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ON SOME PROPERTIES OF THE SOLUTION SET MAP TO VOLTERRA INTEGRAL INCLUSION

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Abstract. For the multivalued Volterra integral equation defined in a Banach space, the set of solutions is proved to be R_δ, without auxiliary conditions imposed in Theorem 6 [J. Math. Anal. Appl. 403 (2013), 643–666]. It is shown that the solution set map, corresponding to this Volterra integral equation, possesses a continuous singlevalued selection; and the image of a convex set under the solution set map is acyclic. The solution set to the Volterra integral inclusion in a separable Banach space and the preimage of this set through the Volterra integral operator are shown to be absolute retracts.

1. Introduction

In [13], the author conducted the study of geometric properties of the solution set to the following Volterra integral inclusion:

$$x(t) \in h(t) + \int_0^t k(t,s)F(s,x(s))\,ds, \quad t \in I = [0,T], \tag{1.1}$$

in a Banach space E, with $h \in C(I,E)$, $k(t,s) \in \mathscr{L}(E)$ and $F\colon I \times E \multimap E$ a convex valued perturbation. It was proven that the solution set $S_F^p(h)$ of integral inclusion (1.1) is acyclic in the space $C(I,E)$ or is even R_δ, provided some additional conditions on the Banach space E or the kernel k and perturbation F are imposed. We will show that these auxiliary assumptions are redundant.

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Thus, the present work complements [13] by strengthening Theorems 6 and 7 of loc. cit. In Section 3, we also give some applications of investigated properties of the solution set to problem (1.1). One of them is the characterization of the solution set of an evolution inclusion in the so-called parabolic case. The second is the result on the existence of periodic trajectories of the integral inclusion under consideration.

Introducing the so-called *solution set map* $S_F^p \colon C(I,E) \multimap C(I,E)$, which associates with each inhomogeneity $h \in C(I,E)$ the set of all solutions to (1.1), we prove that under generic assumptions on F this multimap possesses a continuous singlevalued selection. With this aim we adapt a well-known construction taken from [8].

Already singlevalued examples show that in case of continuous f no more than connectedness of the image $f(M)$ of a connected set M can be expected. Also in the case of the solution set map it is clear only that the set $\bigcup_{h\in M} S_F^p(h)$ is connected, if $M \subset C(I,E)$ is connected. We exploit the admissibility of the solution set map and the result of Vietoris to demonstrate that the image of a compact convex $M \subset C(I,E)$ through S_F^p must be acyclic.

Since the solutions of inclusion (1.1) are understood in the sense of Aumann integral, it is natural to examine the issue of geometric structure of the set of these integrable selections of the perturbation F, which make up the solution set $S_F^p(h)$ being mapped by the Volterra integral operator. It has been shown that these selections form a retract of the space $L^p(I,E)$. In the context of stronger assumptions on the Volterra integral operator kernel, the solution set $S_F^p(h)$ turns out to be also an absolute retract.

2. Preliminaries

Denote by I the interval $[0,T]$ and by Σ the σ-algebra of Lebesgue measurable subsets of I. Let E be a real Banach space with the norm $|\cdot|$ and $\mathscr{B}(E)$ the family of Borel subsets of E. The space of bounded linear endomorphisms of E is denoted by $\mathscr{L}(E)$ and E^* stands for the normed dual of E. Given $S \in \mathscr{L}(E)$, $||S||_{\mathscr{L}}$ is the norm of S. The closure and the closed convex envelope of $A \subset E$ will be denoted by $\overline{A}$ and $\overline{\mathrm{co}}\, A$ and if $x \in E$ we set

$$d(x,A) = \inf\{|x-y| : y \in A\}.$$

Besides, for two nonempty closed bounded subsets A, B in E we denote by $d_H(A,B)$ the Hausdorff distance from A to B, i.e.

$$d_H(A,B) = \max\{\sup\{d(x,B) : x \in A\}, \sup\{d(y,A) : y \in B\}\}.$$

By $(C(I,E), ||\cdot||)$ we mean the Banach space of continuous maps $I \to E$ equipped with the maximum norm. Let $1 \le p < \infty$. Then $(L^p(I,E), ||\cdot||_p)$ is the