INFINITELY MANY SOLUTIONS
FOR A CLASS OF QUASILINEAR EQUATION
WITH A COMBINATION OF CONVEX AND CONCAVE TERMS

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Abstract. We consider the following quasilinear elliptic equation with convex and concave nonlinearities:

$$-\Delta_p u - (\Delta_p u^2)u + V(x)|u|^{p-2}u = \lambda K(x)|u|^{q-2}u + \mu g(x,u), \quad \text{in } \mathbb{R}^N,$$

where $2 \le p < N$, $1 < q < p$, $\lambda, \mu \in \mathbb{R}$, V and K are potential functions, and $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ is a continuous function. Under some suitable conditions on V, K and g, the existence of infinitely many solutions is established.

1. Introduction

In this paper, we study the following quasilinear Schrödinger equation:

$$(1.1) \quad -\Delta_p u - (\Delta_p u^2)u + V(x)|u|^{p-2}u = \lambda K(x)|u|^{q-2}u + \mu g(x,u), \quad \text{in } \mathbb{R}^N,$$

where $-\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u)$, $2 \le p < N$, $1 < q < p$, $\lambda, \mu \in \mathbb{R}$ are two parameters. In order to deal with the concave term we make the following assumptions on the potentials V and K:

(V$_1$) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x \in \mathbb{R}^N} V(x) \ge V_0 > 0$;

(V$_2$) $\int_{\mathbb{R}^N} V(x)^{-1/(p-1)}\,dx < +\infty$;

(K$_0$) $K \in L^\infty(\mathbb{R}^N)$, $K(x) \ge 0$, $K(x) \not\equiv 0$;

(K_1) $K \in L^{2p^*/(2p^*-q)}(\mathbb{R}^N)$, where $p^* = Np/(N-p)$.

Also, we pose the following assumptions on g:

(g_0) $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $\lim_{|s| \to 0} g(x,s)/|s|^{p-2}s = 0$ uniformly for $x \in \mathbb{R}^N$.

(g_1) There exist $c > 0$ and $p < r < 2p^*$ such that $|g(x,u)| \leq c(1 + |u|^{r-1})$ for all $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$.

(g_2) There exists $2p < \theta < 2p^*$ such that $0 < \theta G(x,u) \leq ug(x,u)$ for all $x \in \mathbb{R}^N$ and $u \in \mathbb{R}\backslash\{0\}$.

(g_3) $g(x,u)$ is odd in u.

REMARK 1.1. The assumption $p \geq 2$ is a consequence of the choice of the work space $\mathcal{E}_f$ which requires $|f(t)|^p$ to verify the convexity property and the Δ_2 condition, see Proposition 2.5.

The quasilinear Schrödinger equation of type (1.1) has served for modeling of several physical phenomena. It is related to the existence of standing wave solutions for quasilinear Schrödinger equation of the form

$$(1.2) \qquad iz_t = -\Delta z + W(x)z - f(|z|^2)z - \kappa \Delta h(|z|^2)h'(|z|^2)z, \quad x \in \mathbb{R}^N,$$

where W is a given potential, κ is a real constant, f and h are real functions. For instance, in the case $h(s) = s$, it corresponds to the superfluid film equations in plasma physics, see Kurihara [16]. In the case $h(s) = (1+s)^{1/2}$, it models the self-channeling of a high-power ultra short laser in the matter, see [8]. Equation (1.2) also appears in plasma physics and fluid mechanics, see [16] and [17], in theory of Heisenberg ferromagnets and magnons, see [15], [32]. Considering the case $h(s) = s$, $\kappa = 1$ and setting $z(x,t) = \exp(-iwt)u(x)$, $w \in \mathbb{R}$, it is easy to obtain the corresponding equation

$$(1.3) \qquad\qquad -\Delta u - (\Delta u^2)u + V(x)u = g(u), \quad x \in \mathbb{R}^N,$$

where $V(x) = W(x) - w$, $g(u) = f(|u|^2)u$. Because one of the main difficulties of problem (1.3) is that there is no suitable work space on which the energy functional is of class C^1, the standard critical point theory cannot be applied directly. The existence and multiplicity of solutions to the problems like (1.3) have been considered by many authors in the recent years. To the best of our knowledge, there are some powerful methods developed, such as, the minimizing method [18], [26], the Nehari manifold method [6], [21], the method of change of variables which was independently applied in [20] and [12], the method of nonsmooth critical point theory [22], [23], the perturbation method [25], [19]. By the change of variables, the quasilinear equation (1.3) reduces to a semilinear one, so the usual methods for semilinear Schrödinger equations can be adopted. This method has become the fundamental trick for studying quasilinear problem (1.3). For the recent progress in this regard, we refer the interested readers to [1]–[3], [9] and references therein.