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POROSITY RESULTS FOR SETS OF STRICT CONTRACTIONS ON GEODESIC METRIC SPACES

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Abstract. We consider a large class of geodesic metric spaces, including Banach spaces, hyperbolic spaces and geodesic CAT(κ)-spaces, and investigate the space of nonexpansive mappings on either a convex or a star-shaped subset in these settings. We prove that the strict contractions form a negligible subset of this space in the sense that they form a σ-porous subset. For certain separable and complete metric spaces we show that a generic nonexpansive mapping has Lipschitz constant one at typical points of its domain. These results contain the case of nonexpansive self-mappings and the case of nonexpansive set-valued mappings as particular cases.

1. Introduction

The question of existence of fixed points for nonexpansive mappings

$$f\colon C \to C,$$

where C denotes a certain non-empty closed subset of a complete metric space X, has been well studied. Recall that a mapping f is called *nonexpansive* if it

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satisfies, for all $x, y \in C$, the inequality

$$\rho(f(x), f(y)) \leq \rho(x, y),$$

where ρ denotes the metric on X. If X is a Euclidean space and $C \subset X$ is bounded, closed and convex, Brouwer's fixed point theorem (Satz 4 in [5]) states that every continuous mapping $f\colon C \to C$ has a fixed point. The example

$$T\colon C \to C, \quad Tx := (1, x_1, x_2, \ldots),$$

where $C := \{x \in c_0 : 0 \leq x_n \leq 1\}$, shows that in infinite dimensions there are noncompact C and nonexpansive mappings $f\colon C \to C$ without fixed points. In 1965, F.E. Browder showed in [6] that nonexpansive mappings on the closed unit ball of the Hilbert space ℓ_2 have a fixed point. Detailed discussions of the fixed point property for nonexpansive mappings can be found, for example, in Section 1.6 of [17] and in Chapter 4 of [16]. More recent results are presented, for instance, in [23] and in the references cited therein.

Instead of characterizing the sets C for which every nonexpansive self-mapping has a fixed point, F.S. De Blasi and J. Myjak took a different approach in [9], [10]. They raised the question of whether the typical nonexpansive mapping has a fixed point. To be more precise, let C be a bounded, closed and convex subset of a Banach space X, and denote by

$$\mathcal{M} := \{f\colon C \to C : \|f(x) - f(y)\| \leq \|x - y\| \text{ for all } x, y \in C\}$$

the space of nonexpansive mappings on C equipped with the metric of uniform convergence. In [9] they proved that there is a dense G_δ-set $\mathcal{M}'$ in $\mathcal{M}$ such that each $f \in \mathcal{M}'$ has a unique fixed point which is the pointwise limit of the iterates of f. They improved this result in [10], where they showed that there is a set $\mathcal{M}_* \subset \mathcal{M}$ with a σ-porous complement such that each $f \in \mathcal{M}_*$ has a unique fixed point which is the uniform limit of the iterates of f. Put in different words, these results state that a generic nonexpansive mapping f on a bounded, closed and convex subset of a Banach space has a unique fixed point which is the uniform limit of the iterates of f.

Since Banach's fixed point theorem from 1922, see [1], states that every *strict contraction*, that is, a mapping

$$f\colon C \to C \quad \text{with } \rho(f(x), f(y)) \leq L\rho(x, y) \text{ and } L < 1,$$

has a unique fixed point which is the uniform limit of the iterates of f, the question arises whether a generic nonexpansive mapping on a bounded, closed and convex subset of a Banach space is, in fact, a strict contraction. Using the Kirszbraun–Valentine extension theorem, De Blasi and Myjak answered this natural question in the negative by showing in the aforementioned papers that if X is a Hilbert space, then the set of strict contractions is σ-porous. In the