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MULTIPLE POSITIVE SOLUTIONS FOR A CLASS OF VARIATIONAL SYSTEMS

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Abstract. We consider the variational system $-\Delta u = \lambda(\nabla F)(u)$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded region in $\mathbb{R}^m$ $(m \geq 1)$ with C^1 boundary, λ is a positive parameter, $u\colon \Omega \to \mathbb{R}^N$ $(N > 1)$, and Δ denotes the Laplace operator. Here $F\colon \mathbb{R}^N \to \mathbb{R}$ is a function of class C^2. Using variational methods, we show how changes in the sign of F lead to multiple positive solutions.

1. Introduction

We study the existence of positive solutions to the variational system

$$\begin{cases} -\Delta u = \lambda(\nabla F)(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded region in $\mathbb{R}^m$ $(m \geq 1)$ with C^1 boundary, λ is a positive parameter, $u\colon \Omega \to \mathbb{R}^N$ $(N > 1)$, and Δ denotes the Laplacian operator. Here $F\colon \mathbb{R}^N \to \mathbb{R}$ is a function of class C^2. By a positive solution to (1.1) we mean a function $u = (u_1, \ldots, u_N)$ with each $u_j \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $u_j(x) \geq 0$; Ω, $u_j(x) = 0$; $\partial\Omega$ and $u_l(x_0) > 0$ for some $l \in \{1, \ldots, N\}$, $x_0 \in \Omega$.

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Let $\Sigma = \{z = (z_1, \dots, z_N) \mid z_1 + \dots + z_N = 1$ with $z_i \geq 0$ for all $i = 1, \dots, N\}$ and $Q = \{z = (z_1, \dots, z_N) \mid z_i > 0$ for all $i = 1, \dots, N\}$. For $z \in \mathbb{R}^N$ we will denote $|z|_1 = |z_1| + \dots + |z_N|$ and $P(z) = z/|z|_1$, for $z \neq 0$. We assume:

(H1) $F(z_1, \dots, z_N) = 0$ if $z_1 \dots z_N = 0$, and $F(z_1, \dots, z_N) \leq 0$ if $z_i \leq 0$ for some $i = 1, \dots, N$.

(H2) There exist concave functions $\rho_1, \dots, \rho_{2k} \colon \Sigma \to [0, \infty)$ such that

$$0 < \rho_i(z) < \rho_{i+1}(z), \quad i = 1, \dots, 2k - 1.$$

(H3) For $z \in Q$, $F(z) > 0$ if $|z|_1 \in (\rho_{2i+1}(P(z)), \rho_{2(i+1)}(P(z)))$, and $F(z) < 0$ if $|z|_1 \in (\rho_{2i}(P(z)), \rho_{2i+1}(P(z)))$ for $i = 0, 2, \dots, k - 1$ or $|z|_1 \geq \rho_{2k}(P(z))$.

(H4) For $i = 0, \dots, k-1$ there exists $z^{(i)} \in Q$ such that $|z^{(i)}|_1 \in (\rho_{2i+1}(P(z^{(i)})), \rho_{2(i+1)}(P(z^{(i)})))$ and $F(z^{(i)}) > \max\{F(z) \mid |z|_1 < \rho_{2i+1}(P(z^{(i)}))\}$.

Our main result is the following:

Theorem 1.1. *There exist $\lambda_1 < \dots < \lambda_k$ such that if $\lambda > \lambda_i$, for $i = 1, \dots, k - 1$, then the boundary value problem* (1.1) *has i positive solutions.*

In the single equation case ($N = 1$) there is a rich history on the study of such boundary value problems where the analysis of how the changes of sign of the nonlinear term give rise to the existence of multiple positive solutions. In particular see Brown–Budin [1] where a combination of variational and monotone iteration methods is used, Hess [8] where a combination of variational and topological degree arguments is applied, and De Figueiredo [7] where only variational methods are used. Clement and Sweers in [3] proved that if $f \colon [0, \infty) \to \mathbb{R}$ satisfies

$$\begin{aligned} f(s) &< 0 \quad \text{for } 0 < s < s_1 \text{ or } s > s_2, \\ f(s) &> 0 \quad \text{for } s_1 < s < s_2, \end{aligned} \tag{1.2}$$

for some $0 < s_1 < s_2$, then the (possibly *semipositone*) problem

$$-\Delta u = \lambda f(u) \quad \text{in } \Omega, \qquad u(x) = 0 \quad \text{for } x \in \partial\Omega, \tag{1.3}$$

has a positive solution u for λ large with $\|u\|_\infty \in (s_1, s_2)$ if and only if

$$\int_0^{s_2} f(s)\, ds > 0. \tag{1.4}$$

Also, Dancer and Schmitt in [5] used sub-supersolutions to show that (1.4) is necessary for existence of a positive solution to (1.3). In [6], De Figueiredo showed that for $f(u) = \sin(u)$ and $\lambda > 0$ in (1.3) this equation has only one positive solution. However, to date no extension of this study has been achieved for the system case (when $N > 1$), which we establish in this paper via variational methods.