozwiązana, nie posiadamy nawet dotychczas poprawnego formalnie języka na gruncie którego tę zależność możnaby było rozważać.

that ranges over time and interval variables. The expression $\delta(t,\eta)$ transforms the date denoted by t by an interval denoted by η and results in a date later than t by η. It can be interpreted as "a time later than t by η". For instance, if t represents December 12, 2022, and η represents an interval of 10 days, then $\delta(t,\eta)$ denotes a time instant on December 22, 2022.

The latter, symbol mentioned, $\mathcal{U}$, is a realization operator in its classical sense. It is unclear why this particular letter was chosen by Łoś as it is not an abbreviation of how it should be read in Polish. The realization operator transforms a propositional variable and an instant variable (either a time variable or a result of the δ operator) into a proposition. Therefore, $\mathcal{U}_t p$ represents a proposition in which the time context is fixed at the date denoted by t. Expressions formed with the $\mathcal{U}$ operator were meant to be read as "p occurs at time t". For example, if p denotes the sentence "It is raining in Toruń", and t represents the time instant of December 12, 2022, the expression $\mathcal{U}_t p$ should be read as "It is raining in Toruń occurs at the time of the 12^{th} of December 2022", which could later be translated into "It is raining in Toruń on the 12^{th} of December 2022". A more detailed and accurate presentation of the language of Łoś's system from his 1947 work can be found in (Jarmużek 2019).

It's worth noting that the notation used in our presentation of the system is slightly different from the original notation. Łoś used Łukasiewicz's notation, which was common in his time due to its simplicity and economy. However, for the sake of readability and consistency with other systems of positional logic, we will use a standard modern notation. Moreover, Jarmużek and Tkaczyk noted that although Łoś employed first-order logic with quantification over all variables, he did not explicitly state the specific logic he used, nor did he mention the limitations of the operations[16].

Within this language, Łoś expressed numerous theses and definitions crucial to his goal. Rescher and Garson provided a reconstruction of Łoś core axioms schemes. The following axioms were a part of this reconstruction along with the rule **RG**[17]:

Axiom 1. $\mathcal{U}_t \neg p \leftrightarrow \neg \mathcal{U}_t p$

Axiom 2. $\mathcal{U}_t (p \rightarrow q) \rightarrow (\mathcal{U}_t p \rightarrow \mathcal{U}_t q)$

Axiom 3. $\forall_t (\mathcal{U}_t p) \rightarrow p$

Rescher and Garson observed that the original axioms of Łoś's system, could be divided into three groups: axioms concenring the $\mathcal{U}$ operator, axioms employing all

[16] Jarmużek, Tkaczyk, "Jerzy Łoś Positional Calculus and the Origin of Temporal Logic": 261.

[17] Nicholas Rescher, James Garson, "Topological Logic", in: *Topics in Philosophical Logic*, ed. Nicholas Rescher (Dordrecht: Springer, 1968), 236.

theorems of classical propositional logic, and axioms concerning the time parameter. The first group was preserved in the original version. The second group was condensed into the form of a rule mentioned **RG**. This group was originally composed of three axioms that are analogous to Łukasiewicz's axiom system for classical propositional logic[18]. And the third group was totally omitted by the authors. This last group served to govern the range of the time parameter such that t ranges over the set of real numbers. Consequently, Łoś's system made specific assumptions about time: that it is infinite and dense. The axioms of Łoś in their original shape are as follows:

Axiom 1. $\mathcal{U}_t \neg p \leftrightarrow \neg \mathcal{U}_t p$

Axiom 2. $\mathcal{U}_t (p \to q) \to \mathcal{U}_t p \to \mathcal{U}_t q$

Axiom 3. $\mathcal{U}_t ((p \to q) \to ((q \to r) \to (p \to r)))$

Axiom 4. $\mathcal{U}_t (p \to (\neg p \to q))$

Axiom 5. $\mathcal{U}_t ((\neg p \to p) \to p))$

Axiom 6. $\forall_t (\mathcal{U}_t p) \to p$

Axiom 7. $\forall_{t_1} \forall_{\eta_1} \exists_{t_2} \forall_p (\mathcal{U}_{\delta(t_1, \eta_1)} p \leftrightarrow \mathcal{U}_{t_2} p)$

Axiom 8. $\forall_{t_1} \forall_{\eta_1} \exists_{t_2} \forall_p (\mathcal{U}_{\delta(t_2, \eta_1)} p \leftrightarrow \mathcal{U}_{t_1} p)$

Axiom 9. $\forall_{t_1} \exists_p \forall_{t_2} (\mathcal{U}_{t_2} p \leftrightarrow \forall_q (\mathcal{U}_{t_1} q \leftrightarrow \mathcal{U}_{t_2} q))$

Let us denote this logic by $\mathbf{L}_1$. In terms of the metalogical properties of his logic, Łoś conducted a detailed analysis of consistency and applicability. He presented two proofs of the consistency of his logic. The first one involves trivializing the $\mathcal{U}_\alpha A$ expressions by adding the formula $\mathcal{U}_t p \leftrightarrow p$. This transforms the system into classical propositional calculus with quantifiers that bind all kinds of variables[19]. The second proof is based on the interpretation of the formulas within the real number line, points, and closed sets. The proof is also explained in more detail in (Jarmużek 2019). Next, Łoś provided two proofs demonstrating that Mill's canons could be expressed using the language of his positional logic.

In his first publication on positional logic, Łoś proposed two ways to extend his system of temporal logic for more adequate formalization of Mill's canons. The

[18] Łoś, "Podstawy Analizy Metodologicznej Kanonów Milla": 280.

[19] Jarmużek, Tkaczyk, "Jerzy Łoś Positional Calculus and the Origin of Temporal Logic": 266.

first approach was to broaden the scope of the operator $\mathcal{U}$ to include not only the time variable but also the spatial coordinates, resulting in expressions such as $\mathcal{U}_{x_1,y_1,z_1,t_1} p_1$. Here, t_1 represents a particular point in time, while x_1, y_1 and z_1 represent three-dimensional spatial coordinates. The second approach was to introduce indexical operators of realization, such as $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots$, where each realization operator is interpreted relative to a particular frame of reference. With this method, the expression $\mathcal{U}_{1t_1} p_1$ could be interpreted as "the sentence p_1 is realized in the frame U_1 at time t_1".

2.2. The first assertion logic

In his later work [20], Łoś applied the same positional logic concept to create the first system of epistemic logic[21]. Although he used the same syntactic construction, Łoś used a different symbol for the realization operator than he did previously. Instead of $\mathcal{U}$, he utilized $\mathcal{L}$. This small change in the realization operator was motivated by its semantical features. In contrast to the temporal interpretation with instant variables as a moment of time, the expression $\mathcal{L}_x p$ was now interpreted in an epistemic sense. It is important to note that Łoś's epistemic logic focuses on assertions rather than mental states, where an "assertion" is understood as a disposition to behave in a certain way. Therefore, the expression $\mathcal{L}_x p$ should be understood as "person x accepts that p"[22].

The key differences in the use of the realization operator between the two systems of Łoś are that in his logic of assertion, Łoś used the instant variables that range over persons asserting certain propositions instead of time instants. Additionally, the δ functional symbol and set of interval variables present in the previously presented $\mathbf{L}_1$ system are not included in the assertion logic.

Despite the differences between the two systems, they share some significant similarities. Both logics include the symbols of first-order logic such as sentential variables and operators of classical propositional logic. Additionally, both systems contain sets of instant variables, denoted by $x, x_1, x_2, \ldots$ and a realization operator. They also include quantifiers that bind sentential variables and name variables[23]. Furthermore, the axioms in both logics are quite similar. Although the assertion

[20] Jerzy Łoś, "Logiki Wielowartościowe a Formalizacja Funkcji Intensjonalnych", *Kwartalnik Filozoficzny* 17(1–2) (1948): 59–78.

[21] Marek Lechniak, "Jerzy Łoś's Epistemic Logic and the Origins of Epistemic Logics", *Studia z Historii Filozofii* 11(3) (2020): 22.

[22] Lechniak, "Jerzy Łoś's Epistemic Logic and the Origins of Epistemic Logics": 24.

[23] Ibidem, 24–25.

logic of Łoś does not include axioms concerning the properties of time, most of the axioms used in that system are present in the former one. Mentioned logic contains originally seven axioms, expressed as follows:

Axiom 1. $\mathcal{L}_x p \leftrightarrow \neg\mathcal{L}_x \neg p$

Axiom 2. $\mathcal{L}_x((p \to q) \to ((q \to r) \to (p \to r)))$

Axiom 3. $\mathcal{L}_x((\neg p \to p) \to p)$

Axiom 4. $\mathcal{L}_x(p \to (\neg p \to q))$

Axiom 5. $\mathcal{L}_x(p \to q) \to (\mathcal{L}_x p \to \mathcal{L}_x q)$

Axiom 6. $\forall_x \mathcal{L}_x p \to p$

Axiom 7. $\mathcal{L}_x \mathcal{L}_x p \leftrightarrow \mathcal{L}_x p$

Let us denote this system as $\mathbf{L}_2$. A quick comparison of the two axiom systems reveals that the first six axioms presented above are shared between both logics (with the exception of the axioms that concern time variables). Therefore, Rescher's and Garson's observation concerning the replacement of three Łukasiewicz's axioms by rule **RG** is still valid. Thus, we can reinterpret these axioms in a similar manner as we did in case of the system from the previous section:

Axiom 1. $\mathcal{L}_x \neg p \leftrightarrow \neg\mathcal{L}_x p$

Axiom 2. $\mathcal{L}_x(p \to q) \to (\mathcal{L}_x p \to \mathcal{L}_x q)$

Axiom 3. $\forall_x(\mathcal{L}_x p) \to p$

Axiom 4. $\mathcal{L}_x \mathcal{L}_x p \to \mathcal{L}_x p$

The majority of the axioms are shared between the two logics, with the only exception being axiom 4, which is present in $\mathbf{L}_2$ but not in $\mathbf{L}_1$. Lechniak has demonstrated $\mathcal{L}$ is indeed an intesional operator in the common sense[24].

2.3. The impact of Łoś's positional logic

As previously mentioned, both of Łoś's works were originally written in Polish, and, apart from very brief but informative reviews from Hiż and Suszko written in English, the international audience had nothing more to work with. Both reviews

[24] Ibidem, 26.

included a brief introduction to the philosophical problems that the logics were attempting to solve, as well as to the actual axiom systems. Suszko's review was published just one year after the original article by Łoś, in 1949[25]. However, Hiż's review was published three years after Łoś's work, in 1951[26]. Both works were published in the same journal, *The Journal of Symbolic Logic*.

Fortunately, Łoś's ideas caught the attention of Arthur Prior, and through Hiż's review, they had an impact on the history of tense logics. Łoś' works were cited in significant publications such as Formal Logic, Time and Modality and Past, Present and Future. What is worth mentioning here, is that Prior seemed to recognize Łoś as the creator of the first formalism for temporal logic[27] [28]:

> [...] the 1949 $\mathcal{U}_t p$ logic of Łoś was developed as a part of an attempted formalization of Mill's canons of induction. Even Łoś's logic is not, indeed, a tense-logic but rather an unanalysed date-and-interval logic, but it is at least a logic in which the time-reference is made by an operator which takes a whole "predications" as its arguments.

In addition to his formalism for temporal logic, Prior also acknowledged Łoś's contribution to epistemic logic. He admitted that Łoś was the first logician to find the appropriate formalism for this specific subject matter, stating that[29]:

> [...] he has found, as no one before him seems to have found, an appropriate symbolism for this type of subject-matter and he sees that the logic of dates and intervals and the logic of assertions both require a symbolism of this kind.

After Prior, Łoś's works inspired Nicholas Rescher, Alasdair Urquhart and James Garson and opened up new perspectives for the development and propagation of positional logic.

The widespread development of positional logic was possible due to its versatile applications beyond formalizing notions of time and knowledge. Due to the fact that the realization operator binds the name of the context to the proposition, it allows for interpretation of propositions at various reference points. This semantic feature provides a possible formal framework for many fields of philosophical investiga-

[25] Suszko, "Review: Jerzy Łoś, Many-Valued Logics and Formalization of Intensional Functions".
[26] Hiż, "Review of: J. Łoś, Podstawy Analizy Metodologicznej Kanonów Milla".
[27] Arthur N. Prior, *Time and Modality*, (Oxford: Oxford University Press, 1957): 107.
[28] Which is a more general term than "tense" logic.
[29] Prior, *Time and Modality*, 122.

tions. Consequently, the axioms and language of a base positional logic system can be adjusted to serve many purposes, as observed by Jarmużek[30].

3. Positional logic in the work of Arthur Prior

Arthur Prior was a logician from New Zealand whose most creative period in the area of logic was between the 1940s and 1970s. One of his most notable contributions was the development of tense logic, which is often (though inaccurately) considered the first temporal logic. In his analysis of time, Prior applied positional logic. He possessed a profound familiarity with the concept, to the extent that he built upon Łoś's original system multiple variations of positional logic.

Arthur Prior was born on December 4, 1914, in Masterton, New Zealand. He began his association with Canterbury University immediately after graduating in 1937. In 1959, Prior relocated to the United Kingdom and assumed the position of professor at Manchester University, where he remained until his passing in 1969. While Prior initially focused on logic and ethics until 1953, it was during this year that he embarked on the study of time, which ultimately led to his most significant contributions in the form of his profound works on the logic of time. From that point onward until his demise, Prior dedicated himself to this substantial project for which he is widely recognized.

While Arthur Prior's concept of tense logic differs from that of positional logic, he did incorporate elements of positional logic into some of his works. In fact, Prior not only analyzed positional logic as a tool for formalization of the logic of time but also developed a few systems within this framework. There are several instances where Prior discusses positional logic in his writings. One notable example is his logic textbook, Formal Logic published in 1955, where he mentions Łoś's system in Appendix 1. In his book Time and Modality from 1957, Prior dedicates several sections to positional logic, their applications in temporal and epistemic logic, and Łoś's ideas. This book compiles materials from Prior's lectures at Oxford in 1956 and is particularly significant as it extensively explores Łoś's logic, its axiom system, and its relation to modal logics $\mathbf{S}_4$ and $\mathbf{S}_5$ developed by Lewis. Furthermore, Prior analyzes the relationship between the realization operator and the tense operators, which he himself had devised. Based on this analysis, he presents his two systems of positional logic[31].

[30] Jarmużek, "The Heritage of Jerzy Łoś's Philosophical Logic and the Polish Question. An Introduction to the Volume": 10.

[31] Rescher, Garson, "Topological Logic": 238.

One of the later works on positional logic published during Prior's lifetime was Past, Present and Future in 1967. In this work, particularly in the chapter titled "Metric Tense-Logic" Prior conducted the analysis of positional logic that are built upon two operators. He introduced these operators in his previous publication[32], and he dedicated an entire section in Appendix B of Past, Present and Future to this topic. The same subject reappeared in Papers on Time and Tense published in 1968 and Tensed Propositions as Predicates published in 1969.

3.1. Positional equivalent of S5

In his work Time and Modality, Prior delved into the exploration of expressing temporal aspects of sentences using the framework of modal logic. He built upon Lewis' modal logics as the foundation for his investigations. Initially, he constructed a system similar to Lewis' $\mathbf{S}_4$, where the modal operators $\Diamond$ and $\Box$, denoting notions of possibility and necessity respectively, were translated into the field of tenses. However, when it came to formalizing the temporal counterpart of S5 logic, Prior employed a different logical framework. To achieve this objective, he introduced a system of positional logic based on the realization operator. In Prior's system, the expression $\mathcal{U}_t p$ is interpreted as "the occurrence of p is at time t" or "p at t"[33].

Prior adopted the use of Polish notation, similar to Łoś's system, for expressing the notation and language of those logics. However, for the sake of clarity, we will translate it to a more modern formalism without loss of meaning. Both systems use propositional variables $p,q,r,...$, instance variables $t,t_1,t_2,...$, and standard logical operators, along with the realization operator $\mathcal{U}$ and quantifiers ranging over instance variables and propositional variables. Although Prior did not explicitly state this, an analysis of his use of symbols indicates that this is the case. The first system, which we will denote as $\mathbf{A}_1$ is defined by the following axioms:

Axiom 1. $\mathcal{U}_t \neg p \rightarrow \neg \mathcal{U}_t p$

Axiom 2. $\neg \mathcal{U}_t p \rightarrow \mathcal{U}_t \neg p$

Axiom 3. $\mathcal{U}_t (p \rightarrow q) \rightarrow (\mathcal{U}_t p \rightarrow \mathcal{U}_t q)$

Axiom 4. $\forall_t (\mathcal{U}_t p) \rightarrow p$

Axiom 5. $\mathcal{U}_{t_1} \mathcal{U}_{t_2} p \rightarrow \mathcal{U}_{t_2} p$

[32] Prior, *Time and Modality*.
[33] Ibidem, 19.

Besides all the listed axioms, Prior used a detachment rule, specific rules for quantifiers and a rule **RG**.

The previous paragraph reveals a striking resemblance between the axioms employed by Łoś and Prior. Although there are minor differences in the form of the first two axioms and the last one, the overall structure of the axioms is remarkably similar to those of the $\mathbf{L}_2$ system. This is consistent with Prior's intentions, as he aimed to modify the original system. He explicitly stated that Łoś was a direct source of inspiration for his work. This modification was necessary in Prior's opinion to make the system more adequate for the specific subject matter[34].

Let us rearrange these axioms in a way consistent with Łoś's axioms:

Axiom 1. $\mathcal{U}_t \neg p \leftrightarrow \neg \mathcal{U}_t p$

Axiom 2. $\mathcal{U}_t (p \rightarrow q) \rightarrow (\mathcal{U}_t p \rightarrow \mathcal{U}_t q)$

Axiom 3. $\forall_t (\mathcal{U}_t p) \rightarrow p$

Axiom 4. $\mathcal{U}_{t_1} \mathcal{U}_{t_2} p \rightarrow \mathcal{U}_{t_2} p$

Upon closer examination, it becomes evident that systems $\mathbf{L}_2$ and $\mathbf{A}_1$ share a high degree of similarity, with only one subtle difference between them. This difference lies in the interpretation of the axiom pertaining to the nesting of a realization operator. In $\mathbf{L}_2$, the nesting of the realization operator for the same context can be omitted. However, in $\mathbf{A}_1$, Prior's system considers a scenario in which a realization operator is nested with two different context instances. In this case, Prior's system allows for the removal of the outermost realization operator.

The similarities between Prior's and Łoś's systems are not the fresh topic in the literature. In fact, Rescher and Garson provided a comparison between existing systems of positional logic[35].

3.2. Translating modalities with realization operator

During his exploration of positional logic, Prior achieved a notable feat by translating contemporary modal logics into the language of his positional system. He accomplished this by employing the realization operator to define modal notions. In addition, he presented definitions of well-known modal operators, such as $\Box$ and $\Diamond$ from Lewis' $\mathbf{S}_5$ system, using the vocabulary and constructs of positional logic. Furthermore, Prior demonstrated that systems like Lewis' $\mathbf{S}_5$, Feys' **T**, and Von Wright's **M** could be derived using these defined notions.

[34] Ibidem, 20.

[35] Rescher, Garson, "Topological Logic": 542–543.

Despite presenting a method to derive different modal operators using the language of positional logic, Prior recognized the drawbacks of this approach. In his research, Prior addressed the common concerns regarding the role of the realization operator in formalizing modalities. His argument was comprised of three key points.

The first part of the argument concerned the issue of "open statements", which was raised by Quine and Smart. They argued that such expressions should not be considered complete unless they are given a context that fills the gap, and that only statements whose truth values do not vary over time should be accepted[36]. Thus, they should not be an area of interest for logic. However, from the perspective of modern formal logic, such an objection is invalid. The scope of formal logic is subject to change over time, and there is nothing inherently problematic about expanding the definition of a logical proposition. Logicians often consider the metalogical properties of a system as the limits of logic. Therefore, the ontological status of open statements may not be a significant concern as long as the system used to formalize them remains consistent.

The second part of Prior's argument revolves around the role of the realization operator. This builds on the earlier discussion regarding the status of open statements. Prior suggests that if we agree with the position of Quine and Smart, it should follow that open statements should be rather interpreted as some sort of predicates, not as propositional variables. If we do this, we can simplify $\mathcal{U}_t p$ to the formula $p(t)$. Consequently, the need for the realization operator becomes redundant in this framework, as it is already a part of first-order logic. This aspect of Prior's argument explores the potential consequences of refraining from considering open statements as propositions in a logical meaning. Ultimately, it leads to the conclusion that the realization operator is rendered unnecessary in such cases.

The third part of Prior's argument shifts the focus to purely logical considerations. It examines the implications of combining the use of the realization operator to formalize modal notions with the first-order use of quantifiers, particularly in relation to deriving the Barcan formula as a thesis. This formula yields counterintuitive results in the presented logic, making its inclusion within the system undesirable. This forms the core argument against the use of the realization operator for modal logic in its current form.

The Barcan formula, denoted as $\Diamond\exists_x\phi \rightarrow \exists_x\Diamond\phi$, takes on a particular form. Prior's perspective involves defining $\Diamond p$ as $\exists_t\mathcal{U}_t p$, resulting in the Barcan formula

[36] Prior, *Time and Modality*, 25.

taking the form $\exists_t \mathcal{U}_t (\exists_x \phi) \to \exists_x \exists_t \mathcal{U}_t \phi$. This formulation leads to a specific example that Prior raises to illustrate the implications[37]:

> For example, if it either is or has been or will be the case that someone is flying to the moon, then there is someone who either is flying or has flown or will fly to the moon. And it is not easy to be happy about this. For suppose that in fact someone will fly to the moon some day, but not anyone that who now exists.

The Barcan formula, within the framework of positional logic, poses a challenge as it asserts the existence of objects that may not exist or may no longer exist. Preserving the Barcan formula in its original form would necessitate assuming the sempiternity of all objects, which exceeds the boundaries of logic and researches ontological assumptions[38]. Consequently, the issue of eliminating this formula from the system of positional logic became a significant topic.

The exploration of the relationship between positional logic and quantification theory, particularly regarding the Barcan formula, represents a significant contribution by Prior to the field. His investigations shed light on a problem arising from the definitions of modal operators with the use of realization operator and the application of classical quantification theory. In his work Modality and Quantification in S5, Prior demonstrated that the Barcan formula is provable in Lewis' $\mathbf{S}_5$ when combined with classical quantification theory[39]. Consequently, this result implies that any logic incorporating $\mathbf{S}_5$ would also entail the Barcan formula.

Given Prior's definitions of modal notions incorporating the realization operator, his system $\mathbf{A}_1$ contained $\mathbf{S}_5$ and was thus subject to the implications of the aforementioned result. To address the counter-intuitive nature of the Barcan formula in his system, Prior explored three potential solutions: (1) eliminating the $\mathcal{U}$-operator from the language, (2) employing a nonstandard quantification theory, (3) rejecting the first axiom in Łoś/Prior logics in its original form.

The first option proposed by Prior would be a fundamental change in logic. By eliminating the realization operator, this approach would remove a crucial aspect of positional logic. Additionally, it would effectively resolve the issue of intensionality, transforming the logic into a mere first-order theory. The second option, involving non-standard quantification theories, would necessitate substantial structural changes to existing positional logic systems, potentially leading to the creation of a

37 Ibidem, 26.
38 Ibidem, 29.
39 Prior, "Modality and Quantification in S5", *The Journal of Symbolic Logic* : 60–62.

distinct logic. However, Prior did not undertake this task and did not develop positional logic with non-standard quantification theories. Consequently, this remains an unresolved problem in the field. On the other hand, the third option entails rejecting the law of excluded middle for formulas within the range of the realization operator. Consequently, the formula $\mathcal{U}_t p \vee \mathcal{U}_t \neg p$ would not hold as a thesis in such a case.

Prior, recognizing the advantage of the third solution in terms of minimal changes to the existing system, proposed a modified axiom system to address the issue. This modified system, denoted as $\mathbf{A}_2$, is defined by the following set of axioms:

Axiom 1. $\mathcal{U}_t \neg p \rightarrow \neg \mathcal{U}_t p$

Axiom 2. $\mathcal{U}_t (p \rightarrow q) \rightarrow (\mathcal{U}_t p \rightarrow \mathcal{U}_t q)$

Axiom 3. $\forall_t (\mathcal{U}_t p) \rightarrow p$

Axiom 4. $\mathcal{U}_{t_1} \mathcal{U}_{t_2} p \rightarrow \mathcal{U}_{t_2} p$

Prior's modifications to the axioms of $\mathbf{A}_1$ are evident from their form. Specifically, he altered the first axiom by replacing the equivalence with an implication. This adjustment effectively eliminated the problematic formula that had the potential to generate undesired results[40]:

> [...] $C\mathcal{N}\mathcal{U}_t p\, \mathcal{U}_t \mathcal{N} p$, 'if it is not the case at t that p, then it is the case at t that not p'. It is clearly the formula that must be dropped from a tense-logic with proper names for non-sempiternal objects; for example, from the fact that it was not the case in 1850 that there were facts about me, it does not follow that it was the case in 1850 that there were not then facts about me.

It is noteworthy that in $\mathbf{A}_2$, while the law of excluded middle for formulas within the scope of the realization operator is no longer a theorem, the law for expressions outside the realization operator remains valid. Consequently, $\mathbf{A}_2$ implies $\nvdash \mathcal{U}_t p \vee \mathcal{U}_t \neg p$, indicating that the disjunction of a proposition and its negation within the realization operator is not provable. However, $\mathbf{A}_2$ still implies $\vdash \mathcal{U}_t p \vee \neg \mathcal{U}_t p$, affirming the disjunction of a proposition and its negation outside the scope of the realization operator. This distinction underscores the nuanced behavior of the logical system and the specific implications of the modifications made by Prior.

[40] Prior, *Time and Modality*, 35–36.

3.3. The invention of tense operators

During his exploration of positional logic, Prior displayed great enthusiasm for the idea of tense operators. He regarded them as more fundamental notions in temporal logic compared to the realization operator. Specifically, Prior identified the operators $\mathcal{F}p$ and $\mathcal{P}p$ (also denoted as $\mathcal{F}_\eta p$ and $\mathcal{P}_\eta p$) as expressions denoting "it will be the case that p" and "it was the case that p" respectively ("it will be the case that p [in η time units] from now" and "it was the case that p [η time units] ago"). These operators capture the temporal aspects of propositions by referring to future occurrences or past events relative to a particular point in time. According to Prior, these tense operators embody metaphysically primitive concepts.

Prior's approach to tense operators is intimately connected to the broader philosophical debate surrounding McTaggart's A and B series, which encapsulate two opposing theories of change within the philosophy of time[41]. Tensed propositions, characterized by expressions like "today", "in the past" and "in the future" possess a truth value that dynamically fluctuates over time. This temporal variability plays a crucial role in determining the truth or falsity of propositions such as "Today there is a storm in Gdańsk". In contrast, tenseless propositions fix time as an immutable parameter, resulting in the truth value of such propositions remaining static despite the changing time of assertion. For instance, propositions that rely on specific fixed dates, such as "On November 12, 2022 there is a storm in Gdańsk" demonstrate this characteristic of tenseless propositions.

McTaggart's A series represents a view of time in which events are arranged in terms on the notions of past, present, and future. In contrast, the B series represents a view of time in which events are ordered solely by their temporal relations to one another, such as "before" and "after". The A series is closely associated with tensed propositions that involve temporal dynamics, while the B series aligns with tenseless propositions that emphasize fixed temporal relationships. These contrasting perspectives have given rise to two distinct metaphysical positions regarding the nature of time. It is important to note that the debate between these positions remains within the realm of philosophy, as scientific evidence alone cannot definitively establish the correctness of one view over the other.

Based on the philosophical foundations discussed, it becomes apparent that expressions involving the realization operator are connected to the B series and tenseless propositions, whereas tense operators and the expressions formed with them are linked to the A series and tensed propositions. Determining which approach is more

[41] Nina Emery, Ned Markosian, Meghan Sullivan, "Time", *The Stanford Encyclopedia of Philosophy*, Winter 2020 Edition, Access 9.07.2023. https://plato.stanford.edu/archives/win2020/entries/time/.

fundamental lies beyond the realm of logical argumentation and pertains to an ongoing philosophical debate. Consequently, the selection between the two methods of representing time in logic is primarily driven by metaphysical considerations.

The aforementioned philosophical assumptions, or ontological perspective, were also evident in Prior's work. The rationale behind his views is succinctly captured in the following quote[42]:

> The tense-logical analogue of $\mathbf{S}_5$ can be best approached by introducing the notion of associating an event with a date. This can be defined in terms of the operators already at our disposal – $\mathcal{F}_\eta$ now reverting to its original status as a purely future tense operator.

According to this quote, Prior made use of his tense operators $\mathcal{F}_\eta$ and $\mathcal{P}_\eta$ to define the realization operator $\mathcal{U}_t$. This approach aligns with Prior's philosophical stance as an A-theorist. However, it is important to note that Prior did not explicitly define the realization operator solely through the use of tense operators. Instead, he provided definitions of auxiliary expressions that contributed to the overall understanding of the operator's function and meaning.

Using the aforementioned operators, Prior introduced a new operator $\mathcal{T}p$ which can be interpreted as "It either is or has been or will be the case that p". In addition to this, he presented two auxiliary notions to distinguish between future and past dates: "the date of p's occurrence is plus t" and "the date of p's occurrence is minus t". From these definitions, it can be inferred that Prior intended to define the $\mathcal{U}_t$ operator, which indicates that "the event expressed by p occurs at time t", using these notions. Thus, the realization operator was ostensibly defined through the use of tense operators, rather than the reverse.

It might be the case, that due to the utilization of the $\mathcal{F}_\eta$ operator in defining the realization operator, Rescher and Garson classified one of Prior's $\mathcal{F}_\eta$-operator systems as a positional logic[43]. To gain further insight into the system, which we shall refer to as $\mathbf{F}_1$, let us examine its axioms:

Axiom 1. $\mathbf{F}_\eta \neg p \rightarrow \neg \mathbf{F}_\eta p$

Axiom 2. $\neg \mathbf{F}_\eta p \rightarrow \mathbf{F}_\eta \neg p$

Axiom 3. $\mathbf{F}_\eta (p \rightarrow q) \rightarrow (\mathbf{F}_\eta p \rightarrow \mathbf{F}_\eta q)$

Axiom 4. $\mathbf{F}_0 p \rightarrow p$

[42] Prior, *Time and Modality*, 18.
[43] Rescher, "Topological Logic": 542.

Axiom 5. $\mathbf{F}_{\eta_1}\mathbf{F}_{\eta_2}p \rightarrow \mathbf{F}_{S(\eta_1,\eta_2)}p$

Axiom 6. $\mathbf{F}_{\eta_1}(\exists_{\eta_2}(\mathbf{F}_{\eta_2}p)) \rightarrow \exists_{\eta_2}(\mathbf{F}_{\eta_1}(\mathbf{F}_{\eta_2}p))$

Listed axioms were combined with the inference rule analogous to **RG**.

The structure of these axioms indicates that they express properties comparable to those of an operator $\mathcal{F}$ to those of the operator $\mathcal{U}$ in the positional logic systems mentioned above. By translating the tense operators into the framework of positional logic, it becomes feasible to restate this axiom system in the form of a positional logic system. This observation aligns with the findings presented by Rescher and Garson[44].

Unfortunately, we were unable to locate Rescher's and Garson's results for translating this axiom system into the language of positional logic. Therefore, in order to properly present their findings, we will provide our own outline of such a translation. This will be a syntactical analysis based on Prior's work, and a formal proof of equivalence will not be provided, as this is merely an attempt to reconstruct Rescher's and Garson's ideas. Since Prior had already completed the translation from tense operators to the realization operator, it is possible to offer an inverse definition and present $\mathbf{F}_1$ using the $\mathcal{U}$ operator.

In order to accomplish this, we need to define negative intervals $-\eta$, and once we have this definition, we can straightforwardly define the realization operator using the δ operator from Łoś's system.

$$\mathcal{P}_\eta p \overset{def}{=} \mathcal{U}_{\delta(n,-\eta)}p$$
$$\mathcal{F}_\eta p \overset{def}{=} \mathcal{U}_{\delta(n,\eta)}p$$

The translated version of the axiom system $\mathbf{F}_1$ into the language of positional logic can be denoted as $\mathbf{F}_2$.

Axiom 1. $\mathcal{U}_t\neg p \leftrightarrow \neg\mathcal{U}_t p$

Axiom 2. $\mathcal{U}_t(p \rightarrow q) \rightarrow (\mathcal{U}_t p \rightarrow \mathcal{U}_t q)$

Axiom 3. $\mathcal{U}_n p \rightarrow p$

Axiom 4. $\mathcal{U}_{t_1}\mathcal{U}_{t_2}p \rightarrow \mathcal{U}_{t_1+t_2}p$

Axiom 5. $\mathcal{U}_{t_1}(\exists_{t_2}(\mathcal{U}_{t_2}p)) \rightarrow \exists_{t_1}(\mathcal{U}_{t_2}\mathcal{U}_{t_1}p)$

[44] Ibidem, 542.

When comparing the two systems, $\mathbf{F}_2$ and $\mathbf{A}_1$, significant differences become apparent. While the first two axioms are common to all of Prior's and Łoś's systems, the remaining axioms vary. The third axiom carries a similar meaning to its counterpart in the former system, but the proposition can be asserted if it is satisfied at the present moment, rather than holding true at all moments of time as in the previous system. Regarding the nesting of the realization operator, in the mentioned system, an operation on two dates was introduced when nesting a realization operator. The penultimate axiom addresses the exportation of a quantified realization operator.

3.4. Prior's view on positional logic

Although Prior dedicated significant effort to investigating positional logic and developing variations of Łoś's system, his opinion of it was rather negative. Firstly, Prior noted that some philosophers may contend that the realization operator is redundant. They argue that instead of being an operator, the realization operator could be formalized as a simple predicate to fill open statements. Secondly, some philosophers may argue that open statements should not fall within the realm of formal logic at all. These concerns raised by Prior and others are undoubtedly valid and worthy of consideration.

Furthermore, Prior's critical examination of the realization operator revealed additional issues when applied within the framework of first-order logic. The combination of classical quantification theory and definitions of classical modal notions could lead to counter-intuitive outcomes. In addition to these logical concerns, Prior also highlighted a philosophical divergence between his understanding of time and that of the positional logic's author. Prior regarded the logic of tenses as fundamental, and he saw the realization operator as embodying a distinct ontological perspective.

However, Prior's critique of the structure of positional logic went beyond that. In order to delve deeper into the potential insights provided by the realization operator compared to a first-order predicate, Prior introduced the concept of collapsibility of formulas. He proposed that expressions of the form $\mathcal{T}_x p$ could be reformulated as $\phi(x)$, although this transformation might result in a loss of meaning in certain cases. When the analysis of $\mathcal{T}_x p$ served no logical purpose in any context, Prior referred to it as collapsible. In essence, $\mathcal{T}_x p$ could be collapsed into the form $\phi(x)$ when it no longer conveyed any additional logical information[45]. To define the notion of collapsibility, Prior outlined three necessary conditions[46].

[45] Prior, *Time and Modality*, 118.
[46] Ibidem, 121.

Definition 3.1. *We call a $\mathcal{T}_x p$ formula collapsible into the form $\phi(x)$ if the following conditions are met:*

- $\exists_x \phi(x) \leftrightarrow \mathcal{T}_x p$;
- $\mathcal{T}_y \mathcal{T}_x p \to \exists_z (\mathcal{T}_z p)$;
- $\mathcal{T}_x \zeta(p_1, ..., p_n) \leftrightarrow \zeta(\mathcal{T}_x p_1, ..., \mathcal{T}_x p_n)$, *where ζ is any n-adic truth operator and $p_1, ..., p_n$ does not contain x.*

The first criterion for collapsibility asserts that the expression p should not stand alone as a separate statement, but rather be part of a larger sentence forming the predicate ϕ. In other words, p should only occur within the context of $\mathcal{T}_x$. The second condition states that the nesting of the realization operator $\mathcal{T}$ must be reducible to a single occurrence of the operator. For instance, an expression like $\mathcal{T}_y \mathcal{T}_x p$ should be reducible, through some operation, to a form such as $\mathcal{T}_z p$. The final criterion for collapsibility states that the realization operator must distribute over all logical connectives. This means that if p and q are statements, then $\mathcal{T}_x(p \wedge q)$ is equivalent to $\mathcal{T}_x p \wedge \mathcal{T}_x q$, and likewise for other connectives.

In Prior's examination of various logical systems for collapsibility, he found that both $\mathbf{A}_1$ and $\mathbf{F}_2$ satisfied the specified conditions[47]. However, he demonstrated that $\mathbf{A}_2$ is non-collapsible due to the lack of equivalence between $\neg \mathcal{U}_t p$ and $\mathcal{U}_t \neg p$ within that system. Additionally, Prior extended the notion of collapsibility to Łoś's systems of logic. While $\mathbf{L}_1$ appears to be collapsible, given its similarity to $\mathbf{A}_1$, the same cannot be said for $\mathbf{L}_2$[48].

Prior's analysis yielded a significant and wide-ranging observation – time does not possess any inherent characteristics that would restrict the range of instant variables for realization operators[49]:

> In this analysis the statements are represented by as being of the form $\mathcal{T}_x p$, where $\mathcal{T}$ is a dyadic operator of the form 'It is (was) the case on (in, with) - - - that - - -', having for its first argument some sort of name (of a time, of a place, of an individual) and for a second argument a statement, or something very like a statement.

This remark arose during Prior's examination of the collapsibility of formulas in the format $\mathcal{T}_x p$ into expressions in the format $\phi(x)$. His insight had profound implications, as it was subsequently utilized by researchers like Rescher to extend

[47] Ibidem, 121.
[48] Ibidem, 121.
[49] Ibidem, 117.

the application of positional logic beyond the temporal and epistemic domains of philosophical logic.

Moreover, Prior's analysis led to several intriguing developments, including the formulation of positional place logic and the pioneering use of lambda calculus in conjunction with a realization operator. The language of this logic consisted of propositional variables, interval variables denoted by symbols such as n and m, logical connectives, and operators $\mathcal{T}$, $\mathcal{L}$, $\mathcal{R}$, and λ. Interval variables were used to represent distances in miles, while $\mathcal{T}$ served as a realization operator. The expression $\mathcal{L}_n p$ was interpreted as "It is the case n miles away to the left that p" and $\mathcal{R}_n p$ was understood as "It is the case n miles away to the right that p"[50].

In the presented language, Prior demonstrated that expressions formed using the $\mathcal{L}$ and $\mathcal{R}$ operators could be redefined using the realization operator. He proposed the following equivalence:

$$\mathcal{L}_n p \leftrightarrow \mathcal{T}_{\lambda(n)} p$$
$$\mathcal{R}_n p \leftrightarrow \mathcal{T}_{\lambda(n)} p$$

Here, $\lambda(n)$ denotes "the place n miles to the left" and "the place n miles to the right", respectively[51]. It is worth mentioning that Prior suggested that analogous definitions could be formulated for the $\mathcal{P}_\eta$ and $\mathcal{F}_\eta$ operators using the realization operator $\mathcal{U}$ and the λ operator.

The central argument against positional logic and in favor of tense logic was based on the concept of collapsibility, which Prior extensively explored in Time and Modality. Building upon the framework of place logic and the definitions presented earlier, Prior aimed to demonstrate that, unlike positional logic, his logic of tense operators did not satisfy the second condition of collapsibility. He provided an example with the expression $\mathcal{P}_{S(m,n)}\mathcal{F}_n p$, which could be translated into positional logic as $\mathcal{U}_{\lambda(m+n)}\mathcal{U}_{\lambda(-n)} p$. As Prior explained[52]:

> For even if it was the case m days ago that p, it might not have been true $m+n$ days ago that it was going to be the case n days later that p, for $m+n$ days ago the issue might still have been indeterminate. But if $\mathcal{U}_{\lambda(m+n)}\mathcal{U}_{\lambda(-n)}$ were equivalent to any single $\mathcal{U}_\lambda$ prefix it would clearly be to $\mathcal{U}_{\lambda(m)}$. So this condition of collapsibility cannot be met by this type of tense-logic.

[50] Ibidem, 119.
[51] Ibidem, 119.
[52] Ibidem, 120.

The conclusion reached by Prior in the cited quote may have been a result of an incorrect approach to $\mathcal{P}_{S(m,n)}$. The sentence "It was the case $m+n$ days ago that it will be the case n days hence that p" should not be equated with "It was the case m days ago that p"[53]. It appears that there was an error in Prior's reasoning, as he mistakenly formalized $\mathcal{U}_{\lambda(m+n)}$ instead of $\mathcal{U}_{\lambda(-m-n)}p$ and $\mathcal{U}_{\lambda(-n)}$ instead of $\mathcal{U}_{\lambda(n)}$. This interpretation was inconsistent with his preliminary definitions and could have led him to conclude that tense logic possesses properties that positional logic lacks, such as non-collapsibility in the second case.

4. Topological abstraction of Nicholas Rescher

Nicholas Rescher, born in Germany in 1928, is a highly regarded American philosopher who has made significant contributions to various areas of philosophy and logic. Throughout his career, Rescher has studied subjects such as the history of logic, epistemology and ontology. In his early years, he focused on non-classical logics, exploring areas such as paraconsistent logic, temporal logic, and positional logic, which he encountered through the works of Prior. It was through this exploration that Rescher became acquainted with the works of Łoś. In the latter half of the 1960s, Rescher dedicated himself to the development of positional logic, further advancing the field.

Rescher's exploration of positional logic built upon the foundations laid by Prior. He developed modified systems of temporal logic, drawing from the works of Prior and Łoś. Additionally, Rescher continued to advance Łoś's assertion logic. However, his most significant contribution to positional logic was the creation of topological logic, which closely resembles the positional logic known today. Rescher's inspiration for positional logic stemmed from Prior's Appendix A in "Time and Modality", where Prior suggested that the realization operator could be applied beyond the realms of time and assertions, allowing generalization. Rescher embraced this idea and abstracted the intended interpretation to encompass a wide range of contexts where propositions needed to be considered. Through his work, Rescher expanded the scope of positional logic, making it applicable in various domains.

Rescher made significant contributions to the field of positional logic through a series of notable publications. His first work, "On the Logic of Chronological Propositions" (1966), marked his initial contribution to the field. This was followed by two additional publications in 1968: "Assertion Logic" and "Topological Logic". These three works were subsequently compiled and published together in the book Topics in Philosophical Logic later that same year. Each article approached positional logic

[53] Ibidem.

from a different perspective. The first work continued the application of positional logic within the philosophy of time, while "Assertion Logic" focused on its relevance to epistemology. In the latter article, Rescher explored the abstraction of positional logic detached from specific philosophical interpretations. The application of positional logic in epistemology and the philosophy of time was further expanded upon in two monographs: Temporal Logic (1971) and Epistemic Logic (2005). These books provided extensive analyses and profound insights into the utilization of positional logic within their respective areas of study.

The idea of positional logic was introduced to Rescher through Prior's work, which led him to discover the contributions of Łoś. Interestingly, Rescher's initial exposure to Łoś's work came from a brief mention in the appendix of Prior's textbook Formal Logic[54]. Intrigued by this reference, Rescher further explored the original results of Łoś, relying on English reviews such as "Review of: J. Łoś, Podstawy Analizy Metodologicznej Kanonów Milla" and Suszko's "Review: Jerzy Łoś, Many-Valued Logics and Formalization of Intensional Functions". The mention of the realization operator and Łoś's findings in Prior's textbook served as a catalyst for Rescher and other logicians to investigate the study of positional logic. As Rescher's interests evolved, subsequent research on positional logic was carried out by James Garson, Alasdair Urquhart, and Hirokazu Nishimura in the 1970s.

4.1. Chronological logic

In 1966, Rescher published a paper "On the Logic of Chronological Propositions", which introduced the realization operator. His work can be seen as a continuation of Prior's investigations into the positional reconstructions of Lewis' $\mathbf{S}_4$ and $\mathbf{S}_5$ from Time and Modality[55]. Notably, it is the first work in which symbol $\mathcal{R}$ occurs on the place of the realization operator. Rescher explains that the expression $\mathcal{R}_t p$ should be read as "the proposition p is realized at time t". This symbolic form of the operator serves as an abbreviation for the concept of "realization"[56].

Following in the footsteps of Łoś and Prior, Rescher embarked on a classification of temporal propositions. He discerned between chronologically definite and chronologically indefinite statements, forming the basis for his exploration of two styles of temporal logics corresponding to these categories. The logic of chronologically indefinite statements incorporates a realization operator, enabling the transformation of

[54] Nicholas Rescher, "Assertion Logic", in: *Topics in Philosophical Logic*, ed. Nicholas Rescher (Dordrecht: Springer, 1968), 262.

[55] Nicholas Rescher, "On the Logic of Chronological Propositions", *Mind* 77 (1966): 75.

[56] Ibidem, 80.

such statements into chronologically definite ones by associating them with specific dates.

The aforementioned distinction between chronologically definite and chronologically indefinite statements is just the beginning of Rescher's comprehensive approach. He expanded on this division by introducing another classification based on the procedures of chronological dating. Rescher proposed two distinct procedures: using actual dates and using pseudo-dates. Pseudo-dates encompass expressions like today, tomorrow, day-before yesterday, and others. Rescher argued that employing a chronology based on pseudo-dates gives rise to a logic of chronologically indefinite statements, while a chronology or temporal logic based on actual dates corresponds to a logic of chronologically definite statements.

Based on the aforementioned categorization, various outcomes of the realization operator can be identified. The results depend on whether the sentence within the scope of the operator is a definite or indefinite statement, as well as the categorization of the date instant as either a pseudo-date or a real date. In the simplest scenario, where the variable p represents a definite statement, $\mathcal{R}_t p$ is equivalent to p since the statement is realized at all times, regardless of whether t represents a pseudo-date or a real date. When p represents an indefinite statement, if t corresponds to a pseudo-date, the result of $\mathcal{R}_t p$ remains chronologically indefinite. Conversely, if t represents a real date, the operation yields a chronologically definite statement[57].

Rescher constructed his axiom system based on the analysis presented earlier. Additionally, he considered the historical approaches to nesting the realization operator, which significantly influenced the structure of his systems. Rescher observed that previous logicians had proposed two fundamental solutions to this matter. The first approach involved nesting two realization operators, resulting in an additive operation on the time instants: $\mathcal{R}_{t_1}\mathcal{R}_{t_2} p \leftrightarrow \mathcal{R}_{t_1+t_2} p$. The second approach suggested omitting the outermost operator entirely, leading to the formula $\mathcal{R}_{t_1}\mathcal{R}_{t_2} p \leftrightarrow \mathcal{R}_{t_2} p$.

Rescher's logics were expressed in a standard notation using modern symbols. The language of these logics consisted of propositional variables, denoted as $p, q, r, \ldots$, which encompassed both chronologically definite and chronologically indefinite statements. To represent periods of time elapsed since the origin, period variables such as $t, t_1, t_2, \ldots$ were introduced. Standard logical operators were employed, in addition to the realization operator $\mathcal{R}$. Moreover, the logics incorporated quantifiers that ranged over dates. Both systems employ the rule **RG**.

Rescher's initial system, denoted as $\mathbf{R}_1$, was built upon the first approach to nesting the realization operator. The axioms of this logic can be outlined as follows:

[57] Ibidem.

Axiom 1. $\mathcal{R}_t\neg p \leftrightarrow \neg\mathcal{R}_t p$

Axiom 2. $\mathcal{R}_t(p \wedge q) \leftrightarrow \mathcal{R}_t p \wedge \mathcal{R}_t q$

Axiom 3. $\forall_t(\mathcal{R}_t p) \rightarrow p$

Axiom 4. $\forall_t(\mathcal{R}_t p) \leftrightarrow \mathcal{R}_{t_1}\forall_t(\mathcal{R}_t p)$

Axiom 5. $\mathcal{R}_{t_1}\mathcal{R}_{t_2} p \leftrightarrow \mathcal{R}_{t_1+t_2} p$

The second system, denoted as $\mathbf{R}_2$, adopts the latter approach to nesting the realization operator. The axioms of this logic can be summarized as follows:

Axiom 1. $\mathcal{R}_t\neg p \leftrightarrow \neg\mathcal{R}_t p$

Axiom 2. $\mathcal{R}_t(p \wedge q) \leftrightarrow \mathcal{R}_t p \wedge \mathcal{R}_t q$

Axiom 3. $\forall_t(\mathcal{R}_t p) \rightarrow p$

Axiom 4. $\forall_t(\mathcal{R}_t p) \leftrightarrow \mathcal{R}_{t_1}\forall_t(\mathcal{R}_t p)$

Axiom 5. $\mathcal{R}_{t_1}\mathcal{R}_t p \leftrightarrow \mathcal{R}_t p$

Rescher's objective was to compare his two systems with Prior's $\mathbf{A}_1$ and $\mathbf{F}_2$, and indirectly with Lewis' systems. To obtain those results, he employed the same definitions for the classic modalities as Prior: $\Diamond p$ was defined as $\exists_t\mathcal{R}_t p$, and $\Box p$ was defined as $\forall_t\mathcal{R}_t p$. He showed that $\mathbf{R}_1$ is equivalent to $\mathbf{F}_2$, and thus, by defining specific modalities, the theorems of $\mathbf{S}_4$ can be derived. Similarly, he established analogous connection between $\mathbf{R}_2$ with $\mathbf{A}_1$ and $\mathbf{S}_5$. What is an unexpected result is that Rescher proved that besides $\mathbf{S}_4$, in $\mathbf{R}_1$ it is also possible to prove $\mathbf{S}_5$[58].

In a subsequent publication from 1967 "A Note on Chronological Logic", Rescher and Garson delved further into the metalogical investigation of temporal systems within positional logic. This particular work focused on translating Von Wright's system **T** into the language of $\mathbf{R}_1$, providing a comprehensive analysis of the relationship between these two systems.

This topic is of particular interest due to the inherent differences between the two systems. Von Wright's logic assumes the discreteness of time, while positional logic, including Łoś's, Prior's, and Rescher's systems, does not. In addition, there is a disparity in the expressive power of the languages used in each system. The And-Next system, for instance, is less versatile compared to any positional logic system.

[58] Ibidem, 85–87.

Rescher points out that Von Wright's system lacks the breadth of applications and flexibility that positional logic offers[59].

Rescher and Garson identified several aspects in which positional logic outperforms Von Wright's And-Next system. One notable difference lies in the ontological assumptions of the two systems. Positional logic can be interpreted in both continuous and discrete structures, offering greater flexibility, whereas Von Wright's logic is more limited in this regard. In terms of expressive power, positional logic allows for quantification over time variables, which is not possible in the And-Next system. Additionally, the realization operator in positional logic is not limited to specific time instants or moments, whereas the operator in **T** is bound by adjacent time moments.

To translate Von Wright's system **T** into the language of Rescher's and Garson's positional logic, two translation rules were provided. These rules were designed to capture the expression $\phi T \psi$ in two different ways: $\mathcal{R}_0(\phi) \wedge \mathcal{R}_1(\psi)$ or $\mathcal{R}_0(\phi) \wedge \mathcal{R}_k(\psi)$ for some $k \in \mathbb{N}$. The authors introduced two rules to account for the fact that the interpretation of the And-Next operator is not necessarily limited to the meaning "now and in the time directly following after now", but it can also signify "now and in the time following after now some time in the future". Both of these meanings are captured by the expressions translated into positional logic. The authors asserted that all the axioms of the And-Next system are provable in the positional system after the translation process[60].

During his investigations into positional temporal logic, Rescher addressed the issue raised by Prior regarding the Barcan formula in positional logic with quantifiers. To recap the issue, as Prior demonstrated, the Barcan formula must be a consequence of the axioms present in $\mathbf{A}_1$, $\mathbf{F}_2$, $\mathbf{R}_1$, and $\mathbf{R}_2$. The formula takes the form:

$$\Diamond \exists_x \phi \rightarrow \exists_x \Diamond \phi$$

Prior showed that after providing definitions of classic modalities using the realization operator, the thesis takes the following form:

$$\exists_t \mathcal{R}_t(\exists_x \phi(x)) \rightarrow \exists_x \exists_t \mathcal{R}_t(\phi(x))$$

As previously demonstrated by Prior, the Barcan formula leads to counter-intuitive results, such as the requirement for all objects to be sempiternal. Rescher presented a potential solution to the problem identified by Prior. It involved introducing two types of quantifiers – tensed and tenseless. It is worth noting that Rescher's

[59] Nicholas Rescher, James Garson, "A Note on Chronological Logic", *Teoria* 33(1967): 40.
[60] Ibidem, 42.

solution aligns with one of the general solutions categorized by Prior, namely incorporating non-standard quantification theory.

Tensed quantifiers are represented by the form $\exists_{t,x}\phi(x)$ and $\forall_{t,x}\phi(x)$, which can be interpreted as "there exists an x at time t such that $\phi(x)$" and "for all x that exist at some time t, $\phi(x)$", respectively. On the other hand, tenseless quantifiers take the form $\exists_{n,x}\phi(x)$ and $\forall_{n,x}\phi(x)$, which can be read as "there now-exists an x such that $\phi(x)$" and "for all now-existing x, $\phi(x)$", respectively[61].

Rescher noticed that by incorporating the two distinct meanings of quantifiers, one with temporal restrictions and one without, it became possible to express modalities using both forms of quantification: tensed and tenseless. This allowed for the representation of tenseless possibility and necessity as $\exists_{n,x}\mathcal{R}_t\phi(x)$ and $\forall_{n,x}\mathcal{R}_t\phi(x)$ respectively. On the other hand, their tensed counterparts would be expressed as $\exists_t\exists_{t,x}\mathcal{R}_t\phi(x)$ and $\forall_t\forall_{t,x}\mathcal{R}_t\phi(x)$.

According to Rescher, employing quantifiers in the presented tenseless manner provides a solution to the problem. The Barcan formula, when interpreted this way, is still accepted as a thesis of the system. However, it is evident that the tenseless reading offers an alternative to the Barcan formula in its original form. As a result, it provides a way to avoid the counter-intuitive consequences highlighted by Prior, which imply that all objects are sempiternal.

4.2. Assertion logic

Rescher carried on the tradition of using positional logic for the examination of time, following the path laid out by Łoś and Prior. However, his exploration of positional logic extended beyond the realm of time alone. In his influential book Topics in Philosophical Logic published in 1968, Rescher dedicated a section to demonstrating the application of positional logic in the field of epistemology. This work was a direct continuation of Łoś's earlier contributions to assertion logic in "Logiki Wielowartościowe a Formalizacja Funkcji Intensjonalnych". In "Assertion Logic", Rescher investigated the realm of epistemic logic, constructing five distinct systems that incorporated the realization operator. Drawing upon Łoś's contributions to epistemic logic, Rescher conducted a concise metalogical analysis of this logical framework and established the equivalence between his most robust system and Łoś's original assertion logic.

Rescher's exploration of assertion logic revolved around the examination of the logical connection between asserters and propositions. His logical framework was

[61] Rescher, "On the Logic of Chronological Propositions": 89.

built on the expression $\mathcal{A}_x p$, where $\mathcal{A}$ represents the realization operator, x denotes an assertor, and the expression signifies that "the assertor x asserts the proposition p". Rescher emphasized that assertors can refer to individuals, groups of people, or even abstract entities[62].

The language of assertion logic systems consists of propositional variables such as $p, q, r, \ldots$ and individual variables such as $x, y, z, \ldots$. In addition to standard logical connectives, the realization operator $\mathcal{A}$ is also used. The quantifiers $\forall$ and $\exists$ range over both individual variables and propositional variables in these logics.

Rescher formulated two fundamental postulates within the framework of assertion logic, known as the postulates of a rational assertor[63]. The first postulate, known as the assertor's commitment to assert, can be expressed as a rule, contained in each of his systems:

Rule (RCA)**.**

$$\frac{p \vdash q}{\mathcal{A}_x p \vdash \mathcal{A}_x q}$$

This postulate captures the idea that a rational assertor is committed to the logical consequences of their assertions. If an assertor asserts p, then they are obliged to assert all the logical consequences that follow from p. The second postulate states that no assertor commits themselves to a contradiction, which is represented as the theorem $\neg\exists_x \mathcal{A}_x(p \wedge \neg p)$.

Rescher developed a total of five systems of assertion logic. The first and simplest logic, denoted $\mathbf{E}_1$, is defined by the following set of axioms:

Axiom 1. $\forall_x \exists_p \mathcal{A}_x p$

Axiom 2. $\mathcal{A}_x p \wedge \mathcal{A}_x q \rightarrow \mathcal{A}_x(p \wedge q)$

Axiom 3. $\neg\mathcal{A}_x(p \wedge \neg p)$

The system can be reformulated using a different set of formulas as axioms. Instead of the rule **RCA**, it can be replaced with **RG**, and the other three axioms can be replaced with simpler axioms[64].

Axiom 1. $\mathcal{A}_x \neg p \rightarrow \neg\mathcal{A}_x p$

Axiom 2. $\mathcal{A}_x(p \rightarrow q) \rightarrow (\mathcal{A}_x p \rightarrow \mathcal{A}_x q)$

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62 Rescher, "Assertion Logic", 250.
63 Ibidem, 251.
64 Ibidem, 252.

Moving forward, we will utilize the second axiom system for this logic, as it corresponds with the notation used in Rescher's temporal logics. This will facilitate a more convenient comparison between these systems. From the presented axioms, we can obtain a stronger logic, denoted as $\mathbf{E}_2$, by deriving the previously introduced system and adding a single axiom.

Axiom 1. $\mathcal{A}_x \neg p \rightarrow \neg \mathcal{A}_x p$

Axiom 2. $\mathcal{A}_x (p \rightarrow q) \rightarrow (\mathcal{A}_x p \rightarrow \mathcal{A}_x q)$

Axiom 3. $\forall_x \mathcal{A}_x p \rightarrow p$

The newly introduced axiom can be expressed in first-order logic as the formula $\neg p \rightarrow \exists_x \neg \mathcal{A}_x p$. It is worth noting that within the context of assertion logic, this formulation of the axiom is more convenient in terms of its meaning. Specifically, this formula can be interpreted as stating that for every false proposition, there exists at least one assertor who avoids asserting it[65].

A stronger logic, denoted as $\mathbf{E}_3$, can be obtained by augmenting $\mathbf{E}_1$ with an additional formula in its set of axioms.

Axiom 1. $\mathcal{A}_x \neg p \rightarrow \neg \mathcal{A}_x p$

Axiom 2. $\mathcal{A}_x (p \rightarrow q) \rightarrow (\mathcal{A}_x p \rightarrow \mathcal{A}_x q)$

Axiom 3. $p \rightarrow \exists_x \mathcal{A}_x p$

The form of the introduced formula self-explains the relationship between $\mathbf{E}_3$ and $\mathbf{E}_2$, as the third axiom of the latter system is a special case of the third axiom of the former. Building upon this, we can derive another stronger logic called $\mathbf{E}_4$. To obtain $\mathbf{E}_4$, we simply add an axiom concerning the nesting of realization operators.

Axiom 1. $\mathcal{A}_x \neg p \rightarrow \neg \mathcal{A}_x p$

Axiom 2. $\mathcal{A}_x (p \rightarrow q) \rightarrow (\mathcal{A}_x p \rightarrow \mathcal{A}_x q)$

Axiom 3. $p \rightarrow \exists_x \mathcal{A}_x p$

Axiom 4. $\mathcal{A}_x \mathcal{A}_x p \leftrightarrow \mathcal{A}_x p$

The fifth logic, denoted as $\mathbf{E}_5$, is based on the foundation of $\mathbf{E}_1$ together with an additional axiom specific to $\mathbf{E}_4$ and axioms of completeness of the knowledge of an assertor, expressed as $\mathcal{A}_x p \vee \mathcal{A}_x \neg p$. However, for the sake of clarity and intuition, we present an equivalent axiom system that captures the essence of $\mathbf{E}_5$.

[65] Ibidem, 253.

Axiom 1. $\mathcal{A}_x \neg p \leftrightarrow \neg \mathcal{A}_x p$

Axiom 2. $\mathcal{A}_x (p \wedge q) \leftrightarrow \mathcal{A}_x p \wedge \mathcal{A}_x q$

Axiom 3. $\mathcal{A}_x \mathcal{A}_x p \leftrightarrow \mathcal{A}_x p$

Rescher's system $\mathbf{E}_5$ is of particular significance, as it has been shown to be equivalent to Łoś's $\mathbf{L}_2$, as demonstrated by Rescher himself[66]. These systems introduced and explored a variety of philosophically intriguing concepts and theorems. Among them are notions such as omniscience, veridicality, collective omniscience, universal omniscience, universal veridicality, mutual contradiction, and contesting. These concepts offer profound insight into the nature of knowledge, truth, and the interplay between assertors in the realm of assertion logic.

4.3. Topological logic

After delving into the temporal and epistemological realms of positional logic, Rescher redirected his attention to more abstract domains. Motivated by the successes of Łoś in employing positional logic in different areas of philosophy, Rescher undertook further explorations of the potential of the realization operator as a more general tool.

The initial inklings of this trajectory can be traced back to "Assertion Logic", where Rescher unveiled the discovery of connections between assertion logic and chronological logic. Rescher's insightful observations on these connections are encapsulated in the following quote[67]:

> Assertion logic is thus a fragment of the system $\mathbf{P}_1$ of topological logic. Indeed, we can thus look upon the theory of complete assertors represented by the system $\mathbf{A}_5 = \mathbf{L}_2$ as a halfway house between the logic of assertion proper ($\mathbf{A}_1$), and the system $\mathbf{P}_1$ of topological logic. Since chronological logic (or "tense logic") is a special type of topological logic – for details see Chap. XIII – this finding is also a step towards establishing the kinship of assertion logic with chronological logic.

Building upon these findings, Rescher and Garson collaborated on a related publication in the same year, as documented in "Topological Logic". In this notable work, they undertook a comprehensive exploration of topological logic, which can

[66] Ibidem, 263.
[67] Rescher, Ibidem, 264.

be seen as a more abstract and foundational variant of positional logic. Significantly, this paper marks the earliest known mention of the term "positional logic"[68]. Interestingly, during that period, the term was used interchangeably with topological logic, although this usage did not endure over time. The primary objective of the paper is to provide definitions of positional logic that are detached from specific applications, establishing a generalized family of logics derived from the pioneering contributions of Łoś, Prior, and Rescher.

In their investigations, Rescher and Garson introduced a novel formulation of the realization operator that goes beyond its purely epistemic or temporal semantics. The realization operator, represented as $\mathcal{P}_\alpha p$, indicates that the proposition p is realized at position α. An intriguing aspect of topological logic is that its language closely resembles that of chronological and assertion logics, with one notable difference: position variables $\alpha, \beta, \gamma, \ldots$ do not possess predetermined meanings. Instead, they serve as placeholders for abstract positions that can subsequently be interpreted as diverse entities, such as Cartesian coordinates, seat numbers, integers, real numbers, and so on.

As a result of their work, Rescher and Garson developed two positional axiom systems and provided insights into alternative versions of these systems. Similarly to Rescher's previous work[69], both logics are built on the distinction between two approaches to nesting the realization operator, analogous to $\mathbf{R}_1$ and $\mathbf{R}_2$. Furthermore, the authors identified two distinct approaches to omitting the realization operator.

The first approach suggests that if a proposition is realized in all possible contexts, then the realization operator can be omitted. This idea is captured by the formula $\forall_\alpha \mathcal{P}_\alpha p \rightarrow p$. The second approach focuses on omitting the realization operator when the proposition is realized, within its domain, at the current moment. If we denote a distinguished point in time by ξ, then this idea can be expressed as $\mathcal{P}_\xi p \leftrightarrow p$.

These different approaches, both in terms of nesting and omission of the realization operator, can result in distinct axiom systems. While Rescher explored both approaches, in this presentation, we will focus on the first approach to omitting the realization operator, as it aligns with the chosen axiom system. What is worth noting is that, in both logics, as well as in the case of systems of chronological and assertion logics, rule **RG** is incorporated.

The first system, denoted as $\mathbf{P}_1$, is characterized by the following set of axioms:

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[68] Rescher, Garson, "Topological Logic": 229.

[69] Ibidem.

Axiom 1. $\mathcal{P}_\alpha \neg p \leftrightarrow \neg \mathcal{P}_\alpha p$

Axiom 2. $\mathcal{P}_\alpha (p \wedge q) \leftrightarrow \mathcal{P}_\alpha p \wedge \mathcal{P}_\alpha q$

Axiom 3. $\forall_\alpha (\mathcal{P}_\beta \mathcal{P}_\alpha p) \leftrightarrow \mathcal{P}_\beta \forall_\alpha (\mathcal{P}_\alpha p)$

Axiom 4. $\forall_\alpha \mathcal{P}_\alpha p \rightarrow p$

Axiom 5. $\mathcal{P}_\beta \mathcal{P}_\alpha p \leftrightarrow \mathcal{P}_\alpha p$

The second system, denoted as $\mathbf{P}_2$, is defined by the following set of axioms:

Axiom 1. $\mathcal{P}_\alpha \neg p \leftrightarrow \neg \mathcal{P}_\alpha p$

Axiom 2. $\mathcal{P}_\alpha (p \wedge q) \leftrightarrow \mathcal{P}_\alpha p \wedge \mathcal{P}_\alpha q$

Axiom 3. $\forall_\alpha (\mathcal{P}_\beta \mathcal{P}_\alpha p) \leftrightarrow \mathcal{P}_\beta \forall_\alpha (\mathcal{P}_\alpha p)$

Axiom 4. $\forall_\alpha \mathcal{P}_\alpha p \rightarrow p$

Axiom 5. $\mathcal{P}_\beta \mathcal{P}_\alpha p \leftrightarrow \mathcal{P}_{\beta+\alpha} p$

As we can observe, both of the presented logics, $\mathbf{P}_1$ and $\mathbf{P}_2$, are defined by the same sets of axioms as the two systems of chronological logic, $\mathbf{R}_1$ and $\mathbf{R}_2$. However, semantics played a more significant role in this case. Although the aforementioned systems had a fixed intended temporal interpretation, their applications were limited. In contrast, $\mathbf{P}_1$ and $\mathbf{P}_2$ were designed with a broader scope in mind. The authors provided abstract applications, such as the theory of possible worlds or the logic of spatial coordinates, to demonstrate the versatility of these logics. Furthermore, the authors acknowledged the close connection between the systems[70]. As a result of the development of these logics, investigations were conducted to explore the relationships between various systems of chronological, assertion, and topological logics.

In addition to providing interesting examples of applications for topological logics, the authors conducted a historical comparison of various systems of temporal positional logics. They compared the standard set of axioms in topological logic with several other systems, including Łoś's $\mathbf{L}_1$, Prior's $\mathbf{A}_1$ and $\mathbf{F}_2$, and Rescher's chronological logics $\mathbf{R}_1$ and $\mathbf{R}_2$[71].

[70] Ibidem, 238.
[71] Ibidem, 237–238.

4.4. Other results and continuators

Rescher's investigations into the temporal and epistemological applications of positional logic were further expanded upon in two of his monographs. The first monograph, coauthored with Alasdair Urquhart and published in 1971 Temporal Logic, combined Rescher's findings on chronological logic and topological logic. The second monograph, published much later in 2005 Epistemic Logic, explored the results of assertion logic. These books contain technical details, specific facts, and theorems, including a compilation of previously proven facts in positional logic. Given the extensive nature of these works, we will not cover them in all their entirety. However, we will highlight some of the research areas explored in these books.

The first mentioned book focuses on the application of positional logic as a temporal logic. Within this monograph, Rescher and Urquhart present numerous interesting results. One of their key investigations involves exploring the possibility of defining tense operators within the language of positional logic. They successfully express the minimal system of tense logic, denoted as **K**, using this language. Additionally, the authors examine several metalogical properties of the fundamental system of temporal positional logic, presented in the previous works as $\mathbf{R}_2$. They investigate properties such as completeness and decidability[72]. Furthermore, they provide semantics for positional systems, shedding light on the interpretation and meaning of these logics[73].

Rescher and Urquhart introduced the concept of branching and linear time structures by utilizing the realization operator. This concept was thoroughly examined and applied in the initial reconstruction of Diodorus' Master argument using positional logic. The implications of this result had a significant impact on logicians such as Jarmużek, who dedicated his work On the Sea Battle Tomorrow that May not Happen to exploring various reconstructions of the argument, with a specific focus on positional reconstructions[74]. Notably, intriguing outcomes emerged from these investigations, including attempts to construct multivalued positional logic and the development of logics for processes and world states using the language of positional logic.

On the other hand, in his book, Epistemic Logic: A Survey of the Logic of Knowledge published in 2005, Rescher investigated a system based on the founda-

[72] Rescher, Urquhart, *Temporal Logic*, 44–49.
[73] Ibidem, 56–57.
[74] Tomasz Jarmużek, *On the Sea Battle Tomorrow That May Not Happen. A Logical and Philosophical Analysis of the Master Argument*; (Bern: Peter Lang GmbH, 2018).

tional concept of assertion logic by Łoś[75]. The concept that was further developed into a logic of knowledge. The entire book is dedicated to exploring epistemic logics that are built upon the framework of the realization operator and the modeling of specific epistemological concepts through its application.

Rescher's pioneering work in the field of positional logic paved the way for subsequent researchers in the 1970s. Building upon Rescher's foundations, scholars such as James Garson and Hirokazu Nishimura further developed the concepts of chronological logic and topological logic. Their publications spanning from 1969 to 1979 mark the last known work dedicated to exploring the rich terrain of positional logic until the early 2000s.

Garson's contributions to the field of positional logic began in 1967 when he collaborated with Rescher on their joint work "A Note on Chronological Logic". This collaboration was followed by their influential publication, "Topological Logic". In 1969, Garson furthered his research with his doctoral dissertation The Logics of Space and Time[76]. Building on this foundation, he published his first independent work on positional logic in 1972, "Two New Interpretations of Modality", focusing on the definition of modalities using topological logic[77].

Garson's subsequent works continued to explore the topic of topological logic, which he diligently developed over the years. In his work "The Completeness of an Intensional Logic: Definite Topological Logic"[78], he focused on providing semantic tools for topological logic, specifically for systems $\mathbf{P}_1$ and $\mathbf{P}_2$. Using these semantics, Garson successfully obtained soundness and completeness theorems for these systems. Building on this foundation, his publication "Indefinite Topological Logic" explored the construction of a topological logic for indefinite indexes, based on variations of the $\mathbf{P}_1$ and $\mathbf{P}_2$ systems[79]. Garson introduced the concept of indefinite indexes as analogous to Prior's notion of indefinite sentences, where instead of a fixed index such as a specific date, the index has a floating reference. He described this work as a continuation of his previous publication[80], as he once again provided semantics and a completeness theorem, but this time for topological logic

[75] Nicholas Rescher, *Epistemic Logic: A Survey of the Logic of Knowledge*; (Pittsburgh: University of Pittsburgh Press, 2005).

[76] James Garson, *The Logics of Space and Time*, (Pittsburgh: University of Pittsburgh, 1969).

[77] James Garson, "Two New Interpretations of Modality", *Nouvelle Serie* 15(59–60) (1972): 443–459.

[78] James Garson, "The Completeness of an Intensional Logic: Definite Topological Logic", *Notre Dame Journal of Formal Logic* 14(2) (1973): 175–184.

[79] James Garson, "Indefinite Topological Logic", *Journal of Philosophical Logic* 2(1973): 102–118.

[80] Garson, "The Completeness of an Intensional Logic: Definite Topological Logic".

with indefinite indexes. On the other hand, the work presented in "The Substitution Interpretation in Topological Logic" serves as a continuation of both his previous papers on semantics for topological logic and more specifically, the exploration of indefinite indexes[81].

Nishimura's work primarily focuses on chronological logic, particularly on the incorporation of a modal operator as a primitive term. In his paper titled "On the Completeness of Chronological Logics with Modal Operators", he developed such a logic by combining Rescher and Urquhart's additive positional temporal logic with Von Wright's system $\mathbf{M}''$[82]. The main focus of this paper was to establish a completeness theorem for the logic obtained through the Makinson method. In a subsequent paper, "Is the Semantics of Branching Structures Adequate for Chronological Modal Logics?", Nishimura introduced a similar system that combines Rescher and Urquhart's additive positional temporal logic with Lewis' $\mathbf{S}_5$[83]. He utilized this logic to investigate the relationship between time structures, specifically branching time structures and causal structures.

5. Modern reception of positional logic

After the last works of Garson and Nishimura, positional logic remained largely neglected for almost 40 years. Until 2004, there was a dearth of research in the field, with few notable exceptions. It should be acknowledged that references to Łoś's assertion logic occasionally appeared in works such as Epistemic Logic: A Survey of the Logic of Knowledge or "Logika Epistemiczna Jerzego Łosia a Teoria Racjonalnego Zachowania" as a positional logic had a recognized influence on epistemic logic. However, no significant developments were made specifically within the field of positional logic during this period. This changed in 2004 with the publication of "Completeness of Minimal Positional Calculus" by Jarmużek and Pietruszczak, which marked a revival of interest in this area.

The revival of interest in positional logic can be largely attributed to the groundbreaking work of Tomasz Jarmużek. Many researchers in the post-Rescher period have drawn inspiration from his work, as his mentioned publication marked a new era in the history of positional logic[84]. This era can be viewed as a continuation of

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81 James Garson, "The Substitution Interpretation in Topological Logic", *Journal of Philosophical Logic* 3(1–2) (1974): 109–132.

82 Hirokazu Nishimura, "On the Completeness of Chronological Logics with Modal Operators", *Mathematical Logic Quarterly* 25(31) (1979): 487–496.

83 Hirokazu Nishimura, "Is the Semantics of Branching Structures Adequate for Chronological Modal Logics?", *Journal of Philosophical Logic* 8(1979): 469–475.

84 Tomasz Jarmużek, Pietruszczak Andrzej, "Completeness of Minimal Positional Calculus",

the efforts of previous scholars, particularly in the direction set by Rescher. While Rescher's idea for topological logic involved the abstraction of the intended meaning of context variables, Jarmużek and Pietruszczak took it even further. They not only abstracted from the distinction between definite and indefinite statements and variables, but also developed a more fundamental system of positional logic than Rescher's $\mathbf{P}_1$ and $\mathbf{P}_2$. Moreover, the meaning of context variables was left unspecified in their work. While Rescher mentioned topological spaces, geometry, and focused on temporal and spatial coordinates, Jarmużek's works stripped context variables of any predetermined meaning altogether.

After his initial work on positional logic, Jarmużek shifted his focus to the reconstructions of Diodorus' Master Argument utilizing the realization operator. This can be observed in works such as "Rekonstrukcje Rozumowania Diodora Kronosa w Ontologii Czasu Punktowego [Reconstruction of Diodorus Cronus' Argument in Frame of Ontology of Time Consisted of Points]" from 2006 and the monograph Jutrzejsza Bitwa Morska. Rozumowanie Diodora Kronosa from 2013, which was subsequently translated into English as On the Sea Battle Tomorrow That May Not Happen: A Logical and Philosophical Analysis of the Master Argument in 2018. The research presented in those works encompasses both historical and innovative aspects. The authors thoroughly examined the solutions put forth by influential scholars such as Prior and Rescher. They presented their own original reconstruction that introduces intriguing properties and perspectives to the discourse.

In addition to his research on the reconstruction of Diodorus' Master Argument, Jarmużek also extensively investigated the metalogical properties of positional logic and its extensions, building on the foundation laid in his previous work, Completeness of Minimal Positional Calculus. These publications include notable contributions such as "Minimal Logical Systems with R-operator: Their Metalogical Properties and Ways of Extensions" from 2007, a comprehensive monograph dedicated solely to positional logics titled Normalne Logiki Pozycyjne published in 2015, "Expressive Power of the Positional Operator R: A Case Study in Modal Logic and Modal Philosophy" from 2019, and "On Some Language Extension of Logic MR: A Semantic and Tableau Approach" from 2020.

The research initiated by Jarmużek was continued by Marcin Tkaczyk, his coauthor in Normalne Logiki Pozycyjne. Tkaczyk's focus shifted towards non-normal positional logics. In his work "Negation in Weak Positional Calculi" published in 2013 and "Distribution Laws in Weak Positional Logics", he explored the concept of weak positional logics. Building on this foundation, Anna Karczewska further con-

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Logic and Logical Philosophy 13(2004): 147–162.

tributed to the field with her works “On the Modal Interpretation of the Connective of Realization” and “Set Theoretic Semantics for Many-valued Positional Calculi”, both published in 2021. Furthermore, in her previous work, “Maximality of Minimal $\mathcal{R}$-Logic” from 2017, Karczewska investigated the topic of maximality within the context of positional logic.

The application of positional logic in epistemology has been explored in various works. Marek Lechniak’s 1988 publication “Logika Epistemiczna Jerzego Łosia a Teoria Racjonalnego Zachowania”, analyzes this topic[85]. Lechniak’s work was later supplemented with “Jerzy Łoś’s Epistemic Logic and the Origins of Epistemic Logics” in 2020, providing further insights into the assertion system of Łoś and its historical significance. In addition, Mateusz Klonowski has conducted innovative research in this area. In particular, his works are “The Problem of Logical Omniscience: The Critique of Non-normal Worlds and the Proposition of New Solution” from 2019 and “Epistemic Contextualism and Positional Logic” from 2020.

Positional logic has found successful applications not only in epistemology but also in the philosophy of science and social science. Marcin Tkaczyk, in his monograph, “Logic of Physical Time: The Connective of Temporal Realization in Physical Discourse”, published in 2009, explored the potential application of positional logic to the philosophy of time in the natural sciences. On the other hand, Krzysztof Pietruszczak and Joanna Szalacha-Jarmużek have researched the use of positional logic as a formal framework for social sciences. Their works, including “Logic of Social Ontology and Łoś’s Operator”, “Going Beyond the Dichotomy: Problems of Contemporary Sociology in the Context of the Proposals by Jerzy Łoś” both from 2020, and “Extended MR with Nesting of Predicate Expressions as a Basic Logic for Social Phenomena” from 2021, provide valuable information on the application of positional logic in social science.

Historical research on positional logic has been conducted through individual works as well as collaborative efforts by researchers associated with the Toruń Logic Group. One notable work is the collaborative effort of Jarmużek and Tkaczyk titled “Jerzy Łoś Positional Calculus and the Origin of Temporal Logic” from 2019, which initiated a discussion of the place of Łoś in the history of temporal logic development. Øhrstrom and Hasle’s work, “The Significance of the Contributions of A. N. Prior and Jerzy Łoś in the Early History of Modern Temporal Logic” from 2019, further contributed to this discussion. The work “The Influence of Jerzy Łoś on Early Developments in Temporal Logic” published in 2020 provided a defense of the ini-

[85] Marek Lechniak, “Logika Epistemiczna Jerzego Łosia a Teoria Racjonalnego Zachowania”, *Roczniki Filozoficzne* 36 (1988): 77–89.

tial position presented by Jarmużek and Tkaczyk. Furthermore, a special issue of *Studia z Historii Filozofii* was dedicated to the history of positional logic and Łoś's role in its development. Works such as "The Heritage of Jerzy Łoś's Philosophical Logic and the Polish Question" by Jarmużek and Kupś, as well as contributions by Lechniak, Pietrowicz, Szalacha-Jarmużek, Klonowski, Palczewski, and Parol shed light on the historical significance of Łoś and positional logic.

5.1. Rediscovery of positional logic

In their 2004 work Jarmużek and Pietruszczak revisited the subject of positional logic after more than three decades, reviving this branch of non-classical logics[86]. Building on Rescher's idea of abstracting positional logic from temporal and epistemic interpretations, they pushed the boundaries of positional logic even further.

The title of their work has significant meaning. First, it seems that the authors deliberately chose the term "positional logic" over "topological logic", which had been used in the work of Nishimura and Garson during the 1970s. We cannot know the authors' intentions, but we can certainly discuss the impact of their naming conventions. In doing so, they established a fresh and distinct identity for their research while also acknowledging the historical foundations laid by earlier scholars. It seems that they were also able to avoid certain connotations associated with the mathematical applications of this system, making it more versatile and reusable. Furthermore, the phrase "minimal positional calculus" suggests the exploration of a foundational system within positional logic. In essence, the authors sought to develop a more fundamental logic than even the established logics $\mathbf{P}_1$ and $\mathbf{P}_2$.

It is worth noting that the authors adopted the symbol $\mathcal{R}$, following Rescher's chronological logic. Their intention was to extend and build upon the ideas of Rescher and Urquhart regarding the application of the realization operator in various domains. This intention is encapsulated in the following quote[87]:

> The operator of realization can be applied more widely than only to temporal contexts. A review of these applications one can find in the book of Rescher and Urquhart. It is why we shall, considering some very general axioms in the further part of our paper, write merely about positions, without deciding about their nature.

The generalization of the system proposed by the authors included other aspects as well. In the traditional framework of positional logic, propositional variables were

[86] Jarmużek, Pietruszczak, "Completeness of Minimal Positional Calculus".
[87] Ibidem, 147.

limited to specific sets of propositions, such as temporally indefinite statements. This perspective was prevalent in the work of researchers like Rescher and Garson, who even proposed two approaches to positional logics based on whether definite or indefinite statements were considered. However, for Jarmużek and Pietruszczak's systems, such distinctions became irrelevant. In their approach, propositional variables range over all logically valid statements without being constrained by temporal or any other definiteness.

The authors introduced the language of minimal positional logic (referred to as **MR**) using contemporary notation in a conventional manner. It included propositional variables $p, q, r, \ldots$, positional letters $\alpha, \alpha_1, \alpha_2, \ldots$, standard logical connectives, and the realization operator $\mathcal{R}$.

The logic was defined by the following set of axioms closed under the *modus ponens* rule:

Axiom 1. *$e(A)$, if A is a tautology of* CPL *and e is a substitution of variables with positional formulas.*

Axiom 2. $\mathcal{R}_\alpha \neg A \leftrightarrow \neg \mathcal{R}_\alpha A$

Axiom 3. $\mathcal{R}_\alpha (A \wedge B) \rightarrow \mathcal{R}_\alpha A \wedge \mathcal{R}_\alpha B$

Axiom 4. *$\mathcal{R}_\alpha A$, if A is a tautology of* CPL.

Building on the presented system of logic, Jarmużek and Pietruszczak successfully established the distributivity theorem, which demonstrates that the realization operator distributes over all classical connectives[88]. This theorem was the primary objective in constructing the system of positional logic to achieve a minimal system where this specific theorem holds. Through meticulous definitions of proof theory and semantics for **MR**, the authors were able to establish the correctness and completeness of the system[89].

The work laid the foundation for the development of positional logic by presenting a minimal system that was sufficiently abstract and adaptable for various purposes. The language, along with its axiom system, provided a flexible framework that could be easily extended. The authors also explored the fundamental properties of the system, paving the way for future advancements, including the incorporation of quantification theories into the system.

[88] Ibidem, 151–153.
[89] Ibidem, 155–159.

The system **MR** and its extensions were developed later in such works as "Minimal Logical Systems with R-operator: Their Metalogical Properties and Ways of Extensions", "Expressive Power of the Positional Operator R: A Case Study in Modal Logic and Modal Philosophy", "On Some Language Extension of Logic MR: A Semantic and Tableau Approach", "Logic of Social Ontology and Łoś Operator" and "Extended MR with Nesting of Predicate Expressions as a Basic Logic for Social Phenomena". The most important and extensive work on that topic is undoubtedly Normalne Logiki Pozycyjne. This monograph by Jarmużek and Tkaczyk is devoted solely to positional logics and contains some very interesting results.

First, Jarmużek and Tkaczyk introduced the concept of "normal positional logics". They defined this class of logics as a subclass that satisfies distributivity laws for all logical connectives. Additionally, normal positional logics are characterized by the property that all substitutions for tautologies of classical propositional logic hold. Furthermore, this class of logics is closed under *modus ponens*, allowing for valid inference within the system[90].

Definition 5.1 (Normal Positional Logic)**.** *A normal postional logic is every logic* **X** *in which the following conditions are satisfied:*

- $\vdash_{\mathbf{X}} \neg\mathcal{R}_\alpha A \leftrightarrow \mathcal{R}_\alpha\neg A$
- $\vdash_{\mathbf{X}} \mathcal{R}_\alpha(A \wedge B) \leftrightarrow \mathcal{R}_\alpha A \wedge \mathcal{R}_\alpha B$
- $\vdash_{\mathbf{X}} \mathcal{R}_\alpha(A \vee B) \leftrightarrow \mathcal{R}_\alpha A \vee \mathcal{R}_\alpha B$
- $\vdash_{\mathbf{X}} \mathcal{R}_\alpha(A \rightarrow B) \leftrightarrow \mathcal{R}_\alpha A \rightarrow \mathcal{R}_\alpha B$
- $\vdash_{\mathbf{X}} \mathcal{R}_\alpha(A \leftrightarrow B) \leftrightarrow (\mathcal{R}_\alpha A \leftrightarrow \mathcal{R}_\alpha B)$
- $\vdash_{\mathbf{X}} e(A)$*, if* A *is a tautology of* CPL *and* e *is a a substitution of variables with positional formulas.*
- *logic* **X** *is closed under modus ponens.*

Based on the definition and results presented in "Completeness of Minimal Positional Calculus", we can conclude that **MR** is a normal positional logic. Furthermore, it is a minimal system within this class of logics. It is minimal in several aspects, as highlighted by the authors. Firstly, in terms of syntax complexity, **MR** consists only of propositional connectives and the realization operator. Secondly, it is minimal among the positional logic systems in which the logical connectives preserve their Boolean meanings[91].

[90] Tomasz Jarmużek, Marcin Tkaczyk, *Normalne Logiki Pozycyjne*, (Lublin: Wydawnictwo KUL, 2015), 118.

[91] Ibidem, 118.

After providing a precise definition of the minimal positional logic, the authors explored various ways to extend this system. One notable idea is the addition of an axiom for nesting the realization operator, which is compatible with Rescher's and Prior's systems, such as $\mathbf{R}_1$ and $\mathbf{A}_1$. Jarmużek and Tkaczyk focused on presenting a single syntactic condition that can be incorporated into the axiom system, along with two semantic conditions for two types of nesting of the realization operator. Although a detailed discussion of the semantic investigations is beyond the scope of this paragraph, it is worth noting that the formula presented by the authors has the identical form as the fifth axiom of Rescher's $\mathbf{P}_1$ and $\mathbf{R}_2$, as well as the fifth axiom of Prior's $\mathbf{A}_1$.

$$\mathcal{R}_\alpha(\mathcal{R}_\beta A) \leftrightarrow \mathcal{R}_\beta A$$

In addition to the previous extensions, Jarmużek and Tkaczyk also proposed an extension of positional logics with formulas outside the scope of the realization operator. They presented both semantic and syntactic conditions for two possible solutions. The syntactic condition captures the idea that if a formula outside the realization operator is true, then it is realized in every position.

$$A \rightarrow \mathcal{R}_\alpha A$$

In contrast to Rescher's fourth axiom of $\mathbf{P}_2$ or Prior's fourth axiom of $\mathbf{A}_1$, the opposite implication in **MR** had to be formulated differently, as there is no quantification theory incorporated in the system. Instead, it was expressed as a rule of inference in the following manner[92]:

$$\frac{\{\mathcal{R}_\alpha A : \alpha \in \mathsf{Si}\}}{A}$$

Another way to extend the minimal positional logic is by incorporating a classical quantification theory. Jarmużek and Tkaczyk introduced the system **MRQ**, which is based on first-order logic with analogues of the axioms specific to **MR**. The authors presented a comprehensive semantics for the system and provided proofs of the completeness theorem and other metalogical properties[93]. This extension allows for the incorporation of quantifiers and further expands the expressive power of positional logic.

The research conducted by Jarmużek and Tkaczyk in their book paved the way for further investigations in subsequent works, such as "Expressive Power of the Po-

[92] Ibidem, 127.
[93] Ibidem, 136–148.

sitional Operator R: A Case Study in Modal Logic and Modal Philosophy". In this work, the authors explored the translation of modal operators into the language of positional logic. They demonstrate that the modal system **K** can be properly embedded within **MRQ** through an appropriate translation[94]. In another publication, "On Some Language Extension of Logic MR: A Semantic and Tableau Approach", Jarmużek and Parol extend the **MR** system by expanding the range of the realization operator from a single context symbol to a sequence of context symbols[95]. This extension opens up exciting philosophical possibilities for interpretation and analysis. Another noteworthy work in the field is "Maximality of the Minimal R-Logic", where Karczewska delves into the investigation of the maximality of the **MR** system[96].

Tkaczyk and Karczewska also made significant contributions to the field of non-normal positional logic, expanding the scope of metalogical research surrounding minimal positional logic. Their exploration in this area began with the publication of Tkaczyk Logika Czasu Empirycznego. Funktor Realizacji Czasowej w Językach Teorii Fizykalnych, in which he investigated the development of positional logics that could effectively formalize the physical theory of time and introduced two original systems of positional logic that were weaker than **MR**. As **MR** represents the minimal normal system of positional logic, Tkaczyk's proposed systems were categorized as non-normal positional logics[97].

In the publication "Negation in Weak Positional Calculi", Tkaczyk introduced four systems of non-normal positional logic, each based on a distinct approach to negation[98]. The author's aim was to extend and generalize the findings of Jarmużek and Pietruszczak with respect to **MR**[99]. Tkaczyk continued his research in this field with the work "Distribution Laws in Weak Positional Logics", where he explored the topic of distributivity laws within non-normal positional logics[100]. Karczewska contributed to this area with her works "On the Modal Interpretation of the Connective of Realization" and "Set-Theoretic Semantics for Many-Valued Positional

[94] Tomasz Jarmużek, Marcin Tkaczyk, "Expressive Power of the Positional Operator R: A Case Study in Modal Logic and Modal Philosophy", *Ruch Filozoficzny* 75(2) (2019): 105.

[95] Tomasz Jarmużek, Aleksander Parol, "On Some Language Extension of Logic MR: A Semantic and Tableau Approach", *Roczniki Filozoficzne* 68(4) (2020): 345–366.

[96] Anna Karczewska, "Maximality of the Minimal R-Logic", *Logic and Logical Philosophy* 27(2) (2017): 193–203.

[97] Marcin Tkaczyk, *Logika Czasu Empirycznego. Funktor Realizacji Czasowej w Językach Teorii Fizykalnych*, (Lublin: Wydawnictwo KUL, 2009).

[98] Marcin Tkaczyk, "Negation in Weak Positional Calculi", *Logic and Logical Philosophy* 22(1) (2013): 3–19.

[99] Ibidem, 3–4.

[100] Marcin Tkaczyk, "Distribution Laws in Weak Positional Logics", *Roczniki Filozoficzne* 66(3) (2018): 163–179.

Calculi". In her most recent publication, she described three distinct semantics for weak positional logics[101].

5.2. Historical research on positional logic

The work of Jarmużek and Tkaczyk initiated research in several areas within positional logic[102]. One notable area is historical research, which examines the development of positional logic and Łoś's position in the history of logic as the originator of the system. Of particular interest is the claim made by Jarmużek and Tkaczyk that Łoś's logic $\mathbf{L_1}$ was the first system of temporal logic and should be recognized as such by historians[103].

The claim made in the aforementioned monograph had limited resonance in the scientific community initially because of its publication in Polish. However, in "Jerzy Łoś Positional Calculus and the Origin of Temporal Logic", the authors delved into the topic more thoroughly, providing a further justification for their claim based on historical evidence and the structure of logic $\mathbf{L}_1$[104]. In this work, the authors presented the results of Łoś in a historical context, highlighting the language of the system, its axioms and fundamental metalogical facts. They also highlighted the impact of Łoś's work and the significance of Hiż's review in shaping the reception of his results.

Jarmużek and Tkaczyk put forward a significant claim regarding Łoś's position in the history of temporal logic. This claim relies on two key assumptions. Firstly, they claim that Łoś developed, described, and thoroughly analyzed the first comprehensive calculus of temporal logic, marking a significant milestone in the field. Secondly, they argue that Prior, in his own work, was not only aware of Łoś's ideas but also drew inspiration from them[105].

In fact, the first part of the claim made by Jarmużek and Tkaczyk, asserting that the system $\mathbf{L_1}$ was the first system of temporal logic, seems rather evident when considering the definitions of temporal logic and the intended temporal interpretation of the system. Published in 1947, well before Prior's tense systems, $\mathbf{L_1}$ clearly exhibits temporal characteristics. The authors themselves emphasize this point, underscoring the temporal nature of $\mathbf{L_1}$.

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[101] Anna Karczewska, "Set-Theoretic Semantics for Many-Valued Positional Calculi", *Roczniki Filozoficzne* 68(4) (2021): 367–384.
[102] Jarmużek, Tkaczyk, *Normalne Logiki Pozycyjne*.
[103] Ibidem, 21.
[104] Jarmużek, "Jerzy Łoś Positional Calculus and the Origin of Temporal Logic": 261.
[105] Ibidem, 259.

The second assumption is formulated around the name of Prior due to the fact that in the literature Prior is widely known as the founder of temporal logic. The link between Prior and Łoś could lead to indirect grounding of the work[106] in the tradition of temporal logic. Of course, Prior's inspiration and acknowledgment of the Łoś's results is evident. We can find traces of mentions in such works as Formal Logic, Time and Modality and Past, Present and Future. The same argument is made by Jarmużek and Tkaczyk. Moreover, it is worth highlighting the previous sections of this work, where we discussed the specific connections between Prior's work and the work of Łoś.

The connection between Łoś and Prior is evident, as a significant portion of Prior's Time and Modality is dedicated to positional logics based on the results of Łoś. However, the second part of the claim, which suggests a direct impact of Łoś on the development of Prior's first tense system, is more challenging to substantiate. While it is known that Łoś's work was published in 1947 prior to Prior's important contributions, it is worth noting that it was originally published in Polish, limiting its international impact. It was not until 1951 when Hiż's review was published in the *Journal of Symbolic Logic*, presenting the core ideas from Łoś's work in a language accessible to non-Polish speaking logicians. Therefore, it is plausible that Łoś's ideas had limited influence on Prior's earliest works in tense logic. At least it is debatable.

This claim could be summarized as follows: until 1953, when the idea of tense logic was first articulated, Prior became acquainted with Łoś's work through Hiż's review and found inspiration in it. It is important to note that the authors do not claim that Łoś was the sole source of inspiration for Prior's remarkable contributions. Rather, they argue that Łoś's work served as one of the influences that guided Prior towards his own achievements. This implies a continuity in the development of temporal logic from Łoś to Prior. However, it is crucial to acknowledge that there are no direct proofs either supporting or refuting this claim. The available evidence consists of mere indications, which are insufficient to conclusively support such a proposition.

In fact, it is undeniable that $\mathbf{L}_1$ has the distinction of being the first system of temporal logic. Furthermore, there is evidence to suggest that Łoś's work directly influenced Prior's research, particularly during his lectures in Oxford in 1956, which served as the basis for his book [107].

[106] Łoś, "Podstawy Analizy Metodologicznej Kanonów Milla".

[107] It should be noted that Łoś, which was first mentioned in Prior's work in 1955, Formal Logic.

The claim put forth in the work of Jarmużek and Tkaczyk sparked a debate surrounding Łoś's position in the history of temporal logic[108]. Soon after the publication, Øhrstrøm and Hasle responded with a polemical article[109], where they contested all the assertions made by Jarmużek and Tkaczyk. Their argument is twofold. Firstly, they argue that based on a straightforward and widely accepted understanding of "temporal logic" Prior should unequivocally be recognized as the pioneer of modern temporal logic. Additionally, they point out that Łoś's work was not mentioned in Prior's initial presentation of temporal logic in August 1954, which weakens the claim made by Jarmużek and Tkaczyk[110].

A defense of the modified claim made by Jarmużek and Tkaczyk was presented in (Parol 2020), countering the arguments put forth by Øhrstrøm and Hasle. The objection raised was that the definition employed in the aforementioned argumentation equated temporal logics with tense logics. It was argued that Łoś's system satisfies the conditions of temporal logic, thus establishing it as the first system of temporal logic. The second claim was defended in a weakened form, acknowledging the lack of evidence to pinpoint Prior's awareness of Łoś's results in 1953 or even 1954, as the first references in published works appear in 1955. However, based on Prior's subsequent works from 1955 and 1957, it becomes apparent that he was indeed familiar with Łoś's results and continued his research in positional logic, incorporating it into his own work. The debate culminated in the publication of a special issue of *Studia z Historii Filozofii* exclusively dedicated to historical research on Łoś's contributions to positional logic, titled "The Heritage of Jerzy Łoś's Philosophical Logic". In this publication, notable works such as "Jerzy Łoś's Epistemic Logic and the Origins of Epistemic Logics" can be found, which examines system of assertion logic of Ł oś within its historical context and from the perspective of epistemic logic.

Historical research on positional logic also encompasses its connection to a specific philosophical problem – the reconstruction of Diodorus' Master Argument. This problem is closely tied to temporal logic and, interestingly, to the history of positional logic. Both Prior and Rescher offered reconstructions of the Master Argument, with Rescher explicitly employing the formal framework of positional logic

[108] Jarmużek, Tkaczy "Jerzy Łoś Positional Calculus and the Origin of Temporal Logic".

[109] Peter Øhrstrøm, Per Hasle, "The Significance of the Contributions of A. N. Prior and Jerzy Łoś in the Early History of Modern Temporal Logic", in: *Logic and Philosophy of Time: Further Themes from Prior. Vol. 2*, ed. Patrick Blackburn, Per Hasle, Peter Øhrstrøm, (Aalborg: Aalborg University Press, 2019).

[110] Øhrstrøm, "The Significance of the Contributions of A. N. Prior and Jerzy Łoś in the Early History of Modern Temporal Logic", 33.

and Prior using his tense logic. Noteworthy reconstructions can be found in works such as Prior's Time and Modality and Temporal Logic.

The initial findings of Jarmużek on the topic of the Master Argument were published in "Rekonstrukcje Rozumowania Diodora Kronosa w Ontologii Czasu Punktowego [Reconstruction of Diodorus Cronus' Argument in Frame of Ontology of Time Consisted of Points]". However, these results were further expanded upon in his monograph Jutrzejsza Bitwa Morska. Rozumowanie Diodora Kronosa. Originally written in Polish, an updated and translated version of the content was provided in On the Sea Battle Tomorrow That May Not Happen. In this book, the author presented a comprehensive exploration of various historical reconstructions of the Master Argument, utilizing both positional logic and tense logic. While positional logic is not the sole formal framework presented, it plays a key role in the book, as it serves as the basis for the author's novel reconstruction. Using an extended version of positional logic, Jarmużek proposes a reconstruction that can be interpreted within a branching-time structure. As a result, some philosophers may view it as an indeterministic formalization of the problem.

5.3. Applications

With the resurgence of positional logic, it has found applications in various philosophical domains. Not only have new areas of application emerged, but existing areas have also been revisited. Among the many applications, notable ones include its use in temporal logic, epistemology, the logic of physics, and even as a formal framework in social sciences. The versatility of positional logic has allowed it to find relevance and utility in a wide range of philosophical contexts.

The application of positional logic in the philosophy of time is evident in Jarmużek's extensive research on Diodorus' Master Argument. Noteworthy works in this area include On the Sea Battle Tomorrow That May Not Happen. A Logical and Philosophical Analysis of the Master Argument and preceding works. These works examine various aspects of the philosophy of time, utilizing the realization operator to define important notions such as properties of time structures. Similarly, Tkaczyk's work in Logika Czasu Empirycznego. Funktor Realizacji Czasowej w Językach Teorii Fizykalnych offers a comprehensive analysis of temporal logic, comparing and evaluating different systems of temporal logic. The central theme of Tkaczyk's monograph revolves around the search for a logic of physical time, where positional logic plays a crucial role. In Tkaczyk's aforementioned work, he presented two systems of positional logic specifically designed for their application as temporal logics in the realm of natural sciences.

Another example of the application of positional logic can be found in the works of Lechniak and Klonowski, where it was used to tackle epistemological problems. In Lechniak's works, the focus was on analyzing Łoś's assertion logic as a means of representing knowledge[111]. This assessment was conducted from the standpoint of modern epistemology and the theory of epistemic logics, both of which build upon Łoś's results. While the authors held a negative evaluation of the original system of assertion logic as an adequate model for human beliefs, they proposed certain improvements to the logic that could render it more suitable for the subject. In particular, Lechniak also conducted a comparison among various systems related to Łoś's assertion logic, including those of Rescher and Marciszewski.

The exploration of the epistemic interpretation of positional logic was further advanced by Klonowski. His initial work in this area was presented in "Problem Wszechwiedzy Logicznej. Krytyka Nienormalnych Światów i Propozycja Nowego Rozwiązania [The Problem of Logical Omniscience. The Critique of Non-normal Worlds and the Proposition of New Solution]". In this publication, Klonowski and Krawczyk investigated various solutions to the problem of logical omniscience in epistemic logic. They presented a well-known solution involving the use of non-normal worlds. Additionally, they introduced **MRE** and demonstrated that the problem of logical omniscience does not arise within this framework[112]. Furthermore, they highlighted positional logic as a promising formal framework for epistemic logic.

Subsequent to the aforementioned article, several more works exploring the use of positional logic as a framework for epistemic logic have been published. One notable contribution is the work of Klonowski and Palczewski, Epistemic Contextualism and Positional Logic, which focuses on the formalization of epistemic contextualism using positional logic[113]. Another publication by Klonowski in 2021, "Tableau Systems for Epistemic Positional Logics", presents a tableau method for various epistemic systems within positional logic[114]. These works further expand the application and methodology of positional logic in the realm of epistemology.

A novel and intriguing application of positional logic has emerged from the re-

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[111] Lechniak, "Jerzy Łoś's Epistemic Logic and the Origins of Epistemic Logics".

[112] Mateusz Klonowski, Krzysztof Krawczyk, "Problem Wszechwiedzy Logicznej. Krytyka Nienormalnych Światów i Propozycja Nowego Rozwiązania [The Problem of Logical Omniscience. The Critique of Non-normal Worlds and the Proposition of a New Solution]", *Filozofia Nauki* 27(1) (2019): 46.

[113] Mateusz Klonowski, Rafał Palczewski, "Epistemic Contextualism and Positional Logic". *Studia z Historii Filozofii* 11(3) (2020): 67–104.

[114] Mateusz Klonowski, Krzysztof Krawczyk, Bożena Pięta, "Tableau Systems for Epistemic Positional Logics", *Bulletin of the Section of Logic* 50(2) (2021): 177–204.

search conducted at the logic hub in Toruń. In their interdisciplinary paper "Logic of Social Ontology and Łoś Operator", Malinowski, Szalacha-Jarmużek, and Pietrowicz proposed the use of **MR** logic in the field of social sciences. To enable this application, the authors extended the mentioned system by incorporating sequences of context names instead of singular contexts. With this enhanced system, they presented a formalization approach for capturing social phenomena[115].

The research initiated in the aforementioned publication has since been extended and expanded upon in subsequent works "Going Beyond the Dichotomy. Problems of Contemporary Sociology in the Context of the Proposals by Jerzy Łoś" and "Extended MR with Nesting of Predicate Expressions as a Basic Logic for Social Phenomena". In the first mentioned work, the authors explore the potential of utilizing the realization operator in the field of social sciences, offering a way to bridge the qualitative and quantitative perspectives in social studies[116]. The latter work builds upon the framework introduced previously by Malinowski, Szalacha-Jarmużek, and Pietrowicz and focuses on further extensions of the system. Additionally, the work provides specific examples, social phenomena that have been formalized within this formal framework[117].

The versatility and wide-ranging applicability of positional logic can be observed in various domains. This inherent flexibility was recognized by its early proponents. Łoś, as mentioned earlier, employed positional logic to formalize his understanding of physical time and to express Mill's canons in a methodological context. He also used it to capture his foundational understanding of a system of rational assertions. Building upon Łoś's work, Prior further explored and expanded the scope of positional logic, envisioning its potential for generalization and application in diverse areas. Rescher continued this line of research and developed a system of topological logic, which found application in abstract subjects such as the logic of possible worlds. However, the true appreciation for the generality and flexibility of positional logic emerged with the work of Jarmużek and Pietruszczak, who presented the minimal system of positional logic without imposing any intended interpretation. This realization marked a significant milestone in the development of positional logic as a versatile framework for various domains.

[115] Jacek Malinowski, Krzysztof Pietrowicz, Joanna Szalacha-Jarmużek, "Logic of Social Ontology and Łoś Operator", *Logic and Logical Philosophy* 29 (2020): 239–258.

[116] Joanna Szalacha-Jarmużek, Krzysztof Pietrowicz, "Going Beyond the Dichotomy. Problems of Contemporary Sociology in the Context of the Proposals by Jerzy Łoś", *Studia z Historii Filozofii* 9(3) (2020): 51–65.

[117] Aleksander Parol, Krzysztof Pietrowicz, Joanna Szalacha-Jarmużek, "Extended MR with Nesting of Predicate Expressions as a Basic Logic for Social Phenomena", *Bulletin of the Section of Logic* 50(2) (2021): 205–227.

The applications discussed in this section merely scratch the surface of the vast expressive power of positional logic. Throughout the history of positional logic, we have witnessed a multitude of intriguing interpretations and uses for these systems. Furthermore, the minimal system of positional logic can be effortlessly extended to suit specific requirements and contexts. This versatility and adaptability make positional logic a promising formal framework for various philosophical logics.

6. Further developments

In our work, our objective was to provide a comprehensive overview of the history of positional logic. Our goal was to highlight the significant advancements and noteworthy systems of positional logic, while also providing a broader perspective on less influential research without delving into technical intricacies. We believe that we have achieved this objective by delineating the main trajectory of positional logic, featuring the prominent contributions of Łoś, Prior, Rescher, and subsequently Jarmużek. In addition, while refraining from extensive technical discussions, we aimed to provide a concise summary of more technical works. We believe that we accomplished this by offering an overview of grouped topics, specifically in the areas of topological logic and non-normal positional logics.

The presented perspective on positional logic, although extensive, is by no means exhaustive. There are several areas within the history of positional logic that warrant further study. Firstly, a comparison between the two original systems proposed by Łoś could shed light on their similarities and differences. Secondly, exploring the influence of Łoś's assertion logic on the development of epistemic logic and investigating the relationship between positional systems in epistemic logic would be valuable. Additionally, specific branches of positional logic, such as topological logic, non-normal positional logics, and modal positional logics, deserve attention and exploration.

Another possible topic that deserves a separate work is Prior's argumentation against positional logic. This argumentation should be further analyzed from a logical perspective, and its results should be assessed since it played a role in the transition from the positional system of temporal logic to tense logic. Additionally, the relationship between the works of Prior and those of Łoś should be further investigated to explore which areas of research were, in fact, a continuation of Łoś's results.

Lastly, it should be noted that this article did not provide an extensive study of the relationship between positional logic and similar systems. For instance, we did not explore the connections between positional logic and hybrid logic. Conducting such a study would require expanding the scope of the topic and would likely exceed the limitations of the current format.

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