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ON CRITICAL PSEUDO-RELATIVISTIC HARTREE EQUATION WITH POTENTIAL WELL

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Abstract. The aim of this paper is to investigate the existence and asymptotic behavior of the solutions for the critical pseudo-relativistic Hartree equation

$$\sqrt{-\Delta+m^2}u+(\beta V(x)-\lambda)u=\bigg(\int_{\mathbb{R}^N}\frac{|u(z)|^{2_\mu^*}}{|x-z|^\mu}\,dz\bigg)|u|^{2_\mu^*-2}u$$

for $\mathbb{R}^N$, where $m,\lambda,\beta\in\mathbb{R}^+$, $0<\mu<N$, $N\geq 3$, $2_\mu^*=(2N-\mu)/(N-1)$ plays the role of critical exponent due to the Hardy–Littlewood–Sobolev inequality. By transforming the nonlocal problem into a local one via the Dirichlet-to-Neumann map, we are able to obtain the existence of the solutions by variational methods. Suppose that $0<\lambda<\lambda_1(\Omega)$ with $\lambda_1(\Omega)$ the first eigenvalue and the parameter β is large enough, we can prove the existence of ground state solutions. Furthermore, for any sequences $\beta_n\to\infty$, we can show that the ground state solutions $\{u_n\}$ converges to a solution of

$$\sqrt{-\Delta+m^2}u-\lambda u=\bigg(\int_{\Omega}\frac{|u(z)|^{2_\mu^*}}{|x-z|^\mu}\,dz\bigg)|u|^{2_\mu^*-2}u\quad\text{in }\Omega,$$

where $\Omega:=\operatorname{int}V^{-1}(0)$ is a nonempty bounded set with smooth boundary. By the way we also establish the existence and nonexistence results for the ground state solutions of the problems set on bounded domain.

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