Topological **M**ethods in **N**onlinear **A**nalysis
Volume 51, No. 2, 2018, 609–636
DOI: 10.12775/TMNA.2018.018

ON DOUBLY NONLOCAL p-FRACTIONAL COUPLED ELLIPTIC SYSTEM

Tuhina Mukherjee — Konijeti Sreenadh

Abstract. We study the following nonlinear system with perturbations involving p-fractional Laplacian:

$$(\mathrm{P})\qquad \begin{cases} (-\Delta)^s_p u + a_1(x)u|u|^{p-2} = \alpha(|x|^{-\mu} * |u|^q)|u|^{q-2}u \\ \qquad\qquad +\beta(|x|^{-\mu} * |v|^q)|u|^{q-2}u + f_1(x) \quad \text{in } \mathbb{R}^n, \\ (-\Delta)^s_p v + a_2(x)v|v|^{p-2} = \gamma(|x|^{-\mu} * |v|^q)|v|^{q-2}v \\ \qquad\qquad +\beta(|x|^{-\mu} * |u|^q)|v|^{q-2}v + f_2(x) \quad \text{in } \mathbb{R}^n, \end{cases}$$

where $n > sp$, $0 < s < 1$, $p \geq 2$, $\mu \in (0,n)$, $p(2-\mu/n)/2 < q < p^*_s(2-\mu/n)/2$, $\alpha,\beta,\gamma > 0$, $0 < a_i \in C(\mathbb{R}^n,\mathbb{R})$, $i = 1,2$ and $f_1, f_2\colon \mathbb{R}^n \to \mathbb{R}$ are perturbations. We show existence of at least two nontrivial solutions for (P) using Nehari manifold and minimax methods.

1. Introduction and main results

In this article, we consider the following nonlinear system with perturbations involving p-fractional Laplacian:

$$(\mathrm{P})\qquad \begin{cases} (-\Delta)^s_p u + a_1(x)u|u|^{p-2} = \alpha(|x|^{-\mu} * |u|^q)|u|^{q-2}u \\ \qquad\qquad +\beta(|x|^{-\mu} * |v|^q)|u|^{q-2}u + f_1(x) \quad \text{in } \mathbb{R}^n, \\ (-\Delta)^s_p v + a_2(x)v|v|^{p-2} = \gamma(|x|^{-\mu} * |v|^q)|v|^{q-2}v \\ \qquad\qquad +\beta(|x|^{-\mu} * |u|^q)|v|^{q-2}v + f_2(x) \quad \text{in } \mathbb{R}^n, \end{cases}$$

2010 *Mathematics Subject Classification.* 35R11, 35R09, 35A15.
Key words and phrases. p-fractional Laplacian; Choquard equation; Nehari manifold.

where $p \geq 2, s \in (0,1)$, $n > sp$, $\mu \in (0,n)$, $p(2-\mu/n)/2 < q < p_s^*(2-\mu/n)/2$, $\alpha,\beta,\gamma > 0$, $0 < a_i \in C^1(\mathbb{R}^n,\mathbb{R})$, $i = 1,2$ and $f_1, f_2 \colon \mathbb{R}^n \to \mathbb{R}$ are perturbations. Here $p_s^* = np/(n-sp)$ is the critical exponent associated with the embedding of the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$. The p-fractional Laplace operator is defined on smooth functions as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{|x|>\varepsilon} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\, dy$$

which is nonlinear and nonlocal in nature. This definition matches to linear fractional Laplacian operator $(-\Delta)^s$, *up to* a normalizing constant depending on n and s, when $p = 2$. The operator $(-\Delta)_p^s$ is degenerate when $p > 2$ and singular when $1 < p < 2$. For more details and motivations and the function spaces $W^{s,p}(\Omega)$, we refer to [9], [17]. Researchers are paying a lot of attention to the study of fractional and non-local operators of elliptic type due to concrete real world applications in finance, thin obstacle problem, optimization, quasi-geostrophic flow etc. The eigenvalue problem involving p-fractional Laplace equations has been extensively studied in [7], [8], [32], [34]. The Brezis Nirenberg type problem involving p-fractional Laplacian has been studied in [31] whereas existence has been investigated via Morse theory in [30]. Problems involving p-fractional Laplacian have been studied in [26], [27] using the Nehari manifold. A vast amount of literature can be found for the case $p = 2$, i.e. fractional Laplacian $(-\Delta)^s$, which has been actively investigated in recent years. Separately, we would like to mention work of Servadei and Valdinoci in [42]–[44] on bounded domains.

The study of fractional Schrödinger equations has attracted attention of many researchers nowadays. Frölich et al. studied nonlinear Hartree equations in [19], [20]. In the nonlocal case, using variational methods and the Lusternik–Schnirelmann category theory, Lü and Xu [35] proved existence and multiplicity for the equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = \varepsilon^{-\alpha}(W_\alpha(x) * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^n,$$

where $\varepsilon > 0$ is a parameter, $0 < s < 1$, $N > 2s$, V is a continuous potential, and W_α is the Riesz potential. Wu in [51] proved the existence of standing waves by studying the related constrained minimization problems via the concentration–compactness principle for the following nonlinear fractional Schrödinger equations with Hartree type nonlinearity

$$i\psi_t + (-\Delta)^\alpha \psi - (|\cdot|^{-\gamma} * |\psi|^2)\psi = 0,$$

where $0 < \alpha < 1$, $0 < \gamma < 2\alpha$ and $\psi(x,t)$ is a complex-valued function on $\mathbb{R}^d \times \mathbb{R}$, $d \geq 1$. Some recent works on Schödinger equations with fractional Laplacian equation include [16], [21], [41], [45] with no attempt to provide a complete list.

Existence of solutions for the equation of the type

$$-\Delta u + w(x)u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^n,$$

where w is an appropriate function, I_α is the Reisz potential and $p > 1$ is chosen appropriately, have been studied in [3], [14], [22], [37], [50]. Very recently, Ghimenti, Moroz and Schaftingen [23] proved the existence of least action nodal solution for the above problem taking $w \equiv 1$ and $p = 2$. Alves, Figueiredo and Yang [2] proved existence of a nontrivial solution via the penalization method for the following Choquard equation:

$$-\Delta u + V(x)u = (|x|^{-\mu} * F(u))f(u) \quad \text{in } \mathbb{R}^n,$$

where $0 < \mu < N$, $N = 3$, V is a continuous real function and F is the primitive function of f. Alves and Yang also studied the quasilinear Choquard equation in [4]–[6]. For more results, we also refer to [38]–[40] for interested readers.

Systems of elliptic equations involving fractional Laplacian and homogeneous nonlinearity have been studied in [25], [24], [29] and p-fractional elliptic systems have been studied in [11], [12] using the Nehari manifold techniques. Very recently, Guo et al. [28] studied a nonloca l system involving fractional Sobolev critical exponent and fractional Laplacian. There are not many results on elliptic systems with non-homogeneous nonlinearities in the literature. We also cite [1], [13], [18], [36], [49] as some very recent works on the study of fractional elliptic systems. We also cite [52] where multiplicity of positive solutions for the nonhomogeneous Choquard equation has been shown using the Nehari manifold.

Our work is motivated by the work of Tarantello [47] where the author used the structure of the associated Nehari manifold to obtain the multiplicity of solutions for the following nonhomogeneous Dirichlet problem on bounded domain Ω:

$$-\Delta u = |u|^{2^*-2}u + f \quad \text{in } \Omega, \ u = 0 \text{ on } \partial\Omega.$$

Concerning the nonhomogeneous system, Wang et. al [48] studied the problem (P) in the local case $s = 1$ and obtained partial multiplicity results. In this paper, we improve their results and establish multiplicity results for f_1 and f_2 satisfying a weaker assumption (1.1) below. We describe the topology of the Nehari set and use its structure to obtain solutions which are minimizers of energy functional on its components. We need the following function spaces: For $i = 1, 2$ we introduce the Banach spaces

$$Y_i := \left\{ u \in W^{s,p}(\mathbb{R}^n) : \int_{\mathbb{R}^n} a_i(x)|u|^p \, dx < +\infty \right\}$$

equipped with the norm

$$\|u\|_{Y_i}^p = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy + \int_{\mathbb{R}^n} a_i(x)|u|^p \, dx.$$

We define the product space $Y = Y_1 \times Y_2$ which is a *reflexive Banach space* with the norm

$$\|(u,v)\|^p := \|u\|_{Y_1}^p + \|v\|_{Y_2}^p, \quad \text{for all } (u,v) \in Y.$$

Throughout this paper, we assume the following condition on a_i, for $i = 1,2$:

(A) $a_i \in C(\mathbb{R}^n, \mathbb{R})$, $a_i > 0$ and there exists $M_i > 0$ such that

$$\mu(\{x \in \mathbb{R}^n : a_i \leq M_i\}) < \infty.$$

Then under condition (A) on a_i, for $i = 1,2$, we get $Y_i \hookrightarrow L^r(\mathbb{R}^n)$ continuously for $r \in [p, p_s^*]$.

To obtain our result, we assume the following condition on perturbation terms:

$$\int_{\mathbb{R}^n} (f_1 u + f_2 v) < C_{p,q} \left(\frac{2q+p-1}{4pq} \right) \|(u,v)\|^{p(2q-1)/(2q-p)} \tag{1.1}$$

for all $(u,v) \in Y$ such that

$$\int_{\mathbb{R}^n} (\alpha(|x|^{-\mu} * |u|^q)|u|^q + 2\beta(|x|^{-\mu} * |u|^q)|v|^q + \gamma(|x|^{-\mu} * |v|^q)|v|^q)\, dx = 1$$

and

$$C_{p,q} = \left(\frac{p-1}{2q-1} \right)^{(2q-1)/(2q-p)} \left(\frac{2q-p}{p-1} \right).$$

It is easy to see that

$$2q > p\left(\frac{2n-\mu}{n} \right) > p-1 > \frac{p-1}{2p-1}$$

which implies

$$\frac{2q+p-1}{4pq} < 1.$$

So (1.1) implies that

$$\int_{\mathbb{R}^n} (f_1 u + f_2 v) < C_{p,q} \|(u,v)\|^{p(2q-1)/(2q-p)} \tag{1.2}$$

which we will use more frequently rather than our actual assumption (1.1). The importance of the assumption (1.1) instead of (1.2) can be felt in Lemma 3.5. If $f_1, f_2 = 0$, then we always have a solution for (P) that is the trivial solution. Now, the main results of this paper go as follows.

Theorem 1.1. *Suppose*

$$\frac{p}{2}\left(\frac{2n-\mu}{n} \right) < q < \frac{p}{2}\left(\frac{2n-\mu}{n-sp} \right),$$

$\mu \in (0,n)$ *and* (A) *holds true. Let* $0 \not\equiv f_1, f_2 \in L^{p/(p-1)}(\mathbb{R}^n)$ *satisfy* (1.1) *then* (P) *has a weak solution which is a local minimum of* J *on* Y. *Moreover, if* $f_1, f_2 \geq 0$ *then this solution is a nonnegative weak solution.*

THEOREM 1.2. *Under the hypothesis of Theorem* 1.1, (P) *has second weak solution* (u_1, v_1) *in* Y. *Also, if* $f_1, f_2 \geq 0$, *then the second solution is nonnegative.*

This article is organized as follows: In Section 2, we set up our function space where our weak solution lies and recall some important results especially the Hardy–Littlewood–Sobolev inequality. In Section 3, we analyze fibering maps while defining the Nehari manifold and show that minimization of energy functional on suitable subsets of the Nehari manifold gives us the weak solution to (P). We study the Palais–Smale sequences in Section 4. Finally, we prove our main theorem in Section 5.

2. Preliminary results

In this section, we state some important known results which will be used as tools to prove our main results. The key inequality is the following classical Hardy–Littlewood–Sobolev inequality [33].

PROPOSITION 2.1 (Hardy–Littlewood–Sobolev inequality). *Let* $t, r > 1$ *and* $0 < \mu < n$ *with* $1/t + \mu/n + 1/r = 2$, $f \in L^t(\mathbb{R}^n)$ *and* $h \in L^r(\mathbb{R}^n)$. *There exists a sharp constant* $C(t, n, \mu, r) > 0$, *independent of* f, h, *such that*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)h(y)}{|x-y|^\mu} \, dx \, dy \leq C(t, n, \mu, r) \|f\|_{L^t(\mathbb{R}^n)} \|h\|_{L^r(\mathbb{R}^n)}.$$

REMARK 2.2. In general, let $f = h = |u|^q$ then by the Hardy–Littlewood–Sobolev inequality we get that the quantity

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)|^q |u(y)|^q}{|x-y|^\mu} \, dx \, dy$$

is finite if $|u|^q \in L^t(\mathbb{R}^n)$ for some $t > 1$ satisfying

$$\frac{2}{t} + \frac{\mu}{n} = 2.$$

Since we will be working in the space $W^{s,p}(\mathbb{R}^n)$, by fractional Sobolev embedding theorems (refer [17]), we must have $qt \in [p, p_s^*]$, where $p_s^* = np/(n - sp)$, i.e.

$$\frac{p}{2}\left(\frac{2n-\mu}{n}\right) \leq q \leq \frac{p}{2}\left(\frac{2n-\mu}{n-sp}\right).$$

We define

$$q_l := \frac{p}{2}\left(\frac{2n-\mu}{n}\right) \quad \text{and} \quad q_u := \frac{p}{2}\left(\frac{2n-\mu}{n-sp}\right).$$

Here, q_l and q_u are known as lower and upper critical exponents. We constrain our study to the case

$$\frac{p}{2}\left(\frac{2n-\mu}{n}\right) < q < \frac{p}{2}\left(\frac{2n-\mu}{n-sp}\right).$$

The next result is a basic inequality whose proof can be worked out in similar manner as the proof of Proposition 3.2 in [22, equation (3.3), p. 124].

Lemma 2.3. *For $u, v \in L^{2n/(2n-\mu)}(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)|^q|v(y)|^q}{|x-y|^\mu}\,dx\,dy \leq \bigg(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu}\,dx\,dy\bigg)^{1/2}\bigg(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x)|^q|v(y)|^q}{|x-y|^\mu}\,dx\,dy\bigg)^{1/2},$$

where $\mu \in (0,n)$ and $q \in [q_l, q_u]$.

We now prove following lemma which is a version of the concentration–compactness principle proved in [15, Lemma 2.18].

Lemma 2.4. *Let $n > sp$. Assume that $\{u_k\}$ is bounded in Y_1 and Y_2 and it satisfies*

$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}^n}\int_{B_R(y)} |u_k|^p\,dx = 0,$$

where $R > 0$ and $B_R(y)$ denotes the ball centered at y with radius R. Then $u_k \to 0$ strongly in $L^r(\mathbb{R}^n)$ for $r \in (p, p_s^*)$, as $k \to \infty$.

Proof. We prove the result for $i = 1$, and for $i = 2$ it follows similarly. Let $r \in (p, p_s^*)$, $y \in \mathbb{R}^n$ and $R > 0$. By using the Hölder inequality, for each k we get

$$\|u_k\|_{L^r(B_R(y))} \leq \|u_k\|^{1-\lambda}_{L^p(B_R(y))}\|u_k\|^{\lambda}_{L^{p_s^*}(B_R(y))},$$

where $1/r = (1-\lambda)/p + \lambda/p_s^*$. Then

$$\int_{B_R(y)} |u_k|^r\,dx \leq \|u_k\|^{r(1-\lambda)}_{L^p(B_R(y))}\|u_k\|^{r\lambda}_{L^{p_s^*}(\mathbb{R}^n)}. \tag{2.1}$$

We choose a family of balls $\{B_R(y_i)\}$ where their union covers $\mathbb{R}^n$ and are such that each point of $\mathbb{R}^n$ is contained in at most m of such balls (where m is a prescribed integer). Now, summing (2.1) over this family, we obtain

$$\|u_k\|^r_{L^r(\mathbb{R}^n)} \leq m\sup_{y\in\mathbb{R}^n}\bigg(\int_{B_R(y)}|u_k|^p\,dx\bigg)^{r(1-\lambda)/p}\|u_k\|^{r\lambda}_{L^{p_s^*}(\mathbb{R}^n)}.$$

Using the continuity of the embedding of Y_1 in $L^{p_s^*}(\mathbb{R}^n)$ and our hypothesis, we get $u_k \to 0$ strongly in $L^r(\mathbb{R}^n)$ as $k \to \infty$. □

The following is a compactness result for the space Y_i, $i = 1, 2$, which will be used further.

Lemma 2.5. *Suppose* (A) *holds. Then Y_i is compactly embedded in $L^r(\mathbb{R}^n)$, for $r \in [p, p_s^*)$ and $i = 1, 2$.*

Proof. We prove it for Y_1 (for Y_2 it follows analogously). Let $\{u_k\} \subset Y_1$ be a bounded sequence. Up to a subsequence, we may assume that $u_k \rightharpoonup u_0$ weakly in Y_1 as $k \to \infty$. Then $u_k \to u_0$ in $L^r_{\rm loc}(\mathbb{R}^n)$ as $k \to \infty$, for $r \in [p, p_s^*)$. We first prove that $u_k \to u_0$ strongly in $L^p(\mathbb{R}^n)$. Suppose $\xi_k := \|u_k\|_{L^p(\mathbb{R}^n)}$ and $\xi_k \to \xi$

along a subsequence, as $k \to \infty$. So, $\xi \geq \|u_0\|_{L^p(\mathbb{R}^n)}$. We claim that for each $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_R(0)} |u_k|^p \, dx < \varepsilon \quad \text{uniformly in } k.$$

If this holds then $u_k \to u_0$ strongly in $L^p(\mathbb{R}^n)$. Because we already have $u_k|_{B_R(0)} \to u_0|_{B_R(0)}$ strongly in $L^p(B_R(0))$, as $k \to \infty$,

$$\begin{aligned}\xi \geq \|u_0\|_{L^p(\mathbb{R}^n)} &= \left(\|u_0\|^p_{L^p(B_R(0))} + \|u_0\|^p_{L^p(\mathbb{R}^n \setminus B_R(0))}\right)^{1/p} \\ &\geq \lim_{k\to\infty} \|u_k\|_{L^p(B_R(0))} \\ &\geq \lim_{k\to\infty} \|u_k\|_{L^p(\mathbb{R}^n)} - \lim_{k\to\infty} \|u_k\|_{L^p(\mathbb{R}^n \setminus B_R(0))} \geq \xi - \varepsilon.\end{aligned}$$

To prove our claim, let us fix $\varepsilon > 0$ and choose constants $M, C > 0$ such that

$$M > \frac{2}{\varepsilon} \sup_k \|u_k\|^p_{Y_1} \quad \text{and} \quad C \geq \sup_{u \in Y_1 \setminus \{0\}} \frac{\|u_k\|^p_{L^{pr}(\mathbb{R}^n)}}{\|u_k\|^p_{Y_1}}.$$

Let r' be such that $1/r + 1/r' = 1$. Now condition (A) implies for $R > 0$ large enough,

$$\mu(\{x \in \mathbb{R}^n \setminus B_R(0) : a_1(x) < M\}) \leq \left(\frac{\varepsilon}{2C \sup_k \|u_k\|^p_{Y_1}}\right)^{r'}.$$

We set $A = \{x \in \mathbb{R}^n \setminus B_R(0) : a_1(x) \geq M\}$ and $B = \{x \in \mathbb{R}^n \setminus B_R(0) : a_1(x) < M\}$. Then, we get

$$\int_A |u_k|^p \, dx \leq \int_A \frac{a_1(x)}{M} |u_k|^p \, dx \leq \frac{1}{M} \|u_k\|^p_{Y_1} \leq \frac{\varepsilon}{2}.$$

Also using Hölder's inequality, we get

$$\int_B |u_k|^p \, dx \leq \left(\int_B |u_k|^{pr} \, dx\right)^{1/r} (\mu(B))^{1/r'} \leq C \|u_k\|^p_{Y_1} (\mu(B))^{1/r'} \leq \frac{\varepsilon}{2}.$$

Therefore we can write

$$\int_{\mathbb{R}^n \setminus B_R(0)} |u_k|^p \, dx = \int_A |u_k|^p \, dx + \int_B |u_k|^p \, dx \leq \varepsilon$$

which implies $u_k \to u_0$ strongly in $L^p(\mathbb{R}^n)$. Finally, using Lemma 2.4, it follows that $u_k \to u_0$ strongly in $L^r(\mathbb{R}^n)$, for $r \in [p, p_s^*)$. $\square$

For our convenience, if $u, \phi \in W^{s,p}(\mathbb{R}^n)$, we use the notation $\langle u, \phi \rangle$ to denote

$$\langle u, \phi \rangle := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))|u(x) - u(y)|^{p-2}(\phi(x) - \phi(y))}{|x - y|^{n+sp}} \, dx \, dy.$$

Definition 2.6. A pair of functions $(u,v) \in Y$ is said to be a weak solution to (P) if

$$\begin{aligned}\langle u,\phi_1\rangle &+ \int_{\mathbb{R}^n} a_1(x)u|u|^{p-2}\phi_1\,dx \\ &+ \langle v,\phi_2\rangle + \int_{\mathbb{R}^n} a_2(x)v|v|^{p-2}\phi_2\,dx - \alpha\int_{\mathbb{R}^n}(|x|^{-\mu}*|u|^q)u|u|^{q-2}\phi_1\,dx \\ &- \gamma\int_{\mathbb{R}^n}(|x|^{-\mu}*|v|^q)v|v|^{q-2}\phi_2\,dx - \beta\int_{\mathbb{R}^n}(|x|^{-\mu}*|v|^q)u|u|^{q-2}\phi_1\,dx \\ &- \beta\int_{\mathbb{R}^n}(|x|^{-\mu}*|u|^q)v|v|^{q-2}\phi_2\,dx - \int_{\mathbb{R}^n}(f_1\phi_1+f_2\phi_2)\,dx = 0,\end{aligned}$$

for all $(\phi_1,\phi_2) \in Y$.

Let us define the energy functional corresponding to (P) as

$$\begin{aligned}J(u,v) &= \frac{1}{p}\|(u,v)\|^p - \frac{1}{2q}\int_{\mathbb{R}^n}(\alpha(|x|^{-\mu}*|u|^q)|u|^q + \beta(|x|^{-\mu}*|u|^q)|v|^q)\,dx \\ &- \frac{1}{2q}\int_{\mathbb{R}^n}(\beta(|x|^{-\mu}*|v|^q)|u|^q + \gamma(|x|^{-\mu}*|v|^q)|v|^q)\,dx - \int_{\mathbb{R}^n}(f_1u+f_2v)\,dx.\end{aligned}$$

It is clear that weak solutions to (P) are critical points of J. We have the following symmetric property:

$$\int_{\mathbb{R}^n}(|x|^{\mu}*|u|^q)|v|^q\,dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)|^q|v(y)|^q}{|x-y|^{\mu}}\,dx\,dy = \int_{\mathbb{R}^n}(|x|^{-\mu}*|v|^q)|u|^q\,dx.$$

Therefore J can be written as

$$\begin{aligned}J(u,v) &= \frac{1}{p}\|(u,v)\|^p \\ &- \frac{1}{2q}\int_{\mathbb{R}^n}(\alpha(|x|^{-\mu}*|u|^q)|u|^q + 2\beta(|x|^{-\mu}*|u|^q)|v|^q + \gamma(|x|^{-\mu}*|v|^q)|v|^q)\,dx \\ &\qquad - \int_{\mathbb{R}^n}(f_1u+f_2v)\,dx.\end{aligned}$$

In the context of the Hardy–Littlewood–Sobolev inequality, i.e. Proposition 2.1, we get

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)|^q|u(y)|^q}{|x-y|^{\mu}}\,dx\,dy \le C\|u\|^{2q}_{L^{2nq/(2n-\mu)}(\mathbb{R}^n)}, \tag{2.2}$$

for any $u^q \in L^r(\mathbb{R}^n)$, $r>1$, $\mu \in (0,n)$ and $q \in [q_l,q_u]$. Using (2.2), Lemma 2.3 and $f_1,f_2 \in L^{q/(q-1)}(\mathbb{R}^n)$, we conclude that J is well defined. Moreover, it can be shown that $J \in C^1(Y,\mathbb{R})$.

3. Nehari manifold and Fibering map analysis

To find the critical points of J, we constraint our functional J on the Nehari manifold

$$\mathcal{N} = \{(u,v) \in Y : (J'(u,v),(u,v)) = 0\},$$

where

$$
\begin{aligned}
&(J'(u,v),(u,v)) = \|(u,v)\|^p \\
&\quad - \int_{\mathbb{R}^n} (\alpha(|x|^{-\mu} * |u|^q)|u|^q + 2\beta(|x|^{-\mu} * |u|^q)|v|^q + \gamma(|x|^{-\mu} * |v|^q)|v|^q)\, dx \\
&\qquad\qquad - \int_{\mathbb{R}^n} (f_1 u + f_2 v)\, dx.
\end{aligned}
$$

Clearly, every nontrivial weak solution to (P) belongs to $\mathcal{N}$. Denote $I(u,v) = (J'(u,v),(u,v))$ and subdivide the set $\mathcal{N}$ into three sets: $\mathcal{N}^{\pm} = \{(u,v) \in \mathcal{N} : \pm(I'(u,v),(u,v)) > 0\}$, $\mathcal{N}^0 = \{(u,v) \in \mathcal{N} : (I'(u,v),(u,v)) = 0\}$. Here

$$
\begin{aligned}
&(I'(u,v),(u,v)) = p\|(u,v)\|^p \\
&\quad - 2q \int_{\mathbb{R}^n} (\alpha(|x|^{-\mu} * |u|^q)|u|^q + 2\beta(|x|^{-\mu} * |u|^q)|v|^q + \gamma(|x|^{-\mu} * |v|^q)|v|^q)\, dx \\
&\qquad\qquad - \int_{\mathbb{R}^n} (f_1 u + f_2 v)\, dx.
\end{aligned}
$$

Then $\mathcal{N}^0$ contains the element $(0,0)$ and $\mathcal{N}^+ \cup \mathcal{N}^0$ and $\mathcal{N}^- \cup \mathcal{N}^0$ are closed subsets of Y. In the due course of this paper, we will subsequently give reason to divide $\mathcal{N}$ into above subsets. For $(u,v) \in Y$, we define the fibering map $\varphi\colon (0,\infty) \to \mathbb{R}$ as

$$
\begin{aligned}
&\varphi(t) = J(tu,tv) = \frac{t^p}{p}\, \|(u,v)\|^p \\
&\quad - \frac{t^{2q}}{2q} \int_{\mathbb{R}^n} (\alpha(|x|^{-\mu} * |u|^q)|u|^q + 2\beta(|x|^{-\mu} * |u|^q)|v|^q + \gamma(|x|^{-\mu} * |v|^q)|v|^q)\, dx \\
&\qquad\qquad - t \int_{\mathbb{R}^n} (f_1 u + f_2 v)\, dx.
\end{aligned}
$$

This gives

$$
\begin{aligned}
&\varphi'(t) = t^{p-1}\|(u,v)\|^p \\
&\quad - t^{2q-1} \int_{\mathbb{R}^n} (\alpha(|x|^{-\mu} * |u|^q)|u|^q + 2\beta(|x|^{-\mu} * |u|^q)|v|^q + \gamma(|x|^{-\mu} * |v|^q)|v|^q)\, dx \\
&\qquad\qquad - \int_{\mathbb{R}^n} (f_1 u + f_2 v)\, dx,
\end{aligned}
$$

$$
\begin{aligned}
\varphi''(t) = (p-1)t^{p-2}\|(u,v)\|^p &\\
- (2q-1)t^{2q-2} \int_{\mathbb{R}^n} &(\alpha(|x|^{-\mu} * |u|^q)|u|^q + 2\beta(|x|^{-\mu} * |u|^q)|v|^q \\
&+ \gamma(|x|^{-\mu} * |v|^q)|v|^q)\, dx.
\end{aligned}
$$

It is easy to see that $(tu,tv) \in \mathcal{N}$ if and only if $\varphi'(t) = 0$, for $t > 0$, i.e.

$$\mathcal{N} = \{(tu,tu) \in Y : \varphi'(t) = 0\}.$$

Also, we can check that for $(tu,tv)\in\mathcal{N}$, $(I'(tu,tv),(tu,tv)) >$ or < 0 if and only if $\varphi''(t) >$ or < 0 respectively. Therefore, $\mathcal{N}^+$, $\mathcal{N}^-$ and $\mathcal{N}^0$ can also be written as

$$\mathcal{N}^{\pm} = \{(tu,tv)\in\mathcal{N} : \varphi''(t) \gtrless 0\} \quad \text{and} \quad \mathcal{N}^0 = \{(tu,tv)\in\mathcal{N} : \varphi''(t) = 0\}.$$

We fix $(u,v)\in Y$ and define

$$\begin{aligned} K &:= K(u,v) = \int_{\mathbb{R}^n} (f_1u + f_2v)\,dx, \\ L &= L(u,v) \\ &:= \int_{\mathbb{R}^n} (\alpha(|x|^{-\mu} * |u|^q)|u|^q + 2\beta(|x|^{-\mu} * |u|^q)|v|^q + \gamma(|x|^{-\mu} * |v|^q)|v|^q)\,dx. \end{aligned}$$

Lemma 3.1. *If $f_1, f_2 \in L^{p/(p-1)}(\mathbb{R}^n)$, then J is coercive and bounded from below on $\mathcal{N}$. Hence J is bounded from below on $\mathcal{N}^+$ and $\mathcal{N}^-$.*

Proof. Let $(u,v)\in\mathcal{N}$, then $(J'(u,v),(u,v)) = 0$, i.e.

$$\|(u,v)\|^p - \int_{\mathbb{R}^n} (f_1u + f_2v)\,dx = L(u,v).$$

Using this we obtain

$$\begin{aligned} J(u,v) &= \left(\frac{2q-p}{2qp}\right)\|(u,v)\|^p - \left(\frac{2q-1}{2q}\right)\int_{\mathbb{R}^n}(f_1u+f_2v)\,dx \\ &\geq \left(\frac{2q-p}{2qp}\right)\|(u,v)\|^p \\ &\quad - \left(\frac{2q-1}{2q}\right)\|f_1\|_{L^{p/(p-1)}(\mathbb{R}^n)}\|u\|_{L^p(\mathbb{R}^n)} + \|f_2\|_{L^{p/(p-1)}(\mathbb{R}^n)}\|v\|_{L^p(\mathbb{R}^n)} \\ &\geq \|(u,v)\|\bigg(\left(\frac{2q-p}{2qp}\right)\|(u,v)\|^{p-1} \\ &\quad - \left(\frac{2q-1}{2q}\right)(S_{q,1}+S_{q,2})\max\left\{\|f_1\|_{L^{p/(p-1)}(\mathbb{R}^n)}, \|f_2\|_{L^{p/(p-1)}(\mathbb{R}^n)}\right\}\bigg), \end{aligned}$$

where $S_{q,i}$ denotes the best constant for the embedding $Y \hookrightarrow L^p(\mathbb{R}^n)$, $i = 1,2$. This implies that J is coercive and bounded from below on $\mathcal{N}$. $\square$

Thus it is natural to consider a minimization problem on $\mathcal{N}$ or its subsets. For fixed $(u,v)\in Y$ we define

$$\begin{aligned} m(t) &:= t^{p-1}\|(u,v)\|^p \\ &- t^{2q-1}\int_{\mathbb{R}^n} (\alpha(|x|^{-\mu} * |u|^q)|u|^q + 2\beta(|x|^{-\mu} * |u|^q)|v|^q + \gamma(|x|^{-\mu} * |v|^q)|v|^q)\,dx. \end{aligned}$$

Then $\varphi'(t) = 0$ if and only if $m(t) = K$. Since $p((2n-\mu)/n) < 2q$ and $(2n-\mu)/n > 1$, we get $p<2q$ which implies $\lim_{t\to+\infty} m(t) = -\infty$. Also $\lim_{t\to 0} m(t) = 0$

and it is easy to check that

$$t_0 = \left(\frac{(p-1)\|(u,v)\|^p}{(2q-1)L} \right)^{1/(2q-p)}$$

is a point of global maximum for $m(t)$. For $t > 0$ small enough, $m(t) > 0$. Altogether, this implies that if we choose $K > 0$ sufficiently small then $m(t) = K$ is satisfied in such a way that $\varphi'(t) = 0$ has two positive solutions t_1, t_2 such that $0 < t_1 < t_0 < t_2$. Then, according to the sign of $\varphi''(t_1)$ and $\varphi''(t_2)$, we decide in which subset (i.e. $\mathcal{N}^+$, $\mathcal{N}^-$, $\mathcal{N}^0$) they lie. Hence the sets $\mathcal{N}^+$, $\mathcal{N}^-$ and $\mathcal{N}^0$ contain the point of local maximum, local minimum and point of inflexion of the fibering maps.

We end this section with the following two results.

Lemma 3.2. *If $f_1, f_2 \in L^{p/(p-1)}(\mathbb{R}^n)$ are nonzero and satisfy* (1.1)*, then $\mathcal{N}^0 = \{(0,0)\}$.*

Proof. To prove that $\mathcal{N}^0 = \{(0,0)\}$, we need to show that for $(u,v) \in Y \setminus \{(0,0)\}$, $\varphi(t)$ has no critical point which is a saddle point. Let $(u,v) \in Y \setminus \{(0,0)\}$. From the above analysis, we know that $m(t)$ has a unique point of global maximum at t_0 and

$$m(t_0) = \left(\frac{2q-p}{2q-1} \right) \left(\frac{p-1}{L(2q-1)} \right)^{(p-1)/(2q-p)} \|(u,v)\|^{p(2q-1)/(2q-p)}.$$

From the analysis of the map $m(t)$ done above, we get that if $0 < K < m(t_0)$, then $\varphi'(t) = 0$ has exactly two roots t_1, t_2 such that $0 < t_1 < t_0 < t_2$ and if $K \le 0$ then $\varphi'(t) = 0$ has only one root t_3 such that $t_3 > t_0$. Since $\varphi''(t) = m'(t)$, we get $\varphi''(t_1) > 0$, $\varphi''(t_2) < 0$ and $\varphi''(t_3) < 0$. Hence, if $0 < K < m(t_0)$, then $(t_1 u, t_1 v) \in \mathcal{N}^+$, $(t_2 u, t_2 v) \in \mathcal{N}^-$ and if $K \le 0$ then $(t_3 u, t_3 v) \in \mathcal{N}^-$. This implies $\{(u,v) \in Y : 0 < K < m(t_0)\} \cap \mathcal{N}^\pm \ne \emptyset$ and $\{(u,v) \in Y : K \le 0\} \cap \mathcal{N}^- \ne \emptyset$. As a consequence, $\mathcal{N}^\pm \ne \emptyset$. We saw that for any sign of K, critical point of $\varphi(t)$ is either a point of local maximum or local minimum which implies $\mathcal{N}^0 = \{(0,0)\}$. It remains to show that $0 < K < m(t_0)$ holds. But this is clearly implied by condition (1.1) which we have already assumed. □

Lemma 3.3. *Assume $f_1, f_2 \in L^{p/(p-1)}(\mathbb{R}^n)$ satisfy* (1.1)*, then $\mathcal{N}^-$ is closed.*

Proof. Let $\mathrm{cl}(\mathcal{N}^-)$ be the closure of $\mathcal{N}^-$. Since $\mathrm{cl}(\mathcal{N}^-) \subset \mathcal{N}^- \cup \{(0,0)\}$, it is enough to show that $(0,0) \notin \mathcal{N}^-$ or equivalently $\mathrm{dist}((0,0), \mathcal{N}^-) > 0$. We denote $(\widehat{u}, \widehat{v}) = (u,v)/\|(u,v)\|$ for $(u,v) \in \mathcal{N}^-$, then $\|(\widehat{u}, \widehat{v})\| = 1$. Let us consider the fibering map $\varphi(t)$ corresponding to $(\widehat{u}, \widehat{v})$. From the proof of Lemma 3.2, we get that if $K \le 0$ then $\varphi'(t) = 0$ has exactly one root $t_3 > t_0$ such that $(t_3 \widehat{u}, t_3 \widehat{v}) \in \mathcal{N}^-$. If $(t_3 \widehat{u}, t_3 \widehat{v}) = (u,v) \in \mathcal{N}^-$, then $t_3 = \|(u,v)\|$. Also, if $0 < K < m(t_0)$ then $\varphi'(t) = 0$ has exactly two roots t_1, t_2 satisfying $t_1 < t_0 < t_2$ such that $(t_1 \widehat{u}, t_1 \widehat{v}) \in \mathcal{N}^+$ and $(t_2 \widehat{u}, t_2 \widehat{v}) \in \mathcal{N}^-$. Hence if $(t_2 \widehat{u}, t_2 \widehat{v}) = (u,v) \in \mathcal{N}^-$ then

$t_2 = \|(u,v)\|$. Since $t_2, t_3 > t_0$, we get $\|(u,v)\| > t_0$. Using Lemma 2.3, inequality (2.2), continuous embedding of Y in $L^r(\mathbb{R}^n)$ for $r \in [p, p_s^*]$, $2nq/(2n-\mu) \in (p, p_s^*)$ and $\|(\widehat{u}, \widehat{v})\| = 1$, we get that

$$\begin{aligned}
(3.1)\qquad L \le{}& C\Big(\alpha\|\widehat{u}\|^{2q}_{L^{2nq/(2n-\mu)}(\mathbb{R}^n)} + \gamma\|\widehat{v}\|^{2q}_{L^{2nq/(2n-\mu)}(\mathbb{R}^n)} \\
&+ 2\beta\big(\|\widehat{u}\|^{2q}_{L^{2nq/(2n-\mu)}(\mathbb{R}^n)}\big)^{1/2}\big(\|\widehat{v}\|^{2q}_{L^{2nq/(2n-\mu)}(\mathbb{R}^n)}\big)^{1/2}\Big) \\
\le{}& C_1\big(\alpha\|\widehat{u}\|^{2q}_{Y_1} + \gamma\|\widehat{v}\|^{2q}_{Y_2} + 2\beta\|\widehat{u}\|^{q}_{Y_1}\|\widehat{v}\|^{q}_{Y_2}\big) \le C_2\|(\widehat{u}, \widehat{v})\|^{2q},
\end{aligned}$$

where $C_1, C_2 > 0$ are constants independent of $\widehat{u}$ and $\widehat{v}$. This implies L is bounded from above on the unit sphere of Y. Since $\|\widehat{u}, \widehat{v}\| = 1$, from definition of t_0 it follows that

$$t_0 \ge \left(\frac{p-1}{(2q-1)\sup\limits_{\|(u,v)\|=1} L(u,v)}\right)^{1/(2q-p)} := \theta.$$

Therefore, $\operatorname{dist}((0,0), \mathcal{N}^-) = \inf\limits_{(u,v)\in\mathcal{N}^-}\{\|(u,v)\|\} \ge \theta > 0$ and this proves the lemma. $\square$

Using Lemma 3.1, we can define the following:

$$\Upsilon^+ := \inf_{(u,v)\in\mathcal{N}^+} J(u,v) \quad \text{and} \quad \Upsilon^- := \inf_{(u,v)\in\mathcal{N}^-} J(u,v).$$

If the infimum in the above two equations are achieved, then we can show that they form a weak solution to our problem (P).

Lemma 3.4. *Let (u_1, v_1) and (u_2, v_2) be minimizers of J on $\mathcal{N}^+$ and $\mathcal{N}^-$, respectively. Then (u_1, v_1) and (u_2, v_2) are nontrivial weak solutions to* (P).

Proof. Let $(u_1, v_1) \in \mathcal{N}^+$ be such that $J(u_1, v_1) = \Upsilon^+$ and define $V := \{(u,v) \in Y : (I'(u,v), (u,v)) > 0\}$. So, $\mathcal{N}^+ = \{(u,v) \in V : I(u,v) = 0\}$. Using Theorem 4.1.1 of [10] we deduce that there exists a Lagrangian multiplier $\lambda \in \mathbb{R}$ such that

$$J'(u_1, v_1) = \lambda I'(u_1, v_1).$$

Since $(u_1, v_1) \in \mathcal{N}^+$, $(J'(u_1, v_1), (u_1, v_1)) = 0$ and $(I'(u_1, v_1), (u_1, v_1)) > 0$. This implies $\lambda = 0$. Therefore, (u_1, v_1) is a nontrivial weak solution to (P). Similarly, we can prove that if $(u_2, v_2) \in \mathcal{N}^-$ is such that $J(u_2, v_2) = \Upsilon^-$ then (u_2, v_2) is also a nontrivial weak solution to (P). $\square$

Our next result is an observation regarding the minimizers Υ^+ and Υ^-.

Lemma 3.5. *If $0 \ne f_1, f_2 \in L^{p/(p-1)}(\mathbb{R}^n)$ satisfies* (1.1) *then $\Upsilon^- > 0$ and $\Upsilon^+ < 0$.*

Proof. Let $(u,v) \in Y$ then from the proof of Lemma 3.2, we know that if f_1, f_2 satisfy (1.1) then $K < m(t_0)$. In this case, if $0 < K < m(t_0)$ then corresponding to (u,v), $\varphi'(t) = 0$ has exactly two roots t_1 and t_2 such that $t_1 < t_0 < t_2$, $t_1(u,v) \in \mathcal{N}^+$ and $t_2(u,v) \in \mathcal{N}^-$. Since $\varphi'(t) = \|(u,v)\| t^{p-1} - Lt^{2q-1} - K$, $\lim_{t\to 0^+} \varphi'(t) = -K < 0$. Also $\varphi''(t) > 0$ for all $t \in (0,t_0)$. Since t_1 is a point of local minimum of $\varphi(t)$, $t_1 > 0$ and $\lim_{t\to 0^+} \varphi(t) = 0$, we get $\varphi(t_1) < 0$. Therefore,

$$0 > \varphi(t_1) = J(t_1 u, t_1 v) \geq \Upsilon^+.$$

Now we prove that $\Upsilon^- > 0$. From (3.1), we know that $L \leq C_2 \|(u,v)\|^{2q}$. This implies that there exists a constant $C_3 > 0$ which is independent of (u,v) such that

$$\frac{(\|(u,v)\|^p)^{2q/(2q-p)}}{L^{p/(2q-p)}} \geq C_3.$$

Now, using this and the given hypothesis, we consider $\varphi(t_0)$ corresponding to (u,v) as

$$\begin{aligned}
\varphi(t_0) &= \frac{t_0^p}{p}\|u,v\|^p - L\,\frac{t_0^{2q}}{2q} - Kt_0 = \frac{1}{p}\bigg(\frac{(p-1)\|(u,v)\|^p}{(2q-1)L}\bigg)^{p/(2q-p)} \\
&\quad - \frac{L}{2q}\bigg(\frac{(p-1)\|(u,v)\|^p}{(2q-1)L}\bigg)^{2q/(2q-p)} - K\bigg(\frac{(p-1)\|(u,v)\|^p}{(2q-1)L}\bigg)^{1/(2q-p)} \\
&= \frac{(2q-p)(2q+p-1)}{2qp(2q-1)}\bigg(\frac{p-1}{2q-1}\bigg)^{p/(2q-p)} \frac{(\|(u,v)\|^p)^{2q/(2q-p)}}{L^{p/(2q-p)}} \\
&\quad - K\bigg(\frac{p-1}{2q-1}\bigg)^{1/(2q-p)} \frac{(\|(u,v)\|^p)^{1/(2q-p)}}{L^{1/(2q-p)}} \\
&\geq \bigg(\frac{(2q-p)(2q+p-1)(p-1)^{p/(2q-p)}}{2qp(2q-1)^{2q/(2q-p)}}\bigg) \frac{(\|(u,v)\|^p)^{2q/(2q-p)}}{L^{p/(2q-p)}} \\
&\geq C_3\bigg(\frac{(2q-p)(2q+p-1)(p-1)^{p/(2q-p)}}{2qp(2q-1)^{2q/(2q-p)}}\bigg) := M(\text{say}).
\end{aligned}$$

Hence

$$\Upsilon^- = \inf_{(u,v)\in Y\setminus\{(0,0)\}} \max_t J(tu,tv) \geq \inf_{(u,v)\in Y\setminus\{(0,0)\}} \varphi(t_0) \geq M > 0,$$

which completes the proof. □

4. Palais–Smale analysis

In this section, we study the nature of minimizing sequences for the functional J on the Nehari manifold. First we prove some lemmas which will assert the existence of Palais–Smale sequence for the minimizer of J on $\mathcal{N}$. The following lemma is a consequence of Lemma 3.2.

Lemma 4.1. *Let* $0 \neq f_1, f_2 \in L^{p/(p-1)}(\mathbb{R}^n)$ *satisfy* (1.1). *Given* $(u,v) \in \mathcal{N} \setminus \{(0,0)\}$, *there exist* $\varepsilon > 0$ *and a differentiable function* $\Im \colon B((0,0),\varepsilon) \subset Y \to \mathbb{R}^+ := (0,+\infty)$ *such that* $\Im(0,0) = 1$, $\Im(w_1,w_2)((u,v)-(w_1,w_2)) \in \mathcal{N}$ *and*

$$(4.1)\quad (\Im'(0,0),(w_1,w_2)) = \frac{p(A(w_1,w_2)+A_2(w_1,w_2)) - qR(w_1,w_2) - \displaystyle\int_{\mathbb{R}^n}(f_1w_1+f_2w_2)}{(p-1)\|(u,v)\|^p - (2q-1)L(u,v)},$$

for all $(w_1,w_2) \in B((0,0),\varepsilon)$, *where*

$$\begin{aligned}
A_1(w_1,w_2) &:= \langle u, w_1\rangle + \int_{\mathbb{R}^n} a_1(x)|u|^{p-2}uw_1,\\
A_2(w_1,w_2) &:= \langle v, w_2\rangle + \int_{\mathbb{R}^n} a_2(x)|v|^{p-2}vw_2,\\
R(w_1,w_2) &:= 2\alpha\int_{\mathbb{R}^n}(|x|^{-\mu}*|u|^q)|u|^{q-2}uw_1 + 2\gamma\int_{\mathbb{R}^n}(|x|^{-\mu}*|v|^q)|v|^{q-2}vw_2\\
&\quad + \beta\int_{\mathbb{R}^n}(|x|^{-\mu}*|u|^q)|v|^{q-2}vw_2 + \beta\int_{\mathbb{R}^n}(|x|^{-\mu}*|v|^q)|u|^{q-2}uw_1.
\end{aligned}$$

Proof. Fixing a function $(u,v) \in \mathcal{N}$, we define the map $F \colon \mathbb{R}\times Y \to \mathbb{R}$ as follows:

$$\begin{aligned}
F(t,(w_1,w_2)) &:= t^{p-1}\|(u,v)-(w_1,w_2)\|\\
&\quad - t^{2q-1}L((u,v)-(w_1,w_2)) - \int_{\mathbb{R}^n}(f_1(u-w_1)+f_2(v-w_2)).
\end{aligned}$$

It is easy to see that F is differentiable. Since $F(1,(0,0)) = (J'(u,v),(u,v)) = 0$ and $F_t(1,(0,0)) = (p-1)t^{p-2}\|(u,v)-(w_1,w_2)\|^p - (2q-1)t^{2q-2}L((u,v)-(w_1,w_2)) \neq 0$ by Lemma 3.2, we apply the Implicit Function Theorem at the point $(1,(0,0))$ to get the existence of $\varepsilon > 0$ and a differentiable function $\Im \colon B((0,0),\varepsilon) \subset Y \to \mathbb{R}^+$ such that

$$\Im(0,0) = 1 \quad\text{and}\quad F((w_1,w_2),\Im(w_1,w_2)) = 0, \quad \text{for all } (w_1,w_2) \in B((0,0),\varepsilon).$$

This implies

$$\begin{aligned}
0 &= \Im^{p-1}(w_1,w_2)\|(u,v)-(w_1,w_2)\|^p\\
&\quad - \Im^{2q-1}(w_1,w_2)L((u,v)-(w_1,w_2)) - K((u,v)-(w_1,w_2))\\
&= \frac{1}{\Im(w_1,w_2)}\big[\|\Im(w_1,w_2)(u,v)-(w_1,w_2)\|^p\\
&\quad - L(\Im(w_1,w_2)((u,v)-(w_1,w_2))) - K(\Im(w_1,w_2)((u,v)-(w_1,w_2)))\big].
\end{aligned}$$

Since $\Im(w_1,w_2) > 0$ we get $\Im(w_1,w_2)((u,v)-(w_1,w_2)) \in \mathcal{N}$ whenever $(w_1,w_2) \in B((0,0),\varepsilon)$. Finally, (4.15) can be obtained by differentiating

$$F((w_1,w_2),\Im(w_1,w_2)) = 0$$

with respect to (w_1, w_2). $\square$

Let us define $\Upsilon := \inf_{(u,v)\in\mathcal{N}} J(u,v)$.

Lemma 4.2. *There exists a constant $C_1 > 0$ such that*

$$\Upsilon \leq -\frac{(2q-p)(2qp-2q-p)}{4pq^2}\, C_1.$$

Proof. Let $(\widehat{u},\widehat{v}) \in Y$ be the unique solution to the equations given below

$$\begin{aligned}(-\Delta)_p^s \widehat{u} + a_1(x)|\widehat{u}|^{p-1}\widehat{u} &= f_1 \quad \text{in } \mathbb{R}^n,\\ (-\Delta)_p^s \widehat{v} + a_2(x)|\widehat{v}|^{p-1}\widehat{v} &= f_2 \quad \text{in } \mathbb{R}^n.\end{aligned}$$

So, since $f_1, f_2 \neq 0$,

$$\int_{\mathbb{R}^n} (f_1\widehat{u} + f_2\widehat{v}) = \|(\widehat{u},\widehat{v})\|^p > 0.$$

Then, by Lemma 3.2, we know that there exists $t_1 > 0$ such that $t_1(\widehat{u},\widehat{v}) \in \mathcal{N}^+$. Consequently,

$$\begin{aligned}J(t_1\widehat{u}, t_1\widehat{v}) &= -\left(\frac{p-1}{p}\right) t_1^p \|(\widehat{u},\widehat{v})\|^p + \left(\frac{2q-1}{2q}\right) t_1^{2q} L(\widehat{u},\widehat{v})\\ &< -\left(\frac{p-1}{p}\right) t_1^p \|(\widehat{u},\widehat{v})\|^p + \frac{p(2q-1)}{4q^2}\, t_1^p \|(\widehat{u},\widehat{v})\|^p\\ &= -\frac{(2q-p)(2qp-2q-p)}{4pq^2} t_1^p \|(\widehat{u},\widehat{v})\|^p < 0.\end{aligned}$$

Taking $C_1 = t_1^p \|(\widehat{u},\widehat{v})\|^p$ we get the result. $\square$

We recall the following lemma.

Lemma 4.3 ([46]). *Let $0 < \theta < n$, $1 < r < m < \infty$ and $1/m = 1/r - \theta/n$, then*

$$\left| \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\theta}}\, dy \right|_{L^m(\mathbb{R}^n)} \leq C\|f\|_{L^r(\mathbb{R}^n)},$$

where $C > 0$ is a constant.

This implies that the Reisz potential defines a linear and continuous map from $L^r(\mathbb{R}^n)$ to $L^m(\mathbb{R}^n)$, where r, m are defined in the above theorem.

Lemma 4.4. *For $0 \neq f_1, f_2 \in L^{p/(p-1)}(\mathbb{R}^n)$,*

$$\inf_Q \left(C_{p,q}\|(u,v)\|^{p(2q-1)/(2q-p)} - \int_{\mathbb{R}^n} (f_1u + f_2v)\, dx \right) := \delta$$

is achieved, where $Q = \{(u,v) \in Y : L(u,v) = 1\}$. Also, if f_1, f_2 satisfy (1.1), *then $\delta > 0$.*

Proof. Let us define the functional $T\colon Y \mapsto \mathbb{R}$ as

$$T(u,v) = C_{p,q}\|(u,v)\|^{p(2q-1)/(2q-p)} - \int_{\mathbb{R}^n} (f_1 u + f_2 v)\,dx.$$

This implies

$$\begin{aligned}T(u,v) \geq C_{p,q}&\|(u,v)\|^{p(2q-1)/(2q-p)} \\ &- (S_{q,1}+S_{q,2}) \max\big\{\|f_1\|_{L^{p/(p-1)}(\mathbb{R}^n)}, \|f_2\|_{L^{p/(p-1)}(\mathbb{R}^n)}\big\}\|(u,v)\|,\end{aligned}$$

where $S_{q,i}$ denotes the best constant for the embedding $Y \hookrightarrow L^p(\mathbb{R}^n)$, $i=1,2$. Since $p(2q-1)/(2q-p) > 1$, T is coercive. Let $\{(u_k,v_k)\} \subset Q$ be such that $(u_k,v_k) \rightharpoonup (u,v)$ weakly in Y. Then

$$\begin{gathered}\lim_{k\to\infty} \int_{\mathbb{R}^n} (f_1 u_k + f_2 v_k)\,dx = \int_{\mathbb{R}^n} (f_1 u + f_2 v)\,dx, \\ \|(u,v)\|^{p(2q-1)/(2q-p)} \leq \liminf_{k\to\infty} \|(u_k,v_k)\|^{p(2q-1)/(2q-p)}.\end{gathered}$$

which implies $T(u,v) \leq \liminf_{k\to\infty} T(u_k,v_k)$, i.e. T is weakly lower semicontinuous. Consider

$$\begin{aligned}\int_{\mathbb{R}^n} (|x|^{-\mu} * |u_k|^q)|u_k|^q\,dx &- \int_{\mathbb{R}^n} (|x|^{-\mu} * |u|^q)|u|^q\,dx \\ &= \int_{\mathbb{R}^n} (|x|^{-\mu} * (|u_k|^q - |u|^q))(|u_k|^q - |u|^q)\,dx \\ &\qquad + 2\int_{\mathbb{R}^n} (|x|^{-\mu} * |u|^q)(|u_k|^q - |u|^q)\,dx.\end{aligned} \tag{4.2}$$

Since $2nq/(2n-\mu) < p_s^*$, using Lemma 2.5, we have

$$|u_k|^q - |u|^q \to 0 \quad \text{in } L^{2n/(2n-\mu)}(\mathbb{R}^n) \quad \text{as } k\to\infty, \tag{4.3}$$

and thus, using Theorem 4.3, we have

$$|x|^{-\mu} * (|u_k|^q - |u|^q) \to 0 \quad \text{in } L^{(2n/\mu)(\mathbb{R}^n)} \quad \text{as } k\to\infty. \tag{4.4}$$

From (4.3), (4.4) and using Hölder's inequality in (4.2), we get

$$\int_{\mathbb{R}^n} (|x|^{-\mu} * |u_k|^q)|u_k|^q\,dx \to \int_{\mathbb{R}^n} (|x|^{-\mu} * |u|^q)|u|^q\,dx \quad \text{as } k\to\infty. \tag{4.5}$$

Similarly, we get

$$\int_{\mathbb{R}^n} (|x|^{-\mu} * |v_k|^q)|v_k|^q\,dx \to \int_{\mathbb{R}^n} (|x|^{-\mu} * |v|^q)|v|^q\,dx \quad \text{as } k\to\infty. \tag{4.6}$$

It is easy to see that

$$\begin{aligned}
&\int_{\mathbb{R}^n} (|x|^{-\mu} * |u_k|^q)|v_k|^q \, dx - \int_{\mathbb{R}^n} (|x|^{-\mu} * |u|^q)|v|^q \, dx \\
&= \int_{\mathbb{R}^n} (|x|^{-\mu} * (|u_k|^q - |u|^q))(|v_k|^q - |v|^q) \, dx + \int_{\mathbb{R}^n} (|x|^{-\mu} * (|u_k|^q - |u|^q))|v|^q \, dx \\
&\qquad + \int_{\mathbb{R}^n} (|x|^{-\mu} * (|v_k|^q - |v|^q))|u|^q \, dx,
\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^n} (|x|^{-\mu} * |u_k|^q)|v_k|^q \, dx \to \int_{\mathbb{R}^n} (|x|^{-\mu} * |u|^q)|v|^q \, dx \quad \text{as } k \to \infty. \tag{4.7}$$

Thus using (4.5)–(4.7), we get $\lim_{k\to\infty} L(u_k, u_k) = L(u, v)$. Since $(u_k, v_k) \in Q$ for each k, we get $L(u, v) = 1$ which implies $(u, v) \in Q$. Therefore Q is weakly sequentially closed subset of Y. Since Y forms a reflexive Banach space, there exists $(u_0, v_0) \in Q$ such that

$$\inf_Q T(u, v) = T(u_0, v_0).$$

Furthermore, it is obvious that if f_1, f_2 satisfy (1.1), then $\delta \geq T(u_0, v_0) > 0$. This establishes the result. $\square$

For $(u, v) \in Y \setminus \{(0, 0)\}$, we set

$$G(u, v) := C_{p,q} \frac{(\|(u, v)\|^p)^{(2q-1)/(2q-p)}}{L(u, v)^{(p-1)/(2q-p)}} - K(u, v).$$

Corollary 4.5. *For any $\rho > 0$, $\inf_{L(u,v)\geq\rho} G(u, v) \geq \rho\delta$.*

Proof. For $t > 0$, if $L(u, v) = 1$ for $(u, v) \in Y$ then using Lemma 4.4 we have

$$G(tu, tv) = t\big(C_{p,q}(\|(u, v)\|^p)^{(2q-1)/(2q-p)} - K(u, v)\big) \geq t\delta.$$

This implies for any $\rho > 0$, $\inf_{L(u,v)\geq\rho} G(u, v) \geq \rho\delta^{1/2q}$ which completes the proof. $\square$

In the next result, we show the existence of a Palais–Smale sequence for Υ.

Proposition 4.6. *Let $0 \neq f_1, f_2 \in L^{p/(p-1)}(\mathbb{R}^n)$ be such that (1.1) holds. Then there exists a sequence $(u_k, v_k) \subset \mathcal{N}$ such that*

$$J(u_k, v_k) \to \Upsilon \quad \textit{and} \quad \|J'(u_k, v_k)\|_{Y^*} \to 0 \quad \textit{as } k \to \infty,$$

$\|\cdot\|_{Y^*}$ denotes the operator norm on the dual of Y, i.e. Y^*.

Proof. From Lemma 3.1, we already know that J is bounded from below on $\mathcal{N}$. So by Ekeland's Variational Principle we get a sequence $\{(u_k, v_k)\} \subset \mathcal{N}$

such that

$$
\begin{cases} J(u_k,v_k) \le \Upsilon + \dfrac{1}{k}, \\ J(u,v) \ge J(u_k,v_k) - \dfrac{1}{k}\,\|(u_k-u,v_k-v)\| \quad \text{for all } (u,v)\in\mathcal{N}. \end{cases} \tag{4.8}
$$

By taking $k>0$ large enough we have

$$
J(u_k,v_k) = \frac{(2q-p)}{2qp}\,\|(u_k,v_k)\|^p - \frac{(2q-1)}{2qp}\int_{\mathbb{R}^n}(f_1u+f_2v)\,dx < \Upsilon + \frac{1}{k}.
$$

This along with Lemma 4.2 gives

$$
\int_{\mathbb{R}^n}(f_1u_k+f_2v_k)\,dx \ge \frac{(2q-p)(2qp-2q-p)}{2pq(2q-1)}\,C_1 > 0. \tag{4.9}
$$

Therefore $u_k, v_k \neq 0$ for all k. From (4.8) and definition of Υ, it is clear that $J(u_k,v_k)\to\Upsilon<0$ as $k\to\infty$. Since $\{(u_k,v_k)\}\subset\mathcal{N}$, we get

$$
\|(u_k,v_k)\|^p - \int_{\mathbb{R}^n}(f_1u_k+f_2v_k)\,dx = L. \tag{4.10}
$$

Using definition of J and (4.8)–(4.10), we get

$$
\begin{aligned}
\Upsilon^+ + \frac{1}{k} &\ge \left(\frac{1}{p}-\frac{1}{2q}\right)\|(u_k,v_k)\|^p - \left(1-\frac{1}{2q}\right)\int_{\mathbb{R}^n}(f_1u_k+f_2v_k)\,dx \\
&\ge \left(\frac{1}{p}-\frac{1}{2q}\right)\|(u_k,v_k)\|^p - \left(1-\frac{1}{2q}\right)(S_{q,1}+S_{q,2}) \\
&\quad\cdot \max\left\{\|f_1\|_{L^{p/(p-1)}(\mathbb{R}^n)}, \|f_2\|_{L^{p/(p-1)}(\mathbb{R}^n)}\right\}\|(u_k,v_k)\|.
\end{aligned} \tag{4.11}
$$

This implies $\{(u_k,v_k)\}$ is bounded. Now we claim that $\inf_k\|(u_k,v_k)\|\ge\eta>0$, for some constant η. Suppose not, then, up to a subsequence, $\|(u_k,v_k)\|\to 0$ as $k\to\infty$. This implies $J(u_k,v_k)\to 0$ as $k\to\infty$, using (4.11), which is a contradiction to the first assertion. So there exist constants $d_1,d_2>0$ such that

$$
d_1 \le \|(u_k,v_k)\| \le d_2. \tag{4.12}
$$

Now we aim to show that $\|J'(u_k,v_k)\|_{Y^*}\to 0$ as $k\to\infty$. By Lemma 4.1, for each k we obtain a differentiable function $\Im_k\colon B((0,0),\varepsilon_k)\subset Y\to\mathbb{R}^+ := (0,+\infty)$ for $\varepsilon_k>0$ such that $\Im_k(0,0)=1$, $\Im(w_1,w_2)((u_k,v_k)-(w_1,w_2))\in N$ for all $(w_1,w_2)\in B((0,0),\varepsilon_k)$. Choose $0<\rho<\varepsilon_k$ and $(h_1,h_2)\in Y$ such that $\|(h_1,h_2)\|=1$. Let $(w_1,w_2)_\rho := \rho(h_1,h_2)$ then $\|(w_1,w_2)_\rho\|=\rho<\varepsilon_k$ and $(\theta_1,\theta_2)_\rho := \Im_k((w_1,w_2)_\rho)((u_k,v_k)-(w_1,w_2)_\rho)\in N$ for each k. By Taylor's expansion and (4.8), since $(\theta_1,\theta_2)_\rho \in \mathcal{N}$ we get

$$\begin{aligned}(4.13)\quad \frac{1}{k}\,\|(u_k,v_k)-(\theta_1,\theta_2)_\rho\| &\geq J(u_k,v_k)-J((\theta_1,\theta_2)_\rho)\\ &=(J'((\theta_1,\theta_2)_\rho),(u_k-v_k)-(\theta_1,\theta_2)_\rho)+o(\|(u_k,v_k)-(\theta_1,\theta_2)_\rho\|)\\ &=(1-\Im_k((w_1,w_2)_\rho))(J'((\theta_1,\theta_2)_\rho),(u_k-v_k))\\ &\quad+\rho\Im_k((w_1,w_2)_\rho)(J'((\theta_1,\theta_2)_\rho),(h_1,h_2)).\end{aligned}$$

We observe that

$$\lim_{\rho\to 0}\frac{1}{\rho}\,\|(\theta_1,\theta_2)_\rho-(u_k,v_k)\|=\|(u_k,v_k)(\Im_k'(0,0),(h_1,h_2))-(h_1,h_2)\|.$$

Dividing (4.13) by ρ and passing to the limit as $\rho\to 0$ we derive

$$(J'(u_k,v_k),(h_1,h_2))\leq\frac{1}{k}\,(\|(u_k,v_k)\|\|\Im_k'(0,0)\|_{Y^*}+1).$$

From (4.15) and (4.13), there exists a constant $C_2>0$ such that

$$\|\Im_k'(0,0)\|_{Y^*}\leq\frac{C_2}{(p-1)\|u_k,v_k\|^p-(2q-1)L(u_k,v_k)}.$$

It remains to show that

$$(p-1)\|u_k,v_k\|^p-(2q-1)L(u_k,v_k)=(I'(u_k,v_k),(u_k,v_k))$$

is bounded away from zero. If possible let, for a subsequence,

$$|(I'(u_k,v_k),(u_k,v_k))|=o(1)$$

which implies

$$(4.14)\qquad\begin{aligned}(p-1)\|(u_k,v_k)\|^p-(2q-1)L(u_k,v_k)&=o(1),\\ (2q-p)\|(u_k,v_k)\|^p-(2q-1)K(u_k,v_k)&=o(1).\end{aligned}$$

From (4.13) and (4.14), it follows that there exists a constant $d_3>0$ such that $L(u_k,v_k)\geq d_3$, for each k. Since $(u_k,v_k)\in\mathcal{N}$, we have

$$(p-1)K(u_k,v_k)-(2q-p)L(u_k,v_k)=o(1)$$

and (4.14) gives

$$\left(\frac{p-1}{2q-1}\,\|(u_k,v_k)\|^p\right)^{(2q-1)/(2q-p)}-L(u_k,v_k)^{(2q-1)/(2q-p)}=o(1).$$

Using the above along with Corollary 4.5, we obtain

$$\begin{aligned}0&<\delta d_3^{(p-1)/(2q-p)+1/(2q)}\leq L(u_k,v_k)^{(p-1)/(2q-p)}G(u_k,v_k)\\ &\leq C_{p,q}(\|(u_k,v_k)\|^p)\,\frac{2q-1}{2q-p}-K(u_k,v_k)L(u_k,v_k)^{(p-1)/(2q-p)}\\ &\leq\frac{2q-p}{p-1}\left(\frac{p-1}{2q-1}\,\|(u_k,v_k)\|^p\right)^{(2q-1)/(2q-p)}\\ &\quad-L(u_k,v_k)^{(2q-1)/(2q-p)}=o(1)\end{aligned}$$

which is a contradiction. This proves the claim. Therefore we conclude that

$$\|J'(u_k, v_k)\|_{Y^*} \to 0, \quad \text{as } k \to 0,$$

which proves our lemma. □

Lemma 4.7. *Let $0 \neq f_1, f_2 \in L^{p/(p-1)}(\mathbb{R}^n)$ satisfy (1.1). Given $(u,v) \in \mathcal{N}^- \setminus \{(0,0)\}$, there exist $\varepsilon > 0$ and a differentiable function $\Im^- \colon B((0,0),\varepsilon) \subset Y \to \mathbb{R}^+ := (0,+\infty)$ such that $\Im^-(0,0) = 1$, $\Im^-(w_1,w_2)((u,v)-(w_1,w_2)) \in \mathcal{N}^-$ and*

$$(4.15)\quad ((\Im^-)'(0,0),(w_1,w_2)) = \frac{p(A(w_1,w_2) + A_2(w_1,w_2)) - qR(w_1,w_2) - \displaystyle\int_{\mathbb{R}^n} (f_1 w_1 + f_2 w_2)}{(p-1)\|(u,v)\|^p - (2q-1)L(u,v)},$$

for all $(w_1,w_2) \in B((0,0),\varepsilon)$, where

$$A_1(w_1,w_2) := \langle u, w_1\rangle + \int_{\mathbb{R}^n} a_1(x)|u|^{p-2}uw_1,$$
$$A_2(w_1,w_2) := \langle v, w_2\rangle + \int_{\mathbb{R}^n} a_2(x)|v|^{p-2}vw_2$$

and

$$\begin{aligned} R(w_1,w_2) :=& 2\alpha \int_{\mathbb{R}^n} (|x|^{-\mu} * |u|^q)|u|^{q-2}uw_1 + 2\gamma \int_{\mathbb{R}^n} (|x|^{-\mu} * |v|^q)|v|^{q-2}vw_2 \\ &+ \beta \int_{\mathbb{R}^n} (|x|^{-\mu} * |u|^q)|v|^{q-2}vw_2 + \beta \int_{\mathbb{R}^n} (|x|^{-\mu} * |v|^q)|u|^{q-2}uw_1. \end{aligned}$$

Proof. Fix $(u,v) \in \mathcal{N}^- \setminus \{(0,0)\}$, then obviously $(u,v) \in \mathcal{N} \setminus \{(0,0)\}$. Now arguing similarly as in Lemma 4.1, we obtain the existence of $\varepsilon > 0$ and a differentiable function $\Im^- \colon B((0,0),\varepsilon) \subset Y \to \mathbb{R}^+ := (0,+\infty)$ such that $\Im^-(0,0) = 1$, $\Im^-(w_1,w_2)((u,v)-(w_1,w_2)) \in \mathcal{N}$. Because $(u,v) \in \mathcal{N}^-$, we have

$$(I'(u,v),(u,v)) = (2q-p)\|u,v\|^p - (2q-1)\int_{\mathbb{R}^n} (f_1 u + f_2 v)\,dx < 0.$$

Since I' and $\Im^-$ are both continuous, they will not change sign in a sufficiently small neighbourhood. So if we take $\varepsilon > 0$ small enough then

$$\begin{aligned} &(I'(\Im^-(w_1,w_2)((u,v)-(w_1,w_2))), (\Im^-(w_1,w_2)((u,v)-(w_1,w_2)))) \\ &\quad = (2q-p)\|\Im^-(w_1,w_2)((u,v)-(w_1,w_2))\|^p \\ &\qquad - (2q-1)\Im^-(w_1,w_2)\int_{\mathbb{R}^n} (f_1(u-w_1) + f_2(v-w_2))\,dx < 0 \end{aligned}$$

which proves the lemma. □

Proposition 4.8. *Let $0 \neq f_1, f_2 \in L^{p/(p-1)}(\mathbb{R}^n)$ be such that (1.1) holds. Then there exists a sequence $(\widehat{u}_m, \widehat{v}_m) \subset \mathcal{N}^-$ such that*

$$J(\widehat{u}_m, \widehat{v}_m) \to \Upsilon^- \quad \text{and} \quad \|J'(\widehat{u}_m, \widehat{v}_m)\|_{Y^*} \to 0 \quad \text{as } m \to \infty.$$

Proof. We note that $\mathcal{N}^-$ is closed, by Lemma 3.3. Thus by Ekeland's Variational Principle we obtain a sequence $\{(\widehat{u}_m, \widehat{v}_m)\}$ in $\mathcal{N}^-$ such that

$$\begin{cases} J(\widehat{u}_m, \widehat{v}_m) \leq \Upsilon^- + \dfrac{1}{k}, \\ J(u,v) \geq J(\widehat{u}_m, \widehat{v}_m) - \dfrac{1}{k}\, \|(\widehat{u}_m - u, \widehat{v}_m - v)\| \quad \text{for all } (u,v) \in \mathcal{N}^-. \end{cases}$$

By coercivity of J, $\{\widehat{u}_m, \widehat{v}_m\}$ forms a bounded sequence in Y. Then using Lemma 4.7 and following the proof of Proposition 4.6 we conclude the result.

Our next result shows that J satisfies the $(\mathrm{PS})_c$ condition i.e. the Palais–Smale condition for any $c \in \mathbb{R}$.

Lemma 4.9. *Let $0 \neq f_1, f_2 \in L^{p/(p-1)}(\mathbb{R}^n)$ be such that* (1.1) *holds. Then J satisfies the* $(\mathrm{PS})_c$ *condition. That is, if $\{(u_k, v_k)\}$ is a sequence in Y satisfying*

$$J(u_k, v_k) \to c \quad \text{and} \quad J'(u_k, v_k) \to 0, \quad \text{as } k \to \infty, \tag{4.16}$$

for some $c \in \mathbb{R}$, then $\{(u_k, v_k)\}$ has a convergent subsequence.

Proof. Let $\{(u_k, v_k)\}$ be a sequence in Y satisfying (4.16). Using the same arguments as in Lemma 4.6 (see (4.11)), we can show that $\{(u_k, v_k)\}$ is bounded. There exists $(u,v) \in Y$ such that, up to a subsequence, $\{(u_k, v_k)\} \rightharpoonup (u,v)$ weakly in Y as $k \to \infty$. Using the compactness of the embedding $Y \hookrightarrow L^r(\mathbb{R}^n)$, for $r \in [p, p_s^*)$, i.e. Lemma 2.5, we get $(u_k, v_k) \to (u,v)$ strongly in $L^r(\mathbb{R}^n)$ for $r \in (p, p_s^*)$ as $k \to \infty$. From weak continuity of J' and (4.16) we get $J'(u,v) = 0$.

We claim that $\{(u_k, v_k)\} \to (u,v)$ strongly in Y. Since $\lim\limits_{k\to\infty} J'(u_k, v_k) = 0$, we consider

$$\begin{aligned} o_k(1) = {} & \langle u_k, (u_k - u)\rangle + \langle v_k, (v_k - v)\rangle \\ & + \int_{\mathbb{R}^n} (a_1 u_k(u_k - u) + a_2 v_k (v_k - v)) \\ & - \bigg(\alpha \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_k(x)|^{q-2} u_k(x)(u_k - u)(x)|u_k(y)|^q}{|x-y|^\mu}\, dx\, dy \\ & + \gamma \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v_k(x)|^{q-2} v_k(x)(v_k - v)(x)|v_k(y)|^q}{|x-y|^\mu}\, dx\, dy \\ & + \beta \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v_k(x)|^{q-2} v_k(x)(v_k - v)(x)|u_k(y)|^q}{|x-y|^\mu}\, dx\, dy \\ & + \beta \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_k(x)|^{q-2} u_k(x)(u_k - u)(x)|v_k(y)|^q}{|x-y|^\mu}\, dx\, dy \bigg) \\ & - \int_{\mathbb{R}^n} (f_1(u_k - u) + f_2(v_k - v))\, dx. \end{aligned} \tag{4.17}$$

Since $q \in (q_l, q_u)$, $p < 2nq/(2n-\mu) < p_s^*$. So, using Proposition 2.1, we get

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_k(x)|^{q-2} u_k(x)(u_k - u)(x)|u_k(y)|^q}{|x-y|^\mu}\, dx\, dy \tag{4.18}$$

$$
\begin{aligned}
&\leq \bigg(\int_{\mathbb{R}^n} (|u_k|^{q-1}|u_k-u|)^{2n/(2n-\mu)}\bigg)^{(2n-\mu)/(2n)} \\
&\quad \cdot \bigg(\int_{\mathbb{R}^n} |u_k|^{2nq/(2n-\mu)}\bigg)^{(2n-\mu)/(2n)} \\
&\leq \bigg[\bigg(\int_{\mathbb{R}^n} |u_k|^{2nq/(2n-\mu)}\bigg)^{(q-1)/q} \\
&\quad \cdot \bigg(\int_{\mathbb{R}^n} |(u_k-u)|^{2nq/(2n-\mu)}\bigg)^{1/q}\bigg]^{(2n-\mu)/(2n)} \\
&\quad \cdot \bigg(\int_{\mathbb{R}^n} |u_k|^{2nq/(2n-\mu)}\bigg)^{(2n-\mu)/(2n)} \\
&= \bigg(\int_{\mathbb{R}^n} |(u_k-u)|^{2nq/(2n-\mu)}\bigg)^{(2n-\mu)/(2nq)} \\
&\quad \cdot \bigg(\int_{\mathbb{R}^n} |u_k|^{2nq/(2n-\mu)}\bigg)^{(2n-\mu)(2q-1)/(2nq)} \to 0
\end{aligned}
$$

as $k \to \infty$. Similarly,

$$
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{v_k(x)|^{p-2}v_k(x)(v_k-v)(x)|v_k(y)|^p}{|x-y|^\mu}\,dx\,dy \to 0 \quad \text{as } k\to\infty \tag{4.19}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v_k(x)|^{q-2}v_k(x)(v_k-v)(x)|u_k(y)|^q}{|x-y|^\mu}\,dx\,dy \\
&+ \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u_k(x)|^{q-2}u_k(x)(u_k-u)(x)|v_k(y)|^q}{|x-y|^\mu}\,dx\,dy \to 0 \quad \text{as } k\to\infty.
\end{aligned} \tag{4.20}
$$

Using the hypothesis on f_1, f_2 and Hölder's inequality, we have

$$
\int_{\mathbb{R}^n} (f_1(u_k-u)+f_2(v_k-v))\,dx \to 0 \quad \text{as } k\to\infty. \tag{4.21}
$$

Combining (4.17)–(4.21), we get

$$
o_k(1) = \langle u_k,(u_k-u)\rangle + \langle v_k,(v_k-v)\rangle + \int_{\mathbb{R}^n} (a_1u_k(u_k-u)+a_2v_k(v_k-v)). \tag{4.22}
$$

Similarly, since $J'(u,v)=0$, we get

$$
\begin{aligned}
o_k(1) = \langle u,(u_k-u)\rangle &+ \langle v,(v_k-v)\rangle + \int_{\mathbb{R}^n} (a_1u(u_k-u)+a_2v(v_k-v)) \\
&- \bigg(\alpha\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)|^{q-2}u(x)(u_k-u)(x)|u(y)|^q}{|x-y|^\mu}\,dx\,dy \\
&+ \gamma\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x)|^{q-2}v(x)(v_k-v)(x)|v(y)|^q}{|x-y|^\mu}\,dx\,dy \\
&+ \beta\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x)|^{q-2}v(x)(v_k-v)(x)|u(y)|^q}{|x-y|^\mu}\,dx\,dy
\end{aligned}
$$

$$
\begin{aligned}
&+\beta \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)|^{q-2}u(x)(u_k-u)(x)|v(y)|^q}{|x-y|^\mu}\,dx\,dy\bigg) \\
&-\int_{\mathbb{R}^n} (f_1(u_k-u)+f_2(v_k-v))\,dx.
\end{aligned}
$$

Also, reasoning similarly as in (4.18)–(4.21), we get

$$
o_k(1) = \langle u,(u_k-u)\rangle + \langle v,(v_k-v)\rangle + \int_{\mathbb{R}^n} (a_1 u_k(u_k-u)+a_2 v_k(v_k-v)). \tag{4.23}
$$

Finally, (4.22) and (4.23) imply that

$$
\lim_{k\to\infty} \|(u_k,v_k)-(u,v)\|^2 = 0
$$

which proves our claim and consequently ends the proof. $\square$

5. Existence of minimizers in $\mathcal{N}^+$ and $\mathcal{N}^-$

In this section, we show that the minimums are achieved for Υ and Υ^-.

Theorem 5.1. *Let $0\neq f_1,f_2\in L^{p/(p-1)}(\mathbb{R}^n)$ be such that* (1.1) *holds. Then Υ is achieved at a point $(u_0,v_0)\in\mathcal{N}$ which is a weak solution to* (P).

Proof. From Proposition 4.6, we know that there exists a sequence

$$
\{(u_k,v_k)\}\subset\mathcal{N}
$$

such that $J(u_k,v_k)\to\Upsilon$ and $\|J'(u_k,v_k)\|_{Y^*}\to 0$ as $k\to\infty$. Let (u_0,v_0) be the weak limit of the sequence $\{(u_k,v_k)\}$ in Y. Since (u_k,v_k) satisfies (4.9), we get

$$
\int_{\mathbb{R}^n} (f_1u_0+f_2v_0)\,dx>0. \tag{5.1}
$$

Also $\|J'(u_k,v_k)\|_{Y^*}\to 0$ as $k\to\infty$ implies that

$$
(J'(u_0,v_0),(\phi_1,\phi_2))=0, \quad \text{for all } (\phi_1,\phi_2)\in Y,
$$

i.e. (u_0,v_0) is a weak solution to (P). In particular $(u_0,v_0)\in\mathcal{N}$. Moreover,

$$
\Upsilon\le J(u_0,v_0)\le \liminf_{k\to\infty} J(u_k,v_k)=\Upsilon
$$

which implies that (u_0,v_0) is the minimizer for J over $\mathcal{N}$. $\square$

Corollary 5.2. *Let $(u_0,v_0)\in\mathcal{N}$ be such that $\Upsilon=J(u_0,v_0)$, then $(u_0,v_0)\in\mathcal{N}^+$ and (u_0,v_0) is a local minimum for J in Y.*

Proof. Since (5.1) holds, using Lemma 3.2, we get that there exist $t_1,t_2>0$ such that $(u_1,v_1):=(t_1u_0,t_1v_0)\in\mathcal{N}^+$ and $(t_2u_0,t_2v_0)\in\mathcal{N}^-$. We claim that $t_1=1$, i.e. $(u_0,v_0)\in\mathcal{N}^+$. If $t_1<1$ then $t_2=1$ which implies $(u_0,v_0)\in\mathcal{N}^-$. Now $J(t_1u_0,t_1v_0)\le J(u_0,v_0)=\Upsilon$ which is a contradiction to $(t_1u_0,t_1v_0)\in\mathcal{N}^+$.

To show that (u_0, v_0) is also a local minimum for J in Y, we first notice that for each $(u, v) \in Y$ with $K(u, v) > 0$ we have

$$J(\widehat{t}u, \widehat{t}v) \geq J(t_1 u, t_1 v) \quad \text{whenever } 0 < \widehat{t} < \left(\frac{(p-1)\|(u,v)\|^p}{(2q-1)L(u,v)} \right)^{1/(2q-p)},$$

where t_1 is corresponding to (u, v). In particular, if $(u, v) \in \mathcal{N}^+$ then

$$t_1 = 1 < t_0 = \left(\frac{(p-1)\|(u,v)\|^p}{(2q-1)L(u,v)} \right)^{1/(2q-p)}. \tag{5.2}$$

Using Lemma 4.1, we obtain a differentiable map $\Im\colon B((0,0),\varepsilon) \to \mathbb{R}^+$ for $\varepsilon > 0$ such that $\Im(w_1, w_2)((u_0, v_0) - (w_1, w_2)) \in \mathcal{N}$ whenever $\|(w_1, w_2)\| < \varepsilon$. We choose $\varepsilon > 0$ sufficiently small so that

$$1 < \left(\frac{(p-1)\|((u_0,v_0)-(w_1,w_2))\|^p}{(2q-1)L((u_0,v_0)-(w_1,w_2))} \right)^{1/(2q-p)} \tag{5.3}$$

for every $(w_1, w_2) \in B((0,0),\varepsilon)$. By Lemma 4.1 we know that

$$\Im(w_1, w_2)((u_0, v_0) - (w_1, w_2)) \in \mathcal{N}$$

when $(w_1, w_2) \in B((0,0),\varepsilon)$. Also $\Im(w_1, w_2) \to 1$ as $\|(w_1, w_2)\| \to 0$. So we can assume $\Im(w_1, w_2)((u_0, v_0) - (w_1, w_2)) \in \mathcal{N}^+$ when $(w_1, w_2) \in B((0,0),\varepsilon)$ and thus whenever

$$0 < \widehat{t} < \left(\frac{(p-1)\|((u_0,v_0)-(w_1,w_2))\|^p}{(2q-1)L((u_0,v_0)-(w_1,w_2))} \right)^{1/(2q-p)}$$

we have

$$J(\widehat{t}((u_0, v_0) - (w_1, w_2))) \geq J(\Im(w_1, w_2)((u_0, v_0) - (w_1, w_2))) \geq J((u_0, v_0)).$$

Since (5.2) holds, we can take $\widehat{t} = 1$ and this gives

$$J((u_0, v_0) - (w_1, w_2)) \geq J(u_0, v_0) \quad \text{whenever } \|(w_1, w_2)\| < \varepsilon$$

which proves the last assertion. □

Proof of Theorem 1.1. The proof follows from Theorem 5.1 and Corollary 5.2 except that we need to show that there exists a nonnegative solution if $f_1, f_2 \geq 0$. Suppose $f_1, f_2 \geq 0$ then consider the function $(|u_0|, |v_0|)$. We know that there exists $t_1 > 0$ such that $(t_1|u_0|, t_1|v_0|) \in \mathcal{N}^+$ and $t_1|u_0|, t_1|v_0| \geq 0$. It is easy to see that

$$\|(u_0, v_0)\| \geq \|(|u_0|, |v_0|)\|, \quad L(u_0, v_0) = L(|u_0|, |v_0|), \quad K(u_0, v_0) \leq K(|u_0|, |v_0|).$$

If $\varphi_{u,v}(t)$ denotes the fibering map corresponding to $(u, v) \in Y$ as introduced in Section 3, we get $\varphi'_{|u_0|,|v_0|}(1) \leq \varphi'_{u_0,v_0}(1) = 0$ since t_1 is the point of local minimum of $\varphi_{|u_0|,|v_0|}(t)$ for

$$0 < t < \left(\frac{(p-1)\|(|u_0|,|v_0|)\|^p}{(2q-1)L(|u_0|,|v_0|)} \right)^{1/(2q-p)}, \quad t_1 \geq 1.$$

Necessarily,

$$J(t_1|u_0|, t_1|v_0|) \leq J(|u_0|, |v_0|) \leq J(u_0, v_0)$$

which implies that we can always take $u_0, v_0 \geq 0$ while considering the weak solution (u_0, v_0) to (P). $\square$

Next we prove that the infimum Υ^- is achieved and the minimizer is another weak solution to problem (P).

THEOREM 5.3. *Let $0 \neq f_1, f_2 \in L^{p/(p-1)}(\mathbb{R}^n)$ be such that* (1.1) *holds, then there exists $(u_1, v_1) \in \mathcal{N}^-$ such that $\Upsilon^- = J(u_1, v_1)$.*

PROOF. Using Lemma 4.8, we know that there exists a sequence $\{(\widehat{u}_m, \widehat{v}_m)\} \subset \mathcal{N}^-$ such that

$$J(\widehat{u}_m, \widehat{v}_m) \to \Upsilon^- \quad \text{and} \quad J'(\widehat{u}_m, \widehat{v}_m) \to 0, \quad \text{as } m \to \infty.$$

Applying again Lemma 4.9, we get that there exists $(u_1, v_1) \in Y$ such that, up to a subsequence, $(\widehat{u}_m, \widehat{v}_m) \to (u_1, v_1)$ strongly in Y as $m \to \infty$. This implies

$$\lim_{k\to\infty} J(\widehat{u}_m, \widehat{v}_m) = J(u_1, v_1) = \Upsilon^- \quad \text{and} \quad (u_1, v_1) \in \mathcal{N}^-.$$

Therefore, Lemma 3.4 implies that (u_1, v_1) is a weak solution to (P). $\square$

Finally, we prove Theorem 1.2.

PROOF OF THEOREM 1.2. The existence of the second weak solution (u_1, v_1) to (P) is asserted by Theorem 5.3. So we only need to show that we can obtain a nonnegative weak solution if $f_1, f_2 \geq 0$. Consider the function $(|u_1|, |v_1|)$, then there exists $t_2 > 0$ such that $(t_2|u_1|, t_2|v_1|) \in \mathcal{N}^-$. Let

$$t_0 = \left(\frac{(p-1)\|(u_1, v_1)\|^p}{(2q-1)L(u_1, v_1)} \right)^{1/(2q-p)}$$

then, since $(u_1, v_1) \in \mathcal{N}^-$, we conclude that

$$J(u_1, v_1) = \max_{t \geq t_0} J(tu_1, tv_1) \geq J(t_2u_1, t_2v_1) \geq J(t_1|u_1|, t_1|v_1|).$$

Therefore it remains true to assume $u_1, v_1 \geq 0$ while considering the weak solution (u_1, v_1) in case $f_1, f_2 \geq 0$. $\square$

REFERENCES

[1] ADIMURTHI, J. GIACOMONI AND S. SANTRA, *Positive solutions to a fractional equation with singular nonlinearity*, arXiv:1706.01965.

[2] C.O. ALVES, M.G. FIGUEIREDO AND M. YANG, *Existence of solutions for a nonlinear Choquard equation with potential vanishing at infinity*, Adv. Nonlinear Anal. **5** (2016), 331–345.

[3] C.O. ALVES, A.B. NÓBREGA AND M. YANG, *Multi-bump solutions for Choquard equation with deepening potential well*, Calc. Var. (2016), 55:48.

[4] C.O. Alves and M. Yang, *Existence of semiclassical ground state solutions for a generalized Choquard equation*, J. Differential Equations **257** (2014), 4133–164.

[5] C.O. Alves and M. Yang, *Multiplicity and concentration of solutions for a quasilinear Choquard equation*, J. Math. Phys. **55** (2014), no. 6, 061502, 21 pp.

[6] C.O. Alves and M. Yang, *Investigating the multiplicity and concentration behaviour of solutions for a quasi-linear Choquard equation via the penalization method*, Proc. Roy. Soc. Edinburgh Sect. A **146** (2016), 23–58.

[7] L. Brasco and E. Parini, *The second eigenvalue of the fractional p-Laplacian*, Adv. Calc. Var. **9** (2015), 323–355.

[8] L. Brasco, E. Parini and M. Squassina, *Stability of variational eigenvalues for the fractional p-Laplacian*, Discrete Contin. Dyn. Syst. Ser. A **36** (2016), 1813–1845.

[9] L.A. Caffarelli, *Nonlocal equations, drifts and games*, Nonlinear Partial Differential Equations, Abel Symposia **7** (2012), 37–52.

[10] K.C. Chang, *Methods in Nonlinear Analysis*, Springer Monographs in Mathematics, Springer, Berlin, 2005.

[11] W. Chen and S. Deng, *The Nehari manifold for a fractional p-Laplacian system involving concave-convex nonlinearities*, Nonlinear Anal. Real World Appl. **27** (2016), 80–92.

[12] W. Chen and M. Squassina, *Critical nonlocal systems with concave-convex powers*, Adv. Nonlinear Stud. **16** (2016), 821–842.

[13] W. Choi, *On strongly indefinite systems involving the fractional Laplacian*, Nonlinear Anal. **120** (2015), 127–153.

[14] M. Clapp and D. Salazar, *Positive and sign changing solutions to a nonlinear Choquard equation*, J. Math. Anal. Appl. **407** (2013), 1–15.

[15] V. Coti Zelati and P.H. Rabinowitz, *Homoclinic type solutions for a semilinear elliptic* PDE *on* $\mathbb{R}^n$, Comm. Pure Appl. Math. **45** (1992), 1217–1269.

[16] P. d'Avenia, G. Siciliano and M. Squassina, *On fractional Choquard equation*, Math. Models Methods Appl. Sci. **25** (2015), 1447–1476.

[17] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521–573.

[18] L.F.O. Faria, O.H. Miyagaki, F.R. Pereira, M. Squassina and C. Zhang, *The Brezis–Nirenberg problem for nonlocal systems*, Adv. Nonlinear Anal. **5** (2016), 85–103.

[19] J. Fröhlich, T.-P. Tasi and H.-T. Yau, *On a classical limit of quantum theory and the non-linear Hartree eqution*, Visions in Mathematics. Modern Birkhäuser Classics (N. Alon, J. Bourgain, A. Connes, M. Gromov, V. Milman, eds.), Birkhäuser, Basel, 2010.

[20] J. Fröhlich, T.-P. Tasi and H.-T. Yau, *On the point-particle* (*Newtonian*) *limit of the non-linear Hartree equation*, Commun. Math. Phys. **225** (2002), 223–274.

[21] B. Ge, *Multiple solutions of nonlinear Schrödinger equation with the fractional Laplacian*, Nonlinear Anal. Real World Appl. **30** (2016), 236–247.

[22] M. Ghimenti and J.V. Schaftingen, *Nodal solutions for the Choquard equation*, J. Funct. Anal. **271** (2016), 107–135.

[23] M. Ghimenti, V. Moroz and J.V. Schaftingen, *Least action nodal solutions for the quadratic Choquard equation*, Proc. Amer. Math. Soc. **145** (2017), 737–747.

[24] J. Giacomoni, P.K. Mishra and K. Sreenadh, *Critical growth fractional elliptic systems with exponential nonlinearity*, Nonlinear Anal. **136** (2016), 117–135.

[25] J. Giacomoni, P.K. Mishra and K. Sreenadh, *Fractional elliptic systems with exponential nonlinearity*, Nonlinear Anal. **136** (2016), 117–135.

[26] S. GOYAL, *Multiplicity results of fractional p-Laplace equations with sign-changing and singular nonlinearity*, Complex Var. Elliptic Equ. **62** (2017), 158–183.

[27] S. GOYAL AND K. SREENADH, *Existence of multiple solutions of p-fractional Laplace operator with sign-changing weight function*, Adv. Nonlinear Anal. **4** (2015), 37–58.

[28] Z. GUO, S. LUO AND W. ZOU, *On critical systems involving frcational Laplacian*, J. Math. Anal. Appl. **446** (2017), 681–706.

[29] X. HE, M. SQUASSINA AND W. ZOU, *The Nehari manifold for fractional systems involving critical nonlinearities*, Comm. Pure Appl. Math. **15** (2016), 1285–1308.

[30] A. IANNIZZOTTO, S. LIU, K. PERERA AND M. SQUASSINA, *Existence results for fractional p-Laplacian problems via Morse theory*, Adv. Calc. Var. **9** (2014), no. 2, 101–125.

[31] A. IANNIZZOTTO, S. MOSCONI AND M. SQUASSINA, *Global Hölder regularity for the fractional p-Laplacian*, Rev. Mat. Iberoam. **32** (2016), 1353–1392.

[32] A. IANNIZZOTTO AND M. SQUASSINA, *Weyl-type laws for fractional p-eigenvalue problems*, Asymptot. Anal. **88** (2014), no. 4, 233–245.

[33] E. LIEB AND M. LOSS, *Analysis, Graduate Studies in Mathematics*, Amer. Math. Soc., Providence, Rhode Island, 2001.

[34] E. LINDGREN AND P. LINDQVIST, *Fractional eigenvalues*, Calc. Var. (2014), 49:795.

[35] D. LÜ AND G. XU, *On nonlinear fractional Schrödinger equations with Hartree-type nonlinearity*, Appl. Anal. (2016), DOI:10.1080/00036811.2016.1260708.

[36] G. MOLICA BISCI AND V. RĂDULESCU AND R. SERVADEI, *Variational methods for nonlocal fractional problems*, with a foreword by Jean Mawhin. Encyclopedia of Mathematics and its Applications, Vol. 162, Cambridge University Press, Cambridge, 2016.

[37] V. MOROZ AND J.V. SCHAFTINGEN, *Groundstates of nonlinear Choquard equations: Existence, qualitative properties and decay asymptotics*, J. Func. Anal. **265** (2013), no. 2, 153–184.

[38] V. MOROZ AND J.V. SCHAFTINGEN, *Existence of groundstates for a class of nonlinear Choquard equations*, Trans. Amer. Math. Soc. **367** (2015), 6557–6579.

[39] V. MOROZ AND J.V. SCHAFTINGEN, *Groundstates of nonlinear Choquard equations: Hardy–Littlewood–Sobolev critical exponent*, Commun. Contemp. Math. **17** (2015), no. 5, 1550005, 12 pp.

[40] V. MOROZ AND J.V. SCHAFTINGEN, *Semi-classical states for the Choquard equation*, Calc. Var. **52** (2015), 199–235.

[41] T. MUKHERJEE AND K. SREENADH, *Fractional Choquard equation with critical nonlinearities*, Nonlinear Differential Equations and Applications **24** (2017), Art. 63.

[42] R. SERVADEI, *The Yamabe equation in a non-local setting*, Adv. Nonlinear Anal. **2** (2013), no. 3, 235–270.

[43] R. SERVADEI, *A critical fractional Laplace equation in the resonant case*, Topol. Methods Nonlinear Anal. **43** (2014), 251–267.

[44] R. SERVADEI AND E. VALDINOCI, *The Brezis–Nirenberg result for the fractional Laplacian*, Trans. Amer. Math. Soc. **367** (2015), 67–102.

[45] Z. SHEN, F. GAO AND M. YANG, *Ground states for nonlinear fractional Choquard equations with general nonlinearities*, Math. Methods Appl. Sci. **39** (2016), DOI: 10.1002/mma.3849.

[46] E. STEIN, *Singular integrals and differentiability properties of functions*, Princeton, NJ, Princeton University Press, (1970).

[47] G. TARANTELLO, *On nonhomogeneous elliptic equations involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire **9** (1992), 281–304.

[48] J. Wang, Y. Dong, Q. He and L. Xiao, *Multiple positive solutions for a coupled nonlinear Hartree type equations with perturbations*, J. Math. Anal. Appl. **450** (2017), 780–794.

[49] K. Wang and J. Wei, *On the uniqueness of solutions of a nonlocal elliptic system*, Math. Ann., 365(1-2) (2016), 105–153.

[50] T. Wang, *Existence and nonexistence of nontrivial solutions for Choquard type equations*, Electron. J. Differential Equations **3** (2016), 1–17.

[51] D. Wu, *Existence and stability of standing waves for nonlinear fractional Schrödinger equations with Hartree type nonlinearity*, J. Math. Anal. Appl. **411** (2014), 530–542.

[52] T. Xie, L. Xiao and J. Wang, *Exixtence of multiple positive solutions for Choquard equation with perturbation*, Adv. Math. Phys. **2015** (2015), 760157.

Manuscript received June 6, 2017

accepted October 24, 2017

Tuhina Mukherjee and Konijeti Sreenadh
Department of Mathematics
Indian Institute of Technology Delhi
Hauz Khaz, New Delhi-110016, INDIA

E-mail address: tulimukh@gmail.com, sreenadh@gmail.com