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STRICT C^1-TRIANGULATIONS IN O-MINIMAL STRUCTURES

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Abstract. Inspired by the recent articles of T. Ohmoto and M. Shiota [9], [10] on $\mathcal{C}^1$-triangulations of semialgebraic sets, we prove here by using different methods the following theorem: *Let R be a real closed field and let an expansion of R to an o-minimal structure be given. Then for any closed bounded definable subset A of R^n and a finite family $B_1, \ldots, B_r$ of definable subsets of A there exists a definable triangulation $h\colon |\mathcal{K}| \to A$ of A compatible with $B_1, \ldots, B_r$ such that $\mathcal{K}$ is a simplicial complex in R^n, h is a C^1-embedding of each (open) simplex $\Delta \in \mathcal{K}$ and h extends to a definable C^1-mapping defined on a definable open neighborhood of $|\mathcal{K}|$ in R^n.* This improves Ohmoto–Shiota's theorem in three ways; firstly, h is a $\mathcal{C}^1$-embedding on each simplex; secondly, the simplicial complex $\mathcal{K}$ is in the same space as A and thirdly, our proof is performed for any o-minimal structure. The possibility to have h with the first of these properties was stated by Ohmoto and Shiota as an open problem (see [9]).

1. Introduction

Our present article is inspired by the recent results of T. Ohmoto and M. Shiota [9], [10] on $\mathcal{C}^1$-triangulations of semialgebraic sets. We propose here a different proof giving a stronger theorem.

Assume that R is any real closed field and an expansion of R to some o-minimal structure is given. Throughout the paper we will be talking about definable

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sets and mappings referring to this o-minimal structure; for fundamental definitions and results on o-minimal structures the reader is referred to [12] or [1]. We adopt the following definitions of a simplex and a simplicial complex. Let $k, n \in \mathbb{N}$ and $k \leq n$. A *simplex of dimension k in R^n* is the open convex hull

$$\Delta = (a_0, \ldots, a_k) = \left\{ \sum_{i=0}^{k} \alpha_i a_i : \alpha_i > 0 \ (i = 0, \ldots, k), \ \sum_{i=0}^{k} \alpha_i = 1 \right\}$$

of $k+1$ affinely independent points a_i of R^n which are called the *vertices* of Δ. An *l-dimensional face of Δ* is any of the following simplexes $\Delta' = (a_{\nu_o}, \ldots, a_{\nu_l})$, where $0 \leq \nu_o < \ldots < \nu_l \leq k$.

A *simplicial complex* in R^n is a finite family $\mathcal{K}$ of simplexes in R^n which satisfies the following conditions:

(1) If $\Delta_1, \Delta_2 \in \mathcal{K}$ and $\Delta_1 \neq \Delta_2$, then $\Delta_1 \cap \Delta_2 = \emptyset$.
(2) If $\Delta \in \mathcal{K}$ and Δ' is a face of Δ, then $\Delta' \in \mathcal{K}$.

The closed bounded definable subset $|\mathcal{K}| = \bigcup \mathcal{K}$ of R^n is called the *polyhedron of the symplicial complex $\mathcal{K}$*.

Let A be a closed bounded subset of R^n. A *definable C^1-triangulation of A* is a pair $(\mathcal{K}, h)$, where $\mathcal{K}$ is a simplicial complex in some space R^m, $h\colon |\mathcal{K}| \to A$ is a definable homeomorphism such that for each $\Delta \in \mathcal{K}$, $h(\Delta)$ is a definable C^1-submanifold of R^n and $h|\Delta\colon \Delta \to h(\Delta)$ is a C^1-diffeomorphism. When $B_1, \ldots, B_r$ are definable subsets of A, we say that a *triangulation* $(\mathcal{K}, h)$ *is compatible with the sets* $B_1, \ldots, B_r$ if each of the sets $h^{-1}(B_j)$ is a union of some simplexes of $\mathcal{K}$. A *definable strict C^1-triangulation* is such a definable C^1-triangulation $(\mathcal{K}, h)$ that $h\colon |\mathcal{K}| \to R^n$ is of class C^1; i.e. it has an extension to a C^1-mapping defined on an open definable neighborhood of $|\mathcal{K}|$ in R^m.

Theorem 1.1 (Main Theorem). *Let A be a closed bounded definable subset of R^n and let $B_1, \ldots, B_r$ be a finite family of definable subsets of A. Then there exists a definable strict C^1-triangulation $(\mathcal{K}, h)$ of A compatible with $B_1, \ldots, B_r$ such that $\mathcal{K}$ is a simplicial complex in R^n.*

This result improves a theorem of Ohmoto–Shiota [9], [10] in three ways: firstly, h is a $\mathcal{C}^1$-embedding on every simplex; secondly, the simplicial complex $\mathcal{K}$ is in the same space as A and thirdly, it concerns any o-minimal structure. The possibility to have h with the first of these properties was stated by Ohmoto and Shiota as an open problem in [9]. Our proof of Main Theorem below is divided into two parts; in the first one we prove that there exists a definable $\mathcal{C}^1$-triangulation $(\mathcal{K}, h)$ of A compatible with $B_1, \ldots, B_r$ such that $\mathcal{K}$ is a simplicial complex in R^n, $h\colon |\mathcal{K}| \to R^n$ is Lipschitz and $\{h|\Delta\colon \Delta \in \mathcal{K}\}$ is a $\mathcal{C}^1$-stratification of h with the Whitney (A) condition and in the second part this triangulation will be improved to a strict $\mathcal{C}^1$-triangulation.

In the article we adopt the convention to identify mappings with their graphs by denoting a mapping and its graph by the same letter. If $\varphi, \psi \colon A \to R$ are two functions such that $\varphi(a) < \psi(a)$ for each $a \in A$, then (φ, ψ) is defined as $\{(a,t) \in A \times R : \varphi(a) < t < \psi(a)\}$.

2. Proof of Main Theorem

Part I. First we will prove that there exists a definable $\mathcal{C}^1$-triangulation $(\mathcal{K}, h)$ of A compatible with $B_1, \ldots, B_r$ such that $\mathcal{K}$ is a simplicial complex in R^n, $h \colon |\mathcal{K}| \to R^n$ is Lipschitz and $\{h|\Delta : \Delta \in \mathcal{K}\}$ is a $\mathcal{C}^1$-stratification of h with the Whitney (A) condition.

The proof is by induction on n. Without loss of generality we assume that A is the closure of its interior $A = \overline{\mathrm{int} A}$. By Theorem 3.12 from [3] there exists a definable C^1-triangulation $(\mathcal{K}, f)$ of A compatible with $B_1, \ldots, B_r$ such that $\mathcal{K}$ is a simplicial complex in R^n and $f \colon |\mathcal{K}| \to A$ is a Lipschitz mapping. By the assumption about A, $|\mathcal{K}| = \bigcup\{\overline{\Delta} : \Delta \in \mathcal{K},\ \dim \Delta = n\}$. After a linear change of coordinates in R^n, we can assume that there exists a finite number of affine functions $\varphi_j \colon R^{n-1} \to R \quad (j = 1, \ldots, s)$, such that

$$\bigcup\{\partial\Delta : \dim \Delta = n\} \subset \bigcup_{j=1}^{s} \varphi_j,$$

where φ_j stands for the graph of $\varphi_j = \{(x_1, \ldots, x_n) \in R^n : x_n = \varphi_j(x_1, \ldots, x_{n-1})\}$. Then $\{f|\Delta : \Delta \in \mathcal{K}\}$ is a finite definable C^1-stratification of (the graph of) f. By [6] (see also [5] or [8], or [7]) it admits a finite definable C^1-refinement $\mathcal{S}$ with Whitney (A) condition such that strata from $\mathcal{S}$ of dimension n are exactly $\{f|\Delta : \Delta \in K,\ \dim \Delta = n\}$. There exists a corresponding $\mathcal{C}^1$-stratification $\mathcal{T}$ of $|\mathcal{K}|$ which is a refinement of $\mathcal{K}$ such that $\mathcal{S} = \{f|\Lambda : \Lambda \in \mathcal{T}\}$ and $\mathcal{T}$ contains all open simplexes of $\mathcal{K}$. Then for any pair $M, N \in \mathcal{T}$, such that $M \subset \overline{N}$ and for any $x_o \in M$ and any definable arc $\alpha \colon (0, \varepsilon) \to N$ $(\varepsilon > 0)$ such that $\lim_{t\to 0} \alpha(t) = x_o$, we have

$$\lim_{t\to 0} d_{\alpha(t)}(f|\, N) \supset d_{x_o}(f|\, M). \tag{2.1}$$

Here we use the fact that the limit $\lim_{t\to 0} d_{\alpha(t)}(f|\, N)$ always exists due to the o-minimality condition and the uniform boundedness of the differentials $d_{\alpha(t)}(f|N)$ following from the lipschitzianity condition.

Let $\pi \colon R^n \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}) \in R^{n-1}$ denote the natural projection. $\pi(|\mathcal{K}|)$ is a definable closed and bounded subset of R^{n-1}. Take $\rho > 0$ such that $|\varphi_j(y)| < \rho$, for each $y \in \pi(|\mathcal{K}|)$ and $j \in \{1, \ldots, s\}$. By the induction hypothesis there exists a strict C^1-triangulation $(\mathcal{L}, g)$ of $\pi(|\mathcal{K}|)$ compatible with all the subsets $\pi(N)$, where $N \in \mathcal{T}$, and at the same time with all the subsets $\{y \in R^{n-1} : \varphi_{j_1}(y) = \varphi_{j_2}(y)\}$ and $\{y \in R^{n-1} : \varphi_{j_1}(y) < \varphi_{j_2}(y)\}$, where $j_1 \neq j_2$.

Replacing $\mathcal{L}$ by its barycentric subdivision if necessary, we can assume that

$$\text{(2.2)}\quad \Lambda \in \mathcal{L},\ \varphi_{j_1}\circ g < \varphi_{j_2}\circ g \text{ on } \Lambda \\ \Rightarrow (\varphi_{j_1}\circ g)(c) < (\varphi_{j_2}\circ g)(c),\ \text{for some vertex } c \text{ of } \Lambda.$$

Put $\varphi_o \equiv -\rho$ and $\varphi_{s+1} \equiv \rho$.

Similarly as in the classical proofs of triangulation (compare [12, Chapter 8]), we built a polyhedral complex $\mathcal{P}$ in R^n the polyhedron of which is $|\mathcal{L}| \times [-\rho,\rho]$ and such that its projection under π is $|\mathcal{L}|$. To this end fix any simplex $\Lambda \in \mathcal{L}$. Put

$$\{\psi_o^\Lambda,\dots,\psi_{r+1}^\Lambda\} = \{\varphi_j \circ g|\Lambda : j = 0,\dots,s+1\},$$

where $\psi_o^\Lambda < \dots < \psi_{r+1}^\Lambda$, $r = r_\Lambda$ depending on Λ. Let $c_o,\dots,c_k$ be all vertices of Λ. For each $i \in \{0,\dots,r+1\}$, define also $\Psi_i^\Lambda\colon \Lambda \to R$ by the the formula

$$\Psi_i^\Lambda\biggl(\sum_{\nu=0}^{k}\alpha_\nu c_\nu\biggr) := \sum_{\nu=0}^{k}\alpha_\nu \psi_i^\Lambda(c_\nu),$$

where $\alpha_\nu > 0$, for each $\nu \in \{0,\dots,k\}$, and $\sum_{\nu=0}^{k}\alpha_\nu = 1$. Now we define the polyhedral complex

$$\mathcal{P} := \bigl\{\Psi_i^\Lambda : \Lambda \in \mathcal{L},\ i = 0,\dots,r_\Lambda+1\bigr\} \cup \bigl\{(\Psi_i^\Lambda,\ \Psi_{i+1}^\Lambda) : \Lambda \in \mathcal{L},\ i = 0,\dots,r_\Lambda\bigr\}.$$

The complex is well defined because ψ_i^Λ have continuous extensions to $\overline{\Lambda}$ and because of (2.2) (for more detailed explanation, see Lemma 2.1 below). There exists a unique definable homeomorphism $H\colon |\mathcal{L}| \times [-\rho,\rho] \to |\mathcal{L}| \times [-\rho,\rho]$, such that for each $\Lambda \in \mathcal{L}$ and $i \in \{0,\dots,r_\Lambda+1\}$, $H(u,\Psi_i^\Lambda(u)) = (u,\psi_i^\Lambda(u))$, for each $u \in \Lambda$, and for each $i \in \{0,\dots,r_\Lambda\}$ and $u \in \Lambda$, H is an affine isomorphism of the line segment $[(u,\Psi_i^\Lambda(u)),(u,\Psi_{i+1}^\Lambda(u))]$ onto the line segment $[(u,\psi_i^\Lambda(u)),(u,\psi_{i+1}^\Lambda(u))]$ (see Lemma 2.1). Since each of the functions ψ_i^Λ has a $\mathcal{C}^1$-extension to $\overline{\Lambda}$, according to Lemma 2.1, H is Lipschitz, $\mathcal{C}^1$ on every polyhedron $\Theta \in \mathcal{P}$ and $\{H|\,\Theta : \Theta \in \mathcal{P}\}$ is a $\mathcal{C}^1$-stratification of H with the Whitney (A) condition. By Lemma 2.2 below, all the above properties of H hold when we replace $\mathcal{P}$ by a simplicial complex $\mathcal{P}^*$ which is a barycentric subdivision of $\mathcal{P}$, and since $g\colon |\mathcal{L}| \to \pi(|\mathcal{K}|)$ is $\mathcal{C}^1$, the same properties are inherited by the mapping $\widetilde{H} := (g \times \mathrm{id}_R)\circ H\colon |\mathcal{L}| \times [-\rho,\rho] \to \pi(|\mathcal{K}|) \times [-\rho,\rho]$. It is clear from the definitions that there exists a subcomplex $\mathcal{R}$ of $\mathcal{P}$ such that $\{\widetilde{H}(\Theta) : \Theta \in \mathcal{R}\}$ is a $\mathcal{C}^1$-stratification of $|\mathcal{K}|$ which is a refinement of $\mathcal{K}$ such that $\widetilde{H}$ is Lipschitz and $\{\widetilde{H}|\Theta : \Theta \in \mathcal{R}\}$ is a $\mathcal{C}^1$-stratification of $\widetilde{H}$ with the Whitney (A) condition. Now the mapping $G := f \circ \widetilde{H}$ is the desired Lipschitz triangulation such that $\{G|\Theta : \Theta \in \mathcal{R}\}$ is a $\mathcal{C}^1$-stratification of G with Whitney (A) condition (see Lemma 2.2).

LEMMA 2.1 (cf. [3, Lemma 3.10]). *Let $\Lambda = (c_o, \dots, c_k)$ be a simplex in R^n of dimension k. Let $\mathcal{L}_\Lambda$ be the simplicial complex of all faces of Λ; so $|\mathcal{L}_\Lambda| = \overline{\Lambda}$. Let $\psi_i\colon \overline{\Lambda} \to R$ $(i = 1, 2)$ be definable C^1-functions such that $\psi_1 \leq \psi_2$ and*

$$(2.3)\quad \Delta \in \mathcal{L}_\Lambda,\ \psi_1|\Delta \not\equiv \psi_2|\Delta \quad \Rightarrow \text{ there is a vertex } c_\nu \text{ of } \Delta \text{ such that } \psi_1(c_\nu) < \psi_2(c_\nu).$$

Let $\Psi_i\colon |\overline{\Lambda}| \to R$ $(i = 1, 2)$ be defined by the formula

$$\Psi_i\Big(\sum_{\nu=0}^{k} \alpha_\nu c_\nu\Big) = \sum_{\nu=0}^{k} \alpha_\nu \psi_i(c_\nu),$$

where $\sum_{\nu=0}^{k} \alpha_\nu = 1$, $\alpha_\nu \geq 0$. Consider the following polyhedral complex

$$\mathcal{P} = \big\{\Psi_i|\,\Delta : \Delta \in \mathcal{L}_\Lambda,\ i = 1, 2\big\} \cup \big\{(\Psi_1|\Delta, \Psi_2|\Delta) : \Delta \in \mathcal{L}_\Lambda,\ \Psi_1|\Delta < \Psi_2|\Delta\big\}.$$

Then there exists a unique definable homeomorphism

$$H\colon |\mathcal{P}| \to \{(y, z) \in \overline{\Lambda} \times R : \psi_1(y) \leq z \leq \psi_2(y)\}$$

such that, for each $y \in \overline{\Lambda}$ and $i = 1, 2$, $H(y, \Psi_i(y)) = (y, \psi_i(y))$ and H is an affine isomorphism of the line segment $[(y, \Psi_1(y)), (y, \Psi_2(y))]$ onto the line segment $[(y, \psi_1(y)), (y, \psi_2(y))]$. Moreover, we have that

(a) *H is Lipschitz,*
(b) *H is $\mathcal{C}^1$-mapping on each $\Theta \in \mathcal{P}$ and*
(c) *$\{H|\Theta : \Theta \in \mathcal{P}\}$ is a $\mathcal{C}^1$-stratification of H with the Whitney* (A) *condition.*

PROOF. It is clear that, for each $\Delta \in \mathcal{L}_\Lambda$,

$$H(y, w) = \begin{cases} (y, \psi_1(y)) & \text{if } (y, w) \in \Psi_1|\Delta \\ \Big(y, \dfrac{w - \Psi_1(y)}{\Psi_2(y) - \Psi_1(y)}\, \psi_2(y) + \dfrac{\Psi_2(y) - w}{\Psi_2(y) - \Psi_1(y)}\, \psi_1(y)\Big) & \\ & \text{if } (y, w) \in (\Psi_1|\Delta, \Psi_2|\Delta) \\ (y, \psi_2(y)) & \text{if } (y, w) \in \Psi_2|\Delta. \end{cases}$$

Notice that H is a well-defined bijection due to (2.3), which implies that $\psi_1 < \psi_2$ on Δ if and only if $\Psi_1 < \Psi_2$ on Δ, otherwise $\psi_1 \equiv \psi_2$ on Δ and $\Psi_1 \equiv \Psi_2$ on Δ. To prove (a), (b) and (c), first observe that using the following $\mathcal{C}^1$-diffeomorphism

$$\overline{\Lambda} \times R \ni (y, w) \mapsto (y, w - \psi_1(y)) \in \overline{\Lambda} \times R$$

we can assume without any loss of generality that $\psi_1 \equiv \Psi_1 \equiv 0$. Of course, we can assume that $\psi := \psi_2 > 0$ and $\Psi := \Psi_2 > 0$ on Λ. The condition (b) is clearly fulfilled. Put $\Pi = (0|\Lambda, \Psi|\Lambda)$ and $H(y, w) = (y, H^*(y, w))$. In order to prove (a) it suffices to show that all first-order partial derivatives of H^* are bounded on Π. Since

$$\begin{aligned}\frac{\partial H^*}{\partial y_j}(y,w) &= \frac{w}{\Psi(y)}\cdot\frac{\partial\psi}{\partial y_j}(y) - \frac{w}{\Psi(y)}\cdot\frac{\psi(y)}{\Psi(y)}\cdot\frac{\partial\Psi}{\partial y_j}(y),\\ \frac{\partial H^*}{\partial w}(y,w) &= \frac{\psi(y)}{\Psi(y)},\end{aligned}\tag{2.4}$$

it is enough to show that ψ/Ψ is bounded on Λ. This is clear if $\psi(c_\nu)=\Psi(c_\nu)>0$, for all ν, so assume that $\{c_o,\dots,c_l\}=\{c_\nu:\psi(c_\nu)=0\}$, where $0\le l<k$. By an affine change of coordinates one can assume that $c_o=0$ and c_ν ($\nu=1,\dots,k$) are vectors of the canonical basis. Let $y=(y_1,\dots,y_k)\in\Pi$. Put $u=(y_1,\dots,y_l,0,\dots,0)$. We have

$$\left|\frac{\psi(y)}{\Psi(y)}\right| = \left|\frac{\psi(y)-\psi(u)}{\Psi(y)}\right| \le \frac{M\sum_{\nu=l+1}^{k} y_\nu}{\sum_{\nu=l+1}^{k} y_\nu\psi(c_\nu)} \le \frac{M}{\min\{\psi(c_\nu):\nu=l+1,\dots,k\}},$$

where M is the upper bound for the absolute value of the first-order partial derivatives of ψ. In order to check (c), first observe that H is a $\mathcal{C}^1$-diffeomorphism of $\{(y,w)\in|\mathcal{P}|:\Psi(y)>0\}$ onto $\{(y,z)\in\overline{\Lambda}\times R:0\le z\le\psi(y),\ \psi(y)>0\}$. Therefore, without any loss of generality, it suffices to check the Whitney (A) condition for Π and

$$\begin{aligned}\Theta\subset\{(y,w)\in\overline{\Lambda}\times R:\Psi(y)=0=w\} &= \{(y,w)\in\overline{\Lambda}\times R:\psi(y)=0=w\}\\ &= \operatorname{conv}\{c_o,\dots,c_l\}\times\{0\}.\end{aligned}$$

Hence, without any loss of generality, one can assume that $\Theta=(c_o,\dots,c_p)\times\{0\}$, where $p\le l$. Fix any $(a,0)\in\Theta$. By (2.4), since ψ and Ψ are $\mathcal{C}^1$, we have

$$\frac{\partial H^*}{\partial y_j}(y,w)\to 0,\quad \text{for } j=1,\dots,p,\text{ when } \Pi\ni(y,w)\to(a,0).$$

This ends the proof of (c) and of Lemma 2.1. □

The next lemma is a particular case of the general fact that the Whitney (A) condition is preserved in a transversal intersection (see [2]).

Lemma 2.2. *Let $H\colon A\to R^m$ be a definable Lipschitz mapping defined on a closed subset $A\in R^n$. Let $\mathcal{S}$ be a definable finite $\mathcal{C}^1$-stratification of A such that $H|M$ is $\mathcal{C}^1$ for each $M\in\mathcal{S}$ and $\{H|M:M\in\mathcal{S}\}$ is a $\mathcal{C}^1$-stratification of H with the Whitney* (A) *condition. Let $\mathcal{T}$ be a definable finite C^1-stratification of A with the Whitney* (A) *condition which is a refinement of $\mathcal{S}$. Then $\{H|N:N\in\mathcal{T}\}$ is a $\mathcal{C}^1$-stratification of H with the Whitney* (A) *condition.*

Proof. It follows from the Lipschitz condition that the differentials of $H|M$ are uniformly bounded. Hence the proof is immediate. □

Part II. Let $(\mathcal{K}, f)$ be a definable $\mathcal{C}^1$-triangulation of A compatible with $B_1, \ldots, B_r$ such that $\mathcal{K}$ is a simplicial complex in R^n,

$$f\colon |\mathcal{K}| \to R^n \quad \text{is Lipschitz} \tag{2.5}$$

and

$$\{f|\Delta : \Delta \in \mathcal{K}\} \quad \text{is a } \mathcal{C}^1\text{-stratification with the Whitney (A) condition.} \tag{2.6}$$

Now we will improve f to get a strict $\mathcal{C}^1$-triangulation of A. To this end we will modify f in some tubular neighborhoods of simplexes.

Fix any simplex $\Gamma \in \mathcal{K}$ of dimension $p < n$. Without loss of generality we can assume that $0 \in \Gamma$ and $\Gamma \subset R^p = \{(x_1, \ldots, x_n) \in R^n : x_{p+1} = \ldots = x_n = 0\}$. Let $R^{n-p} = \{(x_1, \ldots, x_n) \in R^n : x_1 = \ldots = x_p = 0\}$. There are affine functionals $\rho_j\colon R^p \to R$ $(j = 0, \ldots, p)$ such that $\Gamma = \{u \in R^p : \rho_j(u) > 0,\ j = 0, \ldots, p\}$.

Consider the star $\mathrm{St}(\Gamma, \mathcal{K})$ of Γ in $\mathcal{K}$; i.e. $\mathrm{St}(\Gamma, \mathcal{K}) = \{\Lambda \in \mathcal{K} : \Gamma \text{ is a face of } \Lambda\}$. Then $\Omega := \bigcup\{\Lambda \in \mathrm{St}(\Gamma, \mathcal{K})\}$ is an open neighborhood of Γ in $|\mathcal{K}|$. There exists $\alpha > 0$ such that, for each $u \in \Gamma$,

$$\mathrm{dist}(u, \partial\Omega) > \alpha \min_j \rho_j(u).$$

Put $\omega(u) := \rho_o^2(u) \cdot \ldots \cdot \rho_p^2(u)$, for each $u \in \Gamma$. There exists $\varepsilon > 0$ such that, for each $u \in \Gamma$,

$$2\,\varepsilon\omega(u) \le \alpha \min_j \rho_j(u) < \mathrm{dist}(u, \partial\Omega). \tag{2.7}$$

Then $G := \{(u, v) \in |\mathcal{K}| : u \in \Gamma,\ v \in R^{n-p},\ |v| \le \varepsilon\omega(u)\}$ is a neighborhood of Γ in $|\mathcal{K}|$ contained in Ω due to (2.7).

Let $\varphi\colon [0, +\infty) \to [0, +\infty)$ be a definable $\mathcal{C}^1$-function such that $\varphi(0) = \varphi'(0) = 0$, $\varphi(t) = 1$, for $t \ge 1$, and $\varphi'(t) > 0$, for $t \in (0, 1)$. Now we define $g\colon \Gamma \times R^{n-p} \to \Gamma \times R^{n-p}$ by the formula

$$g(u, v) := \left(u, \varphi\left(\frac{|v|}{\varepsilon\omega(u)}\right) v\right).$$

Then $g(G) = G$ and g is the identity outside G. Besides, g is a $\mathcal{C}^1$-diffeomorphism of $\Gamma \times R^{n-p} \setminus \Gamma$ onto $\Gamma \times R^{n-p} \setminus \Gamma$, because its inverse on $\Gamma \times R^{n-p} \setminus \Gamma$ is

$$g^{-1}(u, w) = \left(u, \varepsilon\omega(u)\, \psi^{-1}\left(\frac{|w|}{\varepsilon\omega(u)}\right) \frac{w}{|w|}\right),$$

where $\psi\colon (0, +\infty) \to (0, +\infty)$ is a $\mathcal{C}^1$-diffeomorphism defined by the formula $\psi(t) := \varphi(t)t$.

Furthermore, g is C^1 on $\Gamma \times R^{n-p}$, because for any $j \in \{1, \ldots, n-p\}$

$$\frac{\partial g}{\partial v_j}(u, v) = \left(0, \frac{v_j}{|v|} \cdot \frac{1}{\varepsilon\omega(u)} \cdot \varphi'\left(\frac{|v|}{\varepsilon\omega(u)}\right) v + \varphi\left(\frac{|v|}{\varepsilon\omega(u)}\right) e_j\right), \tag{2.8}$$

where $e_j = (0, \ldots, \underset{(j)}{1}, \ldots, 0)$. It follows that $(\partial g/\partial v_j)(u,v) \to (0,0)$, when $(u,v) \to (u_o, 0) \in \Gamma$. Similarly, $(\partial g/\partial u_i)(u,v) \to (e_i, 0)$, when $(u,v) \to (u_0, 0) \in \Gamma$.

Now we define $h\colon |\mathcal{K}| \to |\mathcal{K}|$ by putting $h(x) = g(x)$, for each $x \in G$, and $h(x) = x$ on $|\mathcal{K}| \setminus G$. It is clear that h is a homeomorphism of $|\mathcal{K}|$ onto $|\mathcal{K}|$ and a $\mathcal{C}^1$-diffeomorphism of each simplex $\Lambda \in \mathcal{K}$ onto itself. It follows from (2.8) and the boundedness of first-order partial derivatives of $f|\Lambda$ (due to (2.5)) that

$$\frac{\partial(f|\Lambda \circ h)}{\partial z}(u,v) \to (0,0), \quad \text{when } (u,v) \to (u_o, 0) \in \Gamma, \tag{2.9}$$

where $\Lambda \in \mathrm{St}(\Gamma, \mathcal{K}) \setminus \{\Gamma\}$ and z is any nonzero vector from the intersection of the linear subspace L generated by Λ with R^{n-p}. On the other hand we have for any $i \in \{1, \ldots, p\}$ and $(u,v) \in G \cap \Lambda$

$$\begin{aligned}\frac{\partial(f|\Lambda \circ h)}{\partial u_i}(u,v) &= \frac{\partial(f|\Lambda)}{\partial u_i}\left(u, \varphi\left(\frac{|v|}{\varepsilon\omega(u)}\right)v\right) \\ &+ \sum_{\nu=1}^{q} \frac{\partial(f|\Lambda)}{\partial z_\nu}\left(u, \varphi\left(\frac{|v|}{\varepsilon\omega(u)}\right)v\right)(-1)\,\frac{\partial\omega}{\partial u_i}(u)\,\frac{|v|}{\varepsilon\omega^2(u)}\,\varphi'\left(\frac{|v|}{\varepsilon\omega(u)}\right)v_\nu,\end{aligned} \tag{2.10}$$

where $z_1, \ldots, z_q$ is an orthogonal basis of $L \cap R^{n-p}$ and v_ν are coefficients of v with respect to this basis. It follows from (2.6) and from flatness of ω on $\partial\Gamma$ that

$$\frac{\partial(f|\Lambda \circ h)}{\partial \mu}(u,v) \to \frac{\partial(f|\Delta)}{\partial \mu}(u,0), \tag{2.11}$$

when $\Lambda \ni (u,v) \to (u_o, 0) \in \Delta$, for any simplex $\Delta \in \mathcal{K}$ contained in $\overline{\Gamma}$ and any unit vector μ parallel to Δ. This has two consequences. Firstly, all first-order partial derivatives of $f|\Lambda \circ h$ have finite limits when approaching Γ (see (2.9) and (2.11)). Secondly, the new triangulation $f \circ h$ satisfies the condition (2.6) at faces Δ of Γ where it may fail to be $\mathcal{C}^1$-extendable. But such Δ are of dimension less then $p = \dim \Gamma$, and our procedure works by decreasing induction on $p = \dim \Gamma$.

Consequently, after finite number of steps, we obtain a definable $\mathcal{C}^1$-triangulation $f\colon |\mathcal{K}| \to R^n$ of A which has all first-order partial derivatives continuous on $|\mathcal{K}|$. Hence, by a definable version of Whitney's extension theorem (see [4] or [11]), f can be extended to a definable $\mathcal{C}^1$-mapping defined on the whole space R^n.

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