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ON EXTREME VALUES OF NEHARI MANIFOLD METHOD VIA NONLINEAR RAYLEIGH'S QUOTIENT

Yavdat Ilyasov

Abstract. We study applicability conditions of the Nehari manifold method to the equation of the form $D_uT(u) - \lambda D_uF(u) = 0$ in a Banach space W, where λ is a real parameter. Our study is based on the development of the Rayleigh quotient theory for nonlinear problems. It turns out that the extreme values of parameter λ which define intervals of applicability of the Nehari manifold method can be found through the critical values of the corresponding nonlinear generalized Rayleigh quotient. In the main part of this paper, we provide general results on this relationship. Theoretical results are illustrated by considering several examples of nonlinear boundary value problems. Furthermore, we demonstrate that the introduced tool of nonlinear generalized Rayleigh quotient can also be applied to prove new results on the existence of multiple solutions for nonlinear elliptic equations.

1. Introduction

The Nehari manifold method (NM-method), which was introduced in [28] and [29], by now is a well-established and useful tool in finding solutions of equations in variational form. Let us briefly describe it. Assume W is a real

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Banach space, $\Phi_\lambda \colon W \to \mathbb{R}$ is a Fréchet-differentiable functional with derivative $D_u\Phi_\lambda$ and $\lambda \in \mathbb{R}$ is a parameter. Consider the equation in variational form

$$(1.1) \qquad D_u\Phi_\lambda(u) = 0, \quad u \in W.$$

The Nehari manifold associated with (1.1) is defined as

$$\mathcal{N}_\lambda := \{u \in W \setminus 0 : D_u\Phi_\lambda(u)(u) = 0\}.$$

Since any solution of (1.1) belongs to $\mathcal{N}_\lambda$, a natural idea to solve (1.1) is to consider the constrained minimization problem

$$\Phi_\lambda(u) \to \min, \quad u \in \mathcal{N}_\lambda.$$

Suppose that there exists a local minimizer u of this problem and $\Phi_\lambda \in C^2(U, \mathbb{R})$ for some neighbourhood $U \subset W$ of u. Then by the Lagrange multiplier rule one has $\mu_0 D_u\Phi_\lambda(u) + \mu_1(D_u\Phi_\lambda(u) + D_{uu}\Phi_\lambda(u)(u, \cdot\,)) = 0$ for some μ_0, μ_1 such that $|\mu_0| + |\mu_1| \neq 0$. Testing this equality by u we obtain $\mu_1 D_{uu}\Phi_\lambda(u)(u,u) = 0$. Hence, if $D_{uu}\Phi_\lambda(u)(u,u) \neq 0$, then we have successively $\mu_1 = 0$, $\mu_0 \neq 0$ and therefore $D_u\Phi_\lambda(u) = 0$. Thus, one has the following sufficient condition for the applicability of the NM-method:

$$(1.2) \qquad D_{uu}\Phi_\lambda(u)(u,u) \neq 0 \quad \text{for any } u \in \mathcal{N}_\lambda.$$

The feasibility of this condition often depends on the parameter λ. Thus, we may expect that there exists the set of *extreme values of the NM-method* $\sigma_{\mathcal{N}} := \{\lambda_{\min,i}, \lambda_{\max,i}\}_{i=1}^{\infty}$ such that the sufficient condition (1.2) is satisfied only for $\lambda \in \bigcup_{i=1}^{\infty}(\lambda_{\min,i}, \lambda_{\max,i})$. This brings up the question of *how to find these extreme values.*

In general, this question is related to finding bifurcations for critical points of family fibering functions $\phi_{\lambda,v}(s) := \Phi_\lambda(sv)$, $s \in \mathbb{R}^+$, where $v \in S := \{v \in W : \|v\|_W = 1\}$ and $\lambda \in \mathbb{R}$. Indeed, if $u_\lambda = s_\lambda v_\lambda$ satisfies (1.1), then $d\phi_{\lambda,v_\lambda}(s_\lambda)/ds = 0$ and hence $s_\lambda v_\lambda \in \mathcal{N}_\lambda$, whereas condition (1.2) is equivalent to $d^2\phi_{\lambda,v_\lambda}(s_\lambda)/ds^2 \neq 0$. Thus, in general, an extreme value λ^* of the NM-method may occur only: (1) as a bifurcation at zero or at infinity, when $s_\lambda \to 0$ or $s_\lambda \to +\infty$ as $\lambda \to \lambda^*$, respectively; (2) as a bifurcation at a point (s^*, λ^*, v^*), where $d^2\phi_{\lambda^*,v^*}(s^*)/ds^2 = 0$ and $(s^1_\lambda, v^1_\lambda) \to (s^*, v^*)$, $(s^2_\lambda, v^2_\lambda) \to (s^*, v^*)$ as $\lambda \to \lambda^*$, for some branches of critical points s^1_λ and s^2_λ of $\phi_{\lambda,v}(s)$. In fact, when for each $v \in S$ the function $\phi_{\lambda,v}(s)$ may possess at most one critical point in $\mathbb{R}^+$ the extreme values of the NM-method either do not exist or can be found directly. Essential dependence of equation (1.1) on the parameter λ and the necessity of finding the extreme values of the NM-method take place when $\phi_{\lambda,v}(s)$ may have more than one critical point of various types. Nonlinear partial differential equations with such property have been studied in a number of papers dealing with the multiplicity of solutions (see, e.g. [1]–[3], [15], [30]). Furthermore, to study such problems a general approach, the so-called fibering method, had been introduced by Pohozaev [32], [33]. However, as far as we know, the problem of finding the extreme values of the NM-method in the general setting has not been given much attention to. This problem was studied in [17], [18], [20] where a method (the so-called spectral analysis by the fibering procedure) of finding variational principles corresponding to the extreme values of the NM-method has been introduced. Although this method has been applied to a number of problems (see, e.g. [8], [12], [13], [16], [21], [22], [24], [25]), it has certain disadvantages mainly due to its complexity. Difficulties in finding of the extreme values of the NM-method significantly increase when systems of equations are considered. Actually, in this case the corresponding fibering function $\phi_{\lambda,\overline{v}}(\overline{t}) := \Phi_\lambda(\overline{t}\,\overline{v})$ is a function of several variables $\overline{t} \in (\mathbb{R}^+)^n$ and analysis of its critical points is known to be more complicated as compared with the fibering function of one variable $\phi_{\lambda,v}(s)$ for the scalar problem.

The aim of the present paper is to introduce a new approach to this problem. To specify the principal idea, let us consider equation (1.1) in the particular form

$$D_uT(u) - \lambda D_uG(u) = 0.$$

Let us assume that $D_uG(u)(u) \neq 0$ for any $u \in W \setminus 0$. Testing the equation by $u \in W$ and then solving it with respect to $\lambda =: R(u)$ we obtain the following functional:

$$R(u) = \frac{D_uT(u)(u)}{D_uG(u)(u)}, \quad u \in W \setminus 0, \tag{1.3}$$

which is meaningful to call the Rayleigh quotient. Note that u belongs to $\mathcal{N}_\lambda$ if and only if it lies on the level set $R(u) = \lambda$. Using this fact we derive that

$$D_uR(u)(u) = \frac{1}{D_uG(u)(u)}\, D^2_{uu}\Phi_\lambda(u)(u,u), \quad \text{for all } u \in \mathcal{N}_\lambda, \tag{1.4}$$

which means, in particular, that the sufficient condition (1.2) is satisfied if and only if $D_uR(u)(u) \neq 0$. Note also that $D_uR(u)(u) = dR(tu)/dt|_{t=1}$. This reasoning leads us to the following idea:

The extreme values of the NM-method can be studied by investigation of the critical values of the fibering Rayleigh quotient $r_u(t) := R(tu)$ *in* $\mathbb{R}^+ \setminus 0$*, for* $u \in W \setminus 0$.

To implement this idea, we introduce a new approach to generalization of the Rayleigh quotient that preserves main properties of the original Rayleigh quotient and allows to analyse the nonlinear problems. Basically this way of generalization may be described as follows: In general, one may assume that for every $u \in W\setminus 0$ the function $r_u(t)$ has a countable (or finite) set of extreme points $t_1(u), t_2(u), \ldots \in \mathbb{R}^+ \setminus 0$, which determine the mappings $t_i(\,\cdot\,)\colon W \setminus 0 \to \mathbb{R}^+ \setminus 0$,

$i = 1, 2, \dots$, so that we are able to introduce 0-homogeneous functionals

$$\lambda_i(u) := r_u(t_i(u)), \quad u \in W \setminus 0, \ i = 1, 2, \dots,$$

which we call the *nonlinear generalized Rayleigh quotients*. Thus, we arrive to the main idea of our approach:

The set of the extreme values of the NM-method $\sigma_{\mathcal{N}}$ can be found by studying the critical values of the nonlinear generalized Rayleigh quotients $(\lambda_i(u))_{i=1}^{\infty}$.

On the whole, application of the Nehari manifold method for a fixed value λ means implementation of the following steps:

(1) verify that the Nehari manifold $\mathcal{N}_\lambda$ is not empty;
(2) prove that there exists a minimizing point $u_\lambda \in \mathcal{N}_\lambda$ of problem (2.3);
(3) prove that the minimizing point u_λ corresponds to a solution of equation (2.1).

This work is mainly focused on studying conditions under which step (3) is satisfied. However, some general results on extreme values of λ for step (1) will also be obtained. Furthermore, we believe that the introduced category of nonlinear generalized Rayleigh quotients may be useful not only for determining the extreme values of the NM-method. Some results confirming this are given in Section 5.

The literature on the Nehari manifold method is rather extensive and it would be impossible to provide a reasonably complete references on this. We have reported only the papers more closely related to the material discussed herein. A particular topic which we leave out in order to keep this work reasonably short is the critical point theory and its application in the framework of the Nehari manifold method (see, e.g. [4], [14], [31], [34], [36], [37]). We do not discuss applicability conditions for many other constrained minimization methods as for instance the fibering method [32], [33] or the one employing Pohozaev's identity as a constraint [7], [21], [25], [35].

This paper is organized as follows. Section 2 contains preliminaries on the Nehari manifold method. It should be noted that when we consider a system of equations the Nehari manifold may be introduced by several ways. We discuss two main approaches, the so-called vector and scalar Nehari manifold methods. Section 3 is devoted to the nonlinear generalized Rayleigh quotient and its main properties. In Section 4, we show how one can find the extreme values of NM-method for a system of equations with nonlinearity indefinite in sign applying the NG-Rayleigh quotients theory. In Section 5, using the NG-Raylegh quotients we prove a result on existence of multiple solutions for an abstract system of equations and as a consequence therefrom we obtain a new result on existence of multiple sign-constant solutions for a boundary value problem with a general convex-concave type nonlinearity and p-Laplacian.

Notations. We will denote by $W = W_1 \times \ldots \times W_n$ the product of real Banach spaces W_i with the norms $\|\cdot\|_{W_i}$, $i = 1, \ldots, n$, and the norm $\|\cdot\| = \|\cdot\|_{W_1} + \ldots + \|\cdot\|_{W_n}$ in W. To simplify the notations we write:

- $\dot{W} = (W_1 \setminus 0) \times \ldots \times (W_n \setminus 0)$ and $\dot{\mathbb{R}}^+ = \mathbb{R}^+ \setminus 0$,
- $\bar{t} := (t_1, \ldots, t_n) \in \mathbb{R}^n$,
- $\bar{t}\,\overline{u} := (t_1 u_1, \ldots, t_n u_n)$, $s\,\overline{u} := (su_1, \ldots, su_n)$, $\langle \bar{t}, \overline{u} \rangle = \sum_{i=1}^{n} t_i u_i$, for $\overline{u} \in W$, $\bar{t} \in \mathbb{R}^n$, $s \in \mathbb{R}$,
- $1_n = (1, \ldots, 1)^{\mathrm{T}}$ and $0_n = (0, \ldots, 0)^{\mathrm{T}}$ denote the vectors $1 \times n$ and $0 \times n$ in $\mathbb{R}^n$, respectively.

For $F \in C^1(W, \mathbb{R})$, $\overline{u}, \overline{v} \in W$, $\bar{t} \in \mathbb{R}^n$ we write

- $\nabla_{\overline{u}} F(\overline{u}) := (D_{u_1} F(\overline{u}), \ldots, D_{u_n} F(\overline{u}))^{\mathrm{T}}$,
- $\nabla_{\overline{u}} F(\overline{u})\,\overline{v} := (D_{u_1} F(\overline{u})(v_1), \ldots, D_{u_n} F(\overline{u})(v_n))^{\mathrm{T}}$,
- $D_{\overline{u}} F(\overline{u})(\overline{v}) := \sum_{i=1}^n D_{u_i} F(\overline{u})(v_i)$,
- $\nabla_{\bar{t}} F(\bar{t}\,\overline{u}) := (\partial_{t_1} F(\bar{t}\,\overline{u}), \ldots, \partial_{t_n} F(\bar{t}\,\overline{u}))^{\mathrm{T}}$,
- $\nabla_{\bar{t}} F(\bar{t}\,\overline{u})\,\bar{t} := (\partial_{t_1} F(\bar{t}\,\overline{u}) t_1, \ldots, \partial_{t_n} F(\bar{t}\,\overline{u}) t_n)^{\mathrm{T}}$,
- $\partial F(\bar{t}\,\overline{u})/\partial \bar{t} := \langle \nabla_{\bar{t}} F(\bar{t}\,\overline{u}), \bar{t} \rangle \equiv \sum_{i=1}^{n} \partial_{t_i} F(\bar{t}\,\overline{u}) t_i$.

Here $D_{u_i} F(\overline{u})$ denotes the Fréchet derivative with respect to $u_i \in W_i$ and $D_{u_i} F(\overline{u})(v_i)$ denotes the value of $D_{u_i} F(\overline{u})$ at $v_i \in W_i$, $i = 1, \ldots, n$.

In the sequel, Ω denotes a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $W := W_0^{1,p}(\Omega)$, $1 < p < +\infty$, is the standard Sobolev space with the norm $\|u\|_W = (\int |\nabla u|^p \, dx)^{1/p}$, p^* denotes the critical Sobolev exponent.

2. Preliminaries

In the present paper, we shall deal with the system of equations of the form

$$\nabla_{\overline{u}} \Phi_\lambda(\overline{u}) \equiv \nabla_{\overline{u}} T(\overline{u}) - \lambda \nabla_{\overline{u}} G(\overline{u}) = 0, \quad \overline{u} \in W, \tag{2.1}$$

where $W = \prod_{i=1}^{n} W_i$ is the product of real Banach spaces W_i, $\lambda \in \mathbb{R}$, $T, G \in C^1(\dot{W}, \mathbb{R})$ and $\Phi_\lambda(\overline{u}) = T(\overline{u}) - \lambda G(\overline{u})$. In the case $n = 1$, we call (2.1) the *scalar problem*. We define the vector Nehari manifold associated with (2.1) as follows:

$$\mathcal{N}_\lambda = \{\overline{u} \in \dot{W} : \nabla_{\overline{u}} \Phi_\lambda(\overline{u})\,\overline{u} \equiv \nabla_{\bar{t}} \Phi_\lambda(\bar{t}\,\overline{u})|_{\bar{t}=1_n} = 0\}, \tag{2.2}$$

where $\dot{W} = \prod_{i=1}^{n} (W_i \setminus 0)$. The corresponding Nehari manifold problem is

$$\begin{cases} \Phi_\lambda(\overline{u}) \to \text{crit}, \\ \overline{u} \in \mathcal{N}_\lambda. \end{cases} \tag{2.3}$$

We will say that $\overline{u} \in \mathcal{N}_\lambda$ is a critical point of Φ_λ with respect to the Nehari manifold (a solution of (2.3) for short) if $\mathcal{N}_\lambda$ has a tangent space $T_{\overline{u}}(\mathcal{N}_\lambda)$ at $\overline{u}$

and $\nabla_{\overline{u}}\Phi_\lambda(\overline{u})\,\overline{h} = 0$ for all $\overline{h} \in T_{\overline{u}}(\mathcal{N}_\lambda)$. We denote by $\widehat{\Phi}_\lambda$ the global minimization value in (2.3), i.e. $\widehat{\Phi}_\lambda := \inf\{\Phi_\lambda(\overline{u}) : \overline{u} \in \mathcal{N}_\lambda\}$. A solution $\overline{u} \in \dot{W}$ of (2.1) is said to be a ground state if $\Phi_\lambda(\overline{u}) \le \Phi_\lambda(\overline{w})$ for any solution $\overline{w} \in \dot{W}$ of (2.1). Thus a global minimizer $\overline{u}$ of (2.3) which satisfies equation (2.1) is a ground state.

The following assumption will be needed throughout the paper: $\nabla_{\overline{t}}\Phi_\lambda(\overline{t}\,\overline{u})$ is a map of class C^1 on $(\dot{\mathbb{R}}^+)^n \times \dot{W}$. Notice that this assumption implies that the constraint functional $\nabla_{\overline{u}}\Phi_\lambda(\overline{u})(\overline{u})$ in (2.3) is a map of class C^1 on $\dot{W}$. Evidently, any functional $\Phi_\lambda \in C^2(\dot{W}, \mathbb{R})$ satisfies this assumption. However, for instance, the functional $\Phi(u) = \int_\Omega |u|^p\,dx$ which is defined on $W := L^p(\Omega)$ where $\Omega \subset \mathbb{R}^n$ does not belong to $C^2(\dot{L}^p(\Omega), \mathbb{R})$ if $1 < p < 2$ but $d\Phi(tu)/dt = pt^{p-1}\int_\Omega |u|^p\,dx \in C^1(\dot{\mathbb{R}}^+ \times \dot{L}^p(\Omega), \mathbb{R})$.

Let $\overline{u} \in \dot{W}$. Consider the Hessian matrix $\mathcal{H}(\Phi_\lambda(\overline{t}\,\overline{u}))$ which is defined as follows:

$$\mathcal{H}(\Phi_\lambda(\overline{t}\,\overline{u})) = \left(\frac{\partial^2}{\partial t_i \partial t_j}\,\Phi_\lambda(\overline{t}\,\overline{u})\right)_{\{1\le i,j\le n\}}.$$

To shorten the notation, we write $\mathcal{H}(\Phi_\lambda(\overline{u})) := \mathcal{H}(\Phi_\lambda(\overline{t}\,\overline{u}))|_{\overline{t}=1_n}$. Note that in the case $\Phi_\lambda \in C^2(\dot{W}, \mathbb{R})$, one has

$$\mathcal{H}(\Phi_\lambda(\overline{u})) = \left(D^2_{u_i u_j}\Phi_\lambda(\overline{u})(u_i, u_j)\right)_{\{1\le i,j\le n\}}.$$

Let us prove

Lemma 2.1. *Let $\lambda \in \mathbb{R}$. Assume $\Phi_\lambda \in C^1(\dot{W}, \mathbb{R})$, $\nabla_{\overline{t}}\Phi_\lambda(\overline{t}\,\overline{u})$ is a map of class C^1 on $(\dot{\mathbb{R}}^+)^n \times \dot{W}$. Suppose that $\mathcal{N}_\lambda \ne \emptyset$ and for all $\overline{u} \in \mathcal{N}_\lambda$*

$$\det \mathcal{H}(\Phi_\lambda(\overline{u})) \ne 0. \tag{2.4}$$

Then $\mathcal{N}_\lambda$ is a C^1-manifold of codimension n, $W = T_{\overline{u}}(\mathcal{N}_\lambda) \oplus \mathbb{R}^n\overline{u}$ for every $\overline{u} \in \mathcal{N}_\lambda$ and any solution of (2.3) *satisfies* (2.1).

Proof. Since $\nabla_{\overline{t}}\Phi_\lambda \in C^1((\dot{\mathbb{R}}^+)^n \times \dot{W}, \mathbb{R}^n)$, the vector-valued functional $\Psi(\overline{u}) := \nabla_{\overline{u}}\Phi_\lambda(\overline{u})(\overline{u})$ is a map of class C^1 on $\dot{W}$. Consider the Jacobian

$$J_{\overline{u}}(\Psi(\overline{u})) = [D_{u_1}\Psi(\overline{u}) \ldots D_{u_n}\Psi(\overline{u})], \quad \overline{u} \in W.$$

Observe

$$J_{\overline{u}}(\Psi(\overline{u}))\,\overline{a}\,\overline{u} = J_{\overline{t}}(\Psi(\overline{tu}))\,\overline{a}|_{\overline{t}=1_n}, \quad \text{for all } \overline{a} \in \mathbb{R}^n. \tag{2.5}$$

Furthermore, for $\overline{u}_0 \in \mathcal{N}_\lambda$ there holds

$$J_{\overline{t}}(\Psi(\overline{t}\,\overline{u}_0))|_{\overline{t}=1_n} = \mathcal{H}(\Phi_\lambda(\overline{u}_0)) + \nabla_{\overline{u}_0}\Phi_\lambda(\overline{u}_0)\,\overline{u}_0 = \mathcal{H}(\Phi_\lambda(\overline{u}_0)).$$

This and (2.4) imply that the function $J_{\overline{t}}(\Psi(\overline{t}\,\overline{u}_0))|_{\overline{t}=1_n} : \mathbb{R}^n \to \mathbb{R}^n$ is bijective and therefore by (2.5), $J_{\overline{u}}(\Psi(\overline{u})) : W \to \mathbb{R}^n$ is a surjective map. Hence by the implicit function theorem $\mathcal{N}_\lambda$ is a C^1-manifold in W which codimension is equal n so that $W = T_{\overline{u}}(\mathcal{N}_\lambda) \oplus \mathbb{R}^n\overline{u}$. Clearly, this yields that any solution of (2.3) satisfies (2.1). $\square$

Let us mention that the Nehari manifold (2.2) is actually introduced by means of the vector fibering map $\Phi_\lambda(\overline{t}\,\overline{u})$, $(\overline{t},\overline{u}) \in (\mathbb{R}^+)^n \times W$. But if we use the scalar fibering map $\Phi_\lambda(s\,\overline{u})$, $(s,\overline{u}) \in \mathbb{R}^+ \times W$, we get another kind of Nehari manifold:

$$\mathcal{N}^s_\lambda = \left\{\overline{u} \in W \setminus 0_n : \frac{d}{ds}\,\Phi_\lambda(s\,\overline{u})|_{s=1} \equiv D_{\overline{u}}\Phi_\lambda(\overline{u})(\overline{u}) = 0\right\}, \tag{2.6}$$

to be further called the *scalar Nehari manifold*.

The *scalar Nehari manifold problem* is defined as follows:

$$\begin{cases} \Phi_\lambda(\overline{u}) \to \text{crit}, \\ \overline{u} \in \mathcal{N}^s_\lambda, \end{cases} \tag{2.7}$$

where the definition of a solution is likewise in (2.3). Arguing as above, we have

Lemma 2.2. *Assume $\Phi_\lambda \in C^1(W \setminus 0_n, \mathbb{R})$ and $d\Phi_\lambda(s\,\overline{u})/ds$ is a map of class C^1 on $\dot{\mathbb{R}}^+ \times (W \setminus 0_n)$. Suppose that $\mathcal{N}_\lambda \neq \emptyset$ and for all $\overline{u} \in \mathcal{N}^s_\lambda$*

$$\frac{d}{ds}\,D_{\overline{u}}\,\Phi_\lambda(s\,\overline{u})(\overline{u})|_{s=1} \neq 0. \tag{2.8}$$

Then $\mathcal{N}^s_\lambda$ is a C^1-manifold of codimension 1, $W = T_{\overline{u}}(\mathcal{N}_\lambda) \oplus \mathbb{R}\overline{u}$ for every $\overline{u} \in \mathcal{N}^s_\lambda$ and any solution of (2.7) satisfies (2.1).

In what follows, we call (2.3) and (2.7) the vector and scalar Nehari manifold method (NM-method), respectively. Note that $\mathcal{N}_\lambda \subseteq \mathcal{N}^s_\lambda$.

Definition 2.3. We say that the vector (scalar) *NM-method is applicable in general* (*applicable* for short) to problem (2.1) for a given $\lambda \in \mathbb{R}$ if condition (2.4) ((2.8)) is satisfied for each $\overline{u} \in \mathcal{N}_\lambda$ ($\overline{u} \in \mathcal{N}^s_\lambda$).

Remark 2.4. The Nehari manifold method can be applied even if condition (2.4) is satisfied only in some subset of manifold $\mathcal{N}_\lambda$. In such case it makes sense to speak about *local* applicability of the Nehari manifold method.

Let us remark that the definition of the Nehari manifold (2.2) ((2.6)) and condition (2.4) ((2.8)) are invariant in the following sense:

Proposition 2.5. *Let $\theta\colon (\dot{\mathbb{R}}^+)^n \to (\dot{\mathbb{R}}^+)^n$ be C^1-map such that*

$$\theta(1_n) = 1_n, \qquad \det J_{\overline{\tau}}(\theta(\overline{\tau}))|_{\overline{\tau}=1_n} \neq 0,$$

where $J_{\overline{\tau}}(\theta(\overline{\tau}))$ is the Jacobian matrix of $\theta(\overline{\tau})$. Then

(a) $\nabla_{\overline{\tau}}\Phi_\lambda(\theta(\overline{\tau})\,\overline{u})|_{\overline{\tau}=1_n} = 0$ *if and only if* $\nabla_{\overline{t}}\Phi_\lambda(\overline{t}\,\overline{u})|_{\overline{t}=1_n} = 0$,
(b) $\det \mathcal{H}(\Phi_\lambda(\theta(\overline{\tau})\,\overline{u}))|_{\overline{\tau}=1_n} \neq 0$ *if and only if* $\det \mathcal{H}(\Phi_\lambda(\overline{u})) \neq 0$.

Proof. Indeed, we have $\nabla_{\overline{\tau}}\Phi_\lambda(\theta(\overline{\tau})\,\overline{u})|_{\overline{\tau}=1_n} = J_{\overline{\tau}}(\theta(\overline{\tau}))|_{\overline{\tau}=1_n}\nabla_{\overline{t}}\Phi_\lambda(\overline{t}\,\overline{u})|_{\overline{t}=1_n}$ and $\det \mathcal{H}(\phi_{\lambda,\overline{u}}(\theta(\overline{\tau}))|_{\overline{\tau}=1_n} = \det J_{\overline{\tau}}(\theta(1_n))\det \mathcal{H}(\Phi_\lambda(\overline{u}))$. □

3. Nonlinear generalized Rayleigh quotient

In the sequel, we always assume:

(A$_1$) $D_{\overline{u}}G(\overline{u})(\overline{u}) \neq 0$ for any $\overline{u} \in \dot{W}$.

In the scalar case of NM-method, this condition is represented as $D_{\overline{u}}G(\overline{u})(\overline{u}) \neq 0$ for any $\overline{u} \in W \setminus 0_n$.

A central role in the present paper will play the following fibering Rayleigh quotient:

$$r_{\overline{u}}(\bar{t}) := R(\bar{t}\,\overline{u}) = \frac{D_{\overline{u}}T(\bar{t}\,\overline{u})(\bar{t}\,\overline{u})}{D_{\overline{u}}G(\bar{t}\,\overline{u})(\bar{t}\,\overline{u})}, \quad \bar{t} \in (\dot{\mathbb{R}}^+)^n,\ \overline{u} \in \dot{W},$$

where

$$R(\overline{u}) := \frac{D_{\overline{u}}T(\overline{u})(\overline{u})}{D_{\overline{u}}G(\overline{u})(\overline{u})}$$

is the original Rayleigh quotient. To shorten notation, we use the same letter to designate the scalar fibering Rayleigh quotient $r_{\overline{u}}(s) = R(s\,\overline{u})$, $s \in \dot{\mathbb{R}}^+$, $\overline{u} \in W \setminus 0_n$.

Note, since (A_1), $r_{\overline{u}}(\bar{t})$ in $(\dot{\mathbb{R}}^+)^n \times \dot{W}$ are well defined. Clearly, $T, G \in C^1(W, \mathbb{R})$ implies $R(\,\cdot\,) \in C(\dot{W}, \mathbb{R})$ and $r_{\overline{u}}(\,\cdot\,) \in C((\dot{\mathbb{R}}^+)^n, \mathbb{R})$ for every $\overline{u} \in \dot{W}$. From now on we make the assumption:

(A$_2$) $\nabla_{\bar{t}}T(\bar{t}\,\overline{u}), \nabla_{\bar{t}}G(\bar{t}\,\overline{u})$ are maps of class C^1 on $(\dot{\mathbb{R}}^+)^n \times \dot{W}$.

Observe, that (A$_1$) and (A$_2$) imply that $r_{\overline{u}}(\bar{t})$ and $\nabla_{\bar{t}}\Phi_\lambda(\bar{t}\,\overline{u})$ are maps of class C^1 on $(\dot{\mathbb{R}}^+)^n \times \dot{W}$.

We will also need the following assumption:

(A$_3$) For every fixed $\overline{u} \in \dot{W}$ and $\overline{a}_n \in (\mathbb{R}^+)^n \setminus (\dot{\mathbb{R}}^+)^n$, there exists

$$\lim_{\bar{t} \to \overline{a}_n} r_{\overline{u}}(\bar{t}) = \widehat{r}_{\overline{u}}(\overline{a}_n), \quad \text{where } |\widehat{r}_{\overline{u}}(\overline{a}_n)| \leq \infty.$$

Note that (A$_3$) entails the existence of a continuation of the fibering Rayleigh quotient $r_{\overline{u}}(\bar{t}) := r_{\overline{u}}(\bar{t})$ to $(\mathbb{R}^+)^n \times \dot{W}$ such that $r_{\overline{u}}(\overline{a}_n) := \widehat{r}_{\overline{u}}(\overline{a}_n)$ for each $\overline{u} \in \dot{W}$ and $\overline{a}_n \in (\mathbb{R}^+)^n \setminus (\dot{\mathbb{R}}^+)^n$. Notice that in the scalar case of NM-method, (A_3) is represented as follows: for every fixed $\overline{u} \in \dot{W}$, there exists $\lim_{s \to 0} r_{\overline{u}}(s) = \widehat{r}_{\overline{u}}(0)$, where $|\widehat{r}_{\overline{u}}(0)| \leq \infty$.

Let $\overline{u} \in \dot{W}$, $\bar{t}_0 \in (\dot{\mathbb{R}}^+)^n$. If $\nabla_{\bar{t}} r_{\overline{u}}(\bar{t}_0) = 0_n$, then $\bar{t}_0$ is said to be a *critical point* of $r_{\overline{u}}(\bar{t})$ and $\lambda = r_{\overline{u}}(\bar{t}_0)$ is said to be a *critical value*. We call $\bar{t}_0 \in (\mathbb{R}^+)^n$ the *extreme point* of $r_{\overline{u}}(\bar{t})$ if the function $r_{\overline{u}}(\bar{t})$ attains at $\bar{t}_0$ its local maximum or minimum on $(\mathbb{R}^+)^n$.

Let $\bar{t}\,\overline{u} \in \mathcal{N}_\lambda$, then we can compute

$$\nabla_{\bar{t}}\, r_{\overline{u}}(\bar{t}) = \frac{\mathcal{H}(\Phi_\lambda(\bar{t}\,\overline{u}))\,1_n}{D_{\overline{u}}G(\bar{t}\,\overline{u})(\bar{t}\,\overline{u})}. \tag{3.1}$$

Notice that $\lambda = r_{\overline{u}}(\overline{t})$ for $\overline{t}\,\overline{u} \in \mathcal{N}_\lambda$. Thus, if $\overline{t}$ is a critical point of $r_{\overline{u}}(\overline{t})$ and $\overline{t}\,\overline{u} \in \mathcal{N}_\lambda$, then $\det \mathcal{H}(\Phi_\lambda(\overline{t}\,\overline{u})) = 0$ with $\lambda = r_{\overline{u}}(\overline{t})$. However, the converse assertion is not always satisfied. Proceeding from (3.1), we just may conclude that to have equality $\nabla_{\overline{t}} r_{\overline{u}}(\overline{t}) = 0_n$ for $\overline{t}\,\overline{u} \in \mathcal{N}_\lambda$, the condition $1_n \in \operatorname{Ker} \mathcal{H}(\Phi_\lambda(\overline{u}))$ is required.

Our basic assumption is the following:

(R) For $\overline{u} \in \mathcal{N}_\lambda$, if $\det \mathcal{H}(\Phi_\lambda(\overline{u})) = 0$ then $1_n \in \operatorname{Ker} \mathcal{H}(\Phi_\lambda(\overline{u}))$.

Lemma 2.1 implies:

Corollary 3.1. *Assume* (A_1), (A_2) *and* (R) *are satisfied. Let* $\lambda \in \mathbb{R}$. *Suppose that* $\mathcal{N}_\lambda \neq \emptyset$ *and* $r_{\overline{u}}(\overline{t})$ *does not have critical points in* $(\dot{\mathbb{R}}^+)^n$ *such that* $\overline{t}\,\overline{u} \in \mathcal{N}_\lambda$. *Then* $\mathcal{N}_\lambda$ *is a* C^1*-manifold of codimension* n, $W = T_{\overline{u}}(\mathcal{N}_\lambda) \oplus \mathbb{R}^n \overline{u}$ *for every* $\overline{u} \in \mathcal{N}_\lambda$ *and any solution of* (2.3) *satisfies* (2.1).

Proof. Let $\lambda \in \mathbb{R}$ and $\overline{u} \in \mathcal{N}_\lambda$. To obtain a contradiction, suppose that $\det \mathcal{H}(\Phi_\lambda(\overline{u})) = 0$. Then (3.1) and (R) imply that the point $\overline{t} = 1_n$ is a critical point for $r_{\overline{u}}(\overline{t})$. But $1_n \overline{u} \in \mathcal{N}_\lambda$ and we get a contradiction. Thus $\det \mathcal{H}(\Phi_\lambda(\overline{u})) \neq 0$ and the proof follows from Lemma 2.1. $\square$

Notice that, in the case of scalar NM-method, assumption (R) is always satisfied. Indeed, in this case, $s\,\overline{u} \in N_\lambda$ if and only if $\lambda = r_{\overline{u}}(s)$. Moreover, (3.1) is written as

$$\frac{d}{ds}\, r_{\overline{u}}(s) = \frac{1}{D_{\overline{u}}\, G(s\,\overline{u})(s\,\overline{u})}\, \frac{d}{ds}\, \big(D_{\overline{u}}\, \Phi_\lambda(s\,\overline{u})(\overline{u})\big). \tag{3.2}$$

Thus, in this case, Corollary 3.1 can be written as follows.

Corollary 3.2. *Assume* (A_1), (A_2) *are satisfied. Let* $\lambda \in \mathbb{R}$. *Suppose that* $\mathcal{N}^s_\lambda \neq \emptyset$ *and the level* λ *is not critical of* $r_{\overline{u}}(\overline{t})$ *for all* $\overline{u} \in \dot{W}$. *Then* $\mathcal{N}^s_\lambda$ *is a* C^1*-manifold of codimension* 1, $W = T_{\overline{u}}(\mathcal{N}^s_\lambda) \oplus \mathbb{R}\overline{u}$ *for every* $\overline{u} \in \mathcal{N}^s_\lambda$ *and any solution of* (2.7) *satisfies* (2.1).

In the next propositions we collected some other basic properties of $r_{\overline{u}}(\overline{t})$.

Proposition 3.3. *For any* $\overline{u} \in \dot{W}$ *and* $\overline{t} \in (\dot{\mathbb{R}}^+)^n$ *there hold*:

(a) $r_{\overline{u}}(\overline{t}) = \lambda$ *if and only if* $\partial \Phi_\lambda(\overline{t}\,\overline{u})/\partial \overline{t} = 0$;

(b) *if* $\overline{t}\,\overline{u} \in \mathcal{N}_\lambda$, *then* $\lambda = r_{\overline{u}}(\overline{t})$.

Furthermore, if $\partial G(\overline{t}\,\overline{u})/\partial \overline{t} > 0$ ($\partial G(\overline{t}\,\overline{u})/\partial \overline{t} < 0$) *for* $\overline{u} \in \dot{W}$ *and* $\overline{t} \in (\dot{\mathbb{R}}^+)^n$, *then*:

(c) $r_{\overline{u}}(\overline{t}) > \lambda$ *if and only if* $\partial \Phi_\lambda(\overline{t}\,\overline{u})/\partial \overline{t} > 0$ ($\partial \Phi_\lambda(\overline{t}\,\overline{u})/\partial \overline{t} < 0$);

(d) $r_{\overline{u}}(\overline{t}) < \lambda$ *if and only if* $\partial \Phi_\lambda(\overline{t}\,\overline{u})/\partial \overline{t} < 0$ ($\partial \Phi_\lambda(\overline{t}\,\overline{u})/\partial \overline{t} > 0$).

Proof. Observe that

$$r_{\overline{u}}(\overline{t}) = \frac{\partial T(\overline{t}\,\overline{u})/\partial \overline{t}}{\partial G(\overline{t}\,\overline{u})/\partial \overline{t}}, \quad \overline{u} \in \dot{W},\ \overline{t} \in (\dot{\mathbb{R}}^+)^n.$$

Thus, to obtain the proof it is sufficient to note that $r_{\overline{u}}(\bar t) =: \lambda$ is nothing else but the root of the equation

$$\frac{\partial \Phi_\lambda(\bar t\,\overline{u})}{\partial \bar t} \equiv \frac{\partial T(\bar t\,\overline{u})}{\partial \bar t} - \lambda\,\frac{\partial G(\bar t\,\overline{u})}{\partial \bar t} = 0. \qquad \square$$

For the case of scalar fibering Rayleigh quotient, in addition, we have

Proposition 3.4. *For* $\overline{u} \in \dot W$ *and* $s > 0$ *there holds*:

(a) $s\,\overline{u} \in \mathcal{N}^s_\lambda$ *if and only if* $\lambda = r_{\overline{u}}(s)$.

Furthermore, if $D_{\overline{u}}G(s\,\overline{u})(s\,\overline{u}) > 0$ ($D_{\overline{u}}G(s\,\overline{u})(s\,\overline{u}) < 0$) *for* $\overline{u} \in \dot W$ *and* $s \in \mathbb{R}^+$, *then*:

(b) $dr_{\overline{u}}(s)/ds < 0$ *if and only if* $d^2\Phi_\lambda(s\,\overline{u})/ds^2 < 0$ ($d^2\Phi_\lambda(s\,\overline{u})/ds^2 > 0$);
(c) $dr_{\overline{u}}(s)/ds > 0$ *if and only if* $d^2\Phi_\lambda(s\,\overline{u})/ds^2 > 0$ ($d^2\Phi_\lambda(s\,\overline{u})/ds^2 < 0$).

The proof is evident.

Remark 3.5. In the case of scalar NM-method, the Nehari manifold can be defined also as

$$\mathcal{N}^s_\lambda = \{\overline{u} \in W \setminus 0 : R(\overline{u}) = \lambda\}, \quad \lambda \in \mathbb{R}. \tag{3.3}$$

Remark 3.6. In view of Proposition 2.5, all of the above statements (Propositions 3.3, 3.4 etc.) still hold after making a change of variable $\bar t = \theta(\bar\tau)$, where $\theta\colon (\dot{\mathbb{R}}^+)^n \to (\dot{\mathbb{R}}^+)^n$ is a C^1-map such that the det $J_{\bar\tau}(\theta(\bar\tau)) \neq 0$ for all $\bar\tau \in (\dot{\mathbb{R}}^+)^n$. Furthermore, (R) is satisfied if and only if the same assumption (R) holds after making a change of variable $\bar t = \theta(\bar\tau)$.

In the present paper, we consider the following *nonlinear generalized Rayleigh quotients* (the NG-Rayleigh quotients for short):

$$\lambda(\overline{u}) := \inf_{\bar t \in (\mathbb{R}^+)^n} r_{\overline{u}}(\bar t), \quad \overline{u} \in \dot W,$$
$$\Lambda(\overline{u}) := \sup_{\bar t \in (\mathbb{R}^+)^n} r_{\overline{u}}(\bar t), \quad \overline{u} \in \dot W,$$

and we restrict our main attention to the extremal values:

$$\lambda_{\min} = \inf_{\overline{u} \in \dot W} \lambda(\overline{u}), \qquad \lambda_{\max} = \sup_{\overline{u} \in \dot W} \Lambda(\overline{u}), \tag{3.4}$$

$$\lambda^*_{\min} = \sup_{\overline{u} \in \dot W} \lambda(\overline{u}), \qquad \lambda^*_{\max} = \inf_{\overline{u} \in \dot W} \Lambda(\overline{u}). \tag{3.5}$$

In the scalar case, similar objects we shall denote as $\lambda^s(\overline{u}), \Lambda^s(\overline{u}), \lambda^s_{\min}, \lambda^s_{\max}$, $\lambda^{*,s}_{\min}, \lambda^{*,s}_{\max}$. First, note that values (3.4) allow us to obtain conditions when the Nehari manifold $\mathcal{N}_\lambda$ is not empty, i.e. we have:

Lemma 3.7.

(a) *If* $\lambda_{\min} > -\infty$ ($\lambda_{\max} < +\infty$), *then* $\mathcal{N}_\lambda = \emptyset$ *for any* $\lambda < \lambda_{\min}$ ($\lambda > \lambda_{\max}$);
(b) $\mathcal{N}^s_\lambda \neq \emptyset$ *for* $\lambda \in (\lambda^s_{\min}, \lambda^s_{\max})$, *and* $\mathcal{N}^s_\lambda = \emptyset$ *for* $\lambda \in \mathbb{R} \setminus [\lambda^s_{\min}, \lambda^s_{\max}]$.

PROOF. Since $R(\overline{u}) = \lambda$ for $u \in \mathcal{N}_\lambda$, (a) is satisfied. (b) holds, because $\overline{u} \in \mathcal{N}^s_\lambda$ if and only if $R(\overline{u}) = \lambda$. $\square$

The level of complexity of the problem of finding the extreme values of NM-method depends on the number of critical values of the fibering Rayleigh quotient $r_{\overline{u}}(\overline{t})$. For instance, the simplest are the problems where $r_{\overline{u}}(\overline{t})$ has no critical values for all $u \in \dot{W}$. Indeed, it follows from Corollary 3.1 that in such case NM-method is applicable to problem (2.1) for all λ. The latter means that the set of extreme values of NM-method is empty. Let us mention that, in the scalar case, the absence of critical values of the fibering Rayleigh quotient $r_{\overline{u}}(s)$ entails that the corresponding fibering function $\Phi_\lambda(s\,\overline{u})$ has precisely one critical value (see Figure 1).

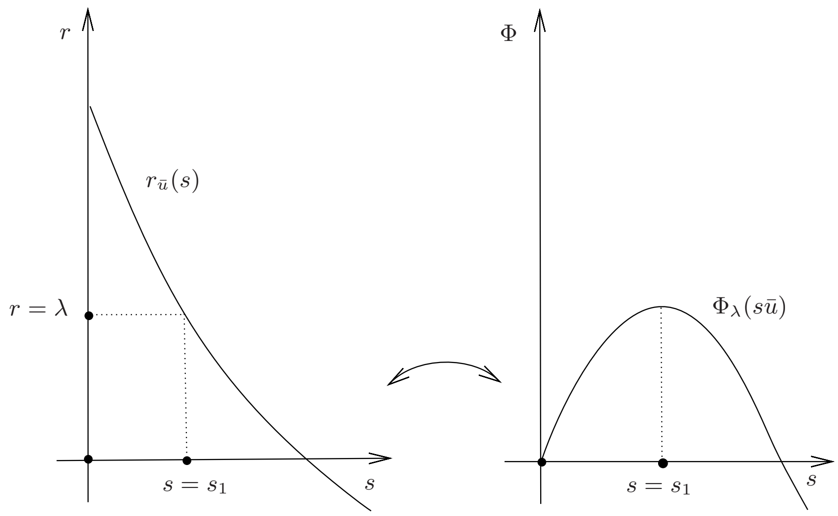

FIGURE 1. $r_{\overline{u}}(s)$ without critical values and the corresponding fibering function $\Phi_\lambda(s\,\overline{u})$.

The present paper is mainly focused on the next level of complexity of such problems when $r_{\overline{u}}(\overline{t})$ allows for existence of one critical value. We indicate this class of problems by the condition:

(S) For all $\overline{u} \in \dot{W}$, $r_{\overline{u}}(\overline{t})$ does not have critical points in $(\dot{\mathbb{R}}^+)^n$ such that $\overline{t}\,\overline{u} \in \mathcal{N}_{r_{\overline{u}}(\overline{t})}$ except points of global minimum or maximum of $r_{\overline{u}}(\overline{t})$ on $(\mathbb{R}^+)^n$.

REMARK 3.8. In the scalar case, since $s\,\overline{u} \in N_{r_{\overline{u}}(s)}$ for any $\overline{u} \in W \setminus 0_n$, $s > 0$, condition (S) can be written as:

(S^s) For all $\overline{u} \in \dot{W}$, $r_{\overline{u}}(s)$ has no critical points in $\dot{\mathbb{R}}^+$ except the points of global minimum or maximum of $r_{\overline{u}}(s)$ on $\mathbb{R}^+$.

We will see in the forthcoming examples that condition (S) could easily be verified. Typical graphs of function $r_{\overline{u}}(s)$ satisfying (S) are presented in Figures 2 and 3.

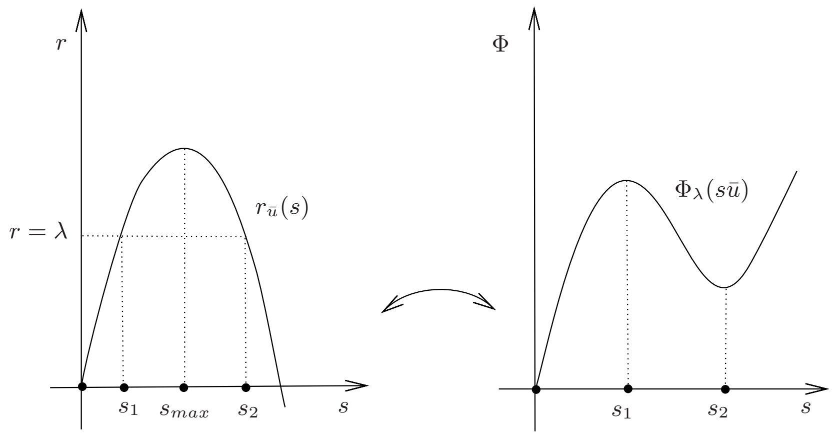

FIGURE 2. $r_{\overline{u}}(s)$ with a unique critical value in $\mathbb{R}^+$ and the corresponding fibering function $\Phi_\lambda(s\,\overline{u})$.

THEOREM 3.9. *Assume* (A_1), (A_2) *and* (A_3) *hold. Suppose* $r_{\overline{u}}(\overline{t})$ *satisfies* (R), (S) *and* $\lambda^*_{\min} < \lambda^*_{\max}$. *Then for each* $\lambda \in (\lambda^*_{\min}, \lambda^*_{\max})$ *the vector Nehari manifold method is applicable to* (2.1) *so that if* $\mathcal{N}_\lambda \neq \emptyset$, *then* $\mathcal{N}_\lambda$ *is a* C^1*-manifold of codimension* n *and any solution of* (2.3) *satisfies* (2.1).

PROOF. Let $\lambda \in (\lambda^*_{\min}, \lambda^*_{\max})$ and $\overline{u} \in \mathcal{N}_\lambda$. Suppose by contradiction that $\det \mathcal{H}(\Phi_\lambda(\overline{u})) = 0$. Then by (3.1) and (R) the point $\overline{t} = 1_n$ is a critical for $r_{\overline{u}}(\overline{t})$, and by (S) the function $r_{\overline{u}}(\overline{t})$ attains its global minimum or/and maximum at $\overline{t} = 1_n$. Assume, for instance, that this is a global minimum point. Since $\lambda > \lambda^*_{\min}$ and $\lambda = r_{\overline{u}}(1_n)$ for $\overline{u} \in \mathcal{N}_\lambda$, (3.5) implies

$$r_{\overline{u}}(1_n) = \min_{\overline{t} \in (\mathbb{R}^+)^n} r_{\overline{u}}(\overline{t}) = \lambda > \lambda^*_{\min} \geq \inf_{\overline{t} \in (\mathbb{R}^+)^n} r_{\overline{u}}(\overline{t}).$$

Thus we get a contradiction and the proof follows from Lemma 2.1. $\square$

EXAMPLE 3.10. Consider the following problem with convex-concave nonlinearity:

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2} u + |u|^{\gamma-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where $1 < q < p < \gamma \le p^*$ and by a solution of (3.6) we mean a weak solution $u \in W := W_0^{1,p}(\Omega)$. The corresponding Rayleigh quotient is given as

$$R(u) = \frac{\displaystyle\int |\nabla u|^p \,dx - \int |u|^\gamma \,dx}{\displaystyle\int |u|^q \,dx}, \quad u \in W \setminus 0.$$

For $u \in W \setminus 0$, $s > 0$, we have

$$r_u(s) = \frac{s^{p-q}\displaystyle\int |\nabla u|^p \,dx - s^{\gamma-q}\int |u|^\gamma \,dx}{\displaystyle\int |u|^q \,dx}. \tag{3.7}$$

Since this is scalar problem, assumption (R) is satisfied (see (3.2)). Compute

$$\frac{d}{ds}\, r_u(s) = \frac{(p-q)s^{p-q-1}\displaystyle\int |\nabla u|^p \,dx - (\gamma-q)s^{\gamma-q-1}\int |u|^\gamma \,dx}{\displaystyle\int |u|^q \,dx}.$$

Hence, $dr_u(s)/ds = 0$ if and only if

$$(p-q)s^{p-q-1}\int |\nabla u|^p \,dx - (\gamma-q)s^{\gamma-q-1}\int |u|^\gamma \,dx = 0.$$

The only nonzero solution of this equation is

$$s_{\max}(u) = \left(\frac{(p-q)\int |\nabla u|^p \,dx}{(\gamma-q)\displaystyle\int |u|^\gamma \,dx}\right)^{1/(\gamma-p)}.$$

From this we conclude that assumption (S) is satisfied. The substituting $s_{\max}(u)$ into $r_u(s)$ yields the following NG-Rayleigh quotient:

$$\Lambda(u) = \sup_{s>0} r_u(s) = c_{p,q}\,\frac{\left(\displaystyle\int |\nabla u|^p \,dx\right)^{(\gamma-q)/(\gamma-p)}}{\displaystyle\int |u|^q dx \left(\int |u|^\gamma \,dx\right)^{(p-q)/(\gamma-p)}},$$

where

$$c_{p,q} = \frac{\gamma-p}{p-q}\left(\frac{p-q}{\gamma-q}\right)^{(\gamma-q)/(\gamma-p)}.$$

Thus

$$\lambda^*_{\max} = c_{p,q} \inf\left\{\frac{\left(\displaystyle\int |\nabla u|^p \,dx\right)^{(\gamma-q)/(\gamma-p)}}{\displaystyle\int |u|^q dx \left(\int |u|^\gamma \,dx\right)^{(p-q)/(\gamma-p)}} : u \in W \setminus 0\right\}. \tag{3.8}$$

Using Sobolev's and Holder's inequalities it is not hard to show that $\lambda^*_{\max} > 0$. From (3.7) it is easily to see that $\lambda(u) = \inf_{s>0} r_u(s) = -\infty$. Thus, $\lambda^*_{\min} = \sup_{u\in W\setminus 0} \lambda(u) = -\infty$ and Theorem 3.9 is applicable with $-\infty = \lambda^*_{\min} < \lambda^*_{\max}$.

In the case of enhancing condition (S) by introducing additional restrictions, one should expect to receive more precise estimations of the extreme values of NM-method. Let us consider the following special case of (S):

(S0) For any $\overline{u} \in \dot{W}$ one of the following holds:
 (a) $r_{\overline{u}}(\overline{t})$ has no critical point $\overline{t} \in (\dot{\mathbb{R}}^+)^n$ such that $\overline{t}\,\overline{u} \in N_{r_{\overline{u}}(\overline{t})}$;
 (b) $\nabla_{\overline{t}} r_{\overline{u}}(\overline{t}) \equiv 0_n$ for all $\overline{t} \in (\dot{\mathbb{R}}^+)^n$.

Typical graphs of the function $r_{\overline{u}}(s)$ satisfying (S) are given in Figure 3.

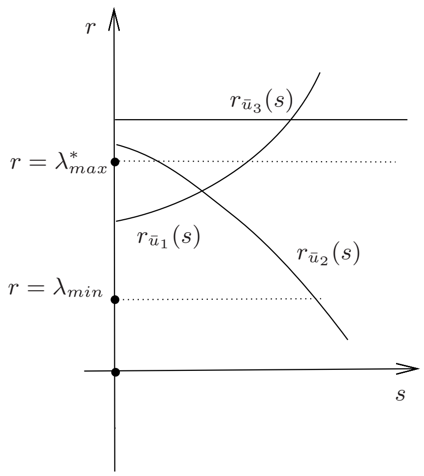

Figure 3. Examples of $r_{\overline{u}}(s)$ which satisfy (S0).

Theorem 3.11. *Assume* (A_1), (A_2) *and* (A_3) *hold. Suppose* $r_{\overline{u}}(\overline{t})$ *satisfies* (R), (S0) *and* $\lambda_{\min} < \lambda^*_{\max}$ ($\lambda^*_{\min} < \lambda_{\max}$). *Then for each* $\lambda \in (\lambda_{\min}, \lambda^*_{\max})$ ($\lambda \in (\lambda^*_{\min}, \lambda_{\max})$) *the vector Nehari manifold method is applicable to* (2.1) *so that if* $\mathcal{N}_\lambda \neq \emptyset$, *then* $\mathcal{N}_\lambda$ *is a* C^1*-manifold of codimension* n *and any solution of* (2.3) *satisfies* (2.1).

Proof. We prove the statement for the case $\lambda_{\min} < \lambda^*_{\max}$. The proof in the case $\lambda^*_{\min} < \lambda_{\max}$ is similar. Suppose by contradiction that $\det \mathcal{H}(\Phi_\lambda(\overline{u})) = 0$. Then (3.1) and (R) yield that $\overline{t} = 1_n$ is a critical point of the function $r_{\overline{u}}(\overline{t})$ and consequently $\nabla_{\overline{t}}\, r_{\overline{u}}(\overline{t})|_{\overline{t}=1_n} = 0_n$. Hence, (S0) entails that the function $r_{\overline{u}}(\overline{t})$ identically equals to the constant λ in $(\mathbb{R}^+)^n$ and attains its global minimum and maximum at any point $t \in (\mathbb{R}^+)^n$. However, the assumption $\lambda < \lambda^*_{\max}$ yields that

$$\lambda < \lambda^*_{\max} \le \sup_{\bar{t}\in(\mathbb{R}^+)^n} r_{\overline{u}}(\bar{t}) = \max_{\bar{t}\in(\mathbb{R}^+)^n} r_{\overline{u}}(\bar{t}) \equiv R(\overline{u}) = \lambda.$$

Thus we get a contradiction and the proof follows from Lemma 2.1. $\square$

Remark 3.12. Clearly, $\lambda_{\min} \le \lambda^*_{\min}$ and $\lambda^*_{\max} \le \lambda_{\max}$. Thus, if assumptions (S0) are satisfied and $\lambda^*_{\min} < \lambda^*_{\max}$, then Theorem 3.11 provides a stronger result than Theorem 3.9.

Example 3.13. Consider the boundary value problem with nonlinearity indefinite in sign:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + f|u|^{\gamma-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

where $\lambda \in \mathbb{R}$, $1 < p < \gamma \le p^*$, $f \in L^\infty(\Omega)$ and by a solution of (3.9) we shall mean a weak solution $u \in W := W^{1,p}_0(\Omega)$. In the case when f may change the sign in Ω, the nonlinearity in right-hand side of (3.9) is called indefinite in sign (cf. [1], [6]). Consider the corresponding Rayleigh quotient

$$R(u) = \frac{\displaystyle\int |\nabla u|^p\,dx - \int f|u|^\gamma\,dx}{\displaystyle\int |u|^p\,dx}, \quad u \in W \setminus 0.$$

For $u \in W \setminus 0$, $s > 0$, we have

$$r_u(s) = \frac{\displaystyle\int |\nabla u|^p\,dx}{\displaystyle\int |u|^p\,dx} - s^{\gamma-p}\frac{\displaystyle\int f|u|^\gamma\,dx}{\displaystyle\int |u|^p\,dx}. \tag{3.10}$$

Evidently, assumption (R) is satisfied (see (3.2)). Furthermore, (3.10) implies that $r_u(s)$ has only extreme point at $s = 0$ or

$$\text{if} \quad \int f|u|^\gamma\,dx = 0, \quad \text{then} \quad r_u(s) \equiv \frac{\displaystyle\int |\nabla u|^p\,dx}{\displaystyle\int |u|^p\,dx}$$

for all $s \ge 0$. Thus, condition (S0) is satisfied and one may apply Theorem 3.11. From (3.10) we have

$$\Lambda(u) = \sup_{s>0} r_u(s) = \begin{cases} \displaystyle\int |\nabla u|^p\,dx \Big/ \int |u|^p\,dx & \text{if } \displaystyle\int f|u|^\gamma\,dx \ge 0, \\ +\infty & \text{if } \displaystyle\int f|u|^\gamma\,dx < 0, \end{cases}$$

and

$$\lambda(u)=\inf_{s>0} r_u(s)=\begin{cases}-\infty & \text{if } \int f|u|^\gamma\,dx>0,\\ \int|\nabla u|^p\,dx\Big/\int|u|^p\,dx & \text{if } \int f|u|^\gamma\,dx\le 0,\end{cases}$$

and thus

$$\lambda^*_{\min}=\sup\left\{\frac{\int|\nabla u|^p\,dx}{\int|u|^p\,dx}:\int f|u|^\gamma\,dx\le 0,\ u\in W\setminus 0\right\}, \tag{3.11}$$

$$\lambda^*_{\max}=\inf\left\{\frac{\int|\nabla u|^p\,dx}{\int|u|^p\,dx}:\int f|u|^\gamma\,dx\ge 0,\ u\in W\setminus 0\right\}. \tag{3.12}$$

Observe, $\lambda_{\min}=-\infty$, $\lambda^*_{\min}=+\infty$ if the set $\{x\in\Omega: f(x)\le 0\}$ contains an open domain up to a subset of Lebesgue measure zero. Thus, in this case, Theorem 3.11 is applicable with $-\infty=\lambda_{\min}$ and $\lambda^*_{\max}$ given by (3.12). On the other hand, if $\int_\Omega f(x)|u|^\gamma\,dx>0$ for all $u\in W\setminus 0$, then $\lambda^*_{\min}=-\infty$ and Theorem 3.11 is applicable with $-\infty=\lambda^*_{\min}<\lambda_{\max}=+\infty$.

As far as we are aware, the extreme value (3.12) was first discovered by Ouyang [30], who apparently used a direct reasoning method.

Remark 3.14. In the present paper, we do not deal with the applicability of Nehari manifold method at its extreme values like $\lambda^*_{\max},\lambda_{\max}$ or $\lambda_{\min}$. This is a subject of another research.

Let us stress that

$$\Lambda^s(\overline{u}):=\sup_{s\in\mathbb{R}^+} r_{\overline{u}}(s)\le\sup_{\overline{t}\in(\mathbb{R}^+)^n} r_{\overline{u}}(\overline{t})=:\Lambda(\overline{u}),$$

and therefore,

$$\lambda^{s,*}_{\max}:=\inf_{\overline{u}\in\dot{W}}\Lambda^s(\overline{u})\le\inf_{\overline{u}\in\dot{W}}\Lambda(\overline{u})=:\lambda^*_{\max}. \tag{3.13}$$

Similarly,

$$\lambda^s_{\max}\le\lambda_{\max},\qquad\lambda_{\min}\le\lambda^s_{\min},\qquad\lambda^*_{\min}\le\lambda^{s,*}_{\min}. \tag{3.14}$$

Thus, $(\lambda^{s,*}_{\min},\lambda^{s,*}_{\max})\subseteq(\lambda^*_{\min},\lambda^*_{\max})$, $(\lambda^s_{\min},\lambda^{s,*}_{\max})\subseteq(\lambda_{\min},\lambda^*_{\max})$ and $(\lambda^{s,*}_{\min},\lambda^s_{\max})\subseteq(\lambda^*_{\min},\lambda_{\max})$, that is, the vector NM-method is preferable.

Remark 3.15. The assumptions (S), (S0) and definition of extreme values $\lambda_{\min},\lambda_{\max},\lambda^*_{\min},\lambda^*_{\max}$ etc. obviously do not depend on changing variables as in Remark 3.6.

4. Extreme values of the NM-method for system of equations with nonlinearity indefinite in sign

In this section, we apply the above theory for a system of equations with nonlinearity indefinite in sign. In the sequel, $\lambda_1 := \lambda_{1,p}$, $\phi_1 := \phi_{1,p}$ denote the first eigenpair of the operator $-\Delta_p$ in Ω, $1 < p < +\infty$ with the zero boundary conditions. Eigenvalue λ_1 is known to be positive, simple and isolated, the corresponding eigenfunction ϕ_1 to be positive and and it can be normalized so that $\|\phi_1\|_W = 1$, see [5], [27].

Consider system of equations with indefinite nonlinearity

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u + \alpha f|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ -\Delta_q v = \lambda|v|^{q-2}v + \beta f|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad v|_{\partial\Omega} = 0, \end{cases} \tag{4.1}$$

where $\lambda \in \mathbb{R}$, $1 < p < +\infty$, $1 < q < +\infty$ and

$$\alpha, \beta > 0, \qquad \frac{\alpha}{p} + \frac{\beta}{q} > 1, \qquad \frac{\alpha}{p^*} + \frac{\beta}{q^*} \le 1. \tag{4.2}$$

We suppose

(f1) $f \in L^d(\Omega)$, where $d \ge p^*q^*/(p^*q^* - \alpha q^* - \beta p^*)$ if $p < N$ or/and $q < N$, $\alpha/p^* + \beta/q^* < 1$; $d = +\infty$ if $p < N$, $q < N$ and $\alpha/p^* + \beta/q^* = 1$; $d > 1$ if $p \ge N$, $q \ge N$.

Furthermore, the function f may change the sign in Ω, i.e. problem (4.1) has the nonlinearity indefinite in sign. By a solution of (4.1) we shall mean a weak solution $(u, v) \in W := W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

Let us study (4.1) using the vector NM-method. Consider the corresponding Nehari manifold problem:

$$\begin{cases} \Phi_\lambda(u, v) \to \text{crit}, \\ (u, v) \in \mathcal{N}_\lambda, \end{cases} \tag{4.3}$$

where

$$\Phi_\lambda(u, v) = \frac{1}{p}\int (|\nabla u|^p - \lambda|u|^p)\, dx + \frac{1}{q}\int (|\nabla v|^q - \lambda|v|^q)\, dx - F(u, v),$$

and

$$\mathcal{N}_\lambda := \Big\{(u, v) \in \dot{W} : \int (|\nabla u|^p - \lambda|u|^p)\, dx - \alpha F(u, v) = 0, \\ \int (|\nabla v|^q - \lambda|v|^q) dx - \beta F(u, v) = 0\Big\}. \tag{4.4}$$

Here $F(u, v) = \int f|u|^{\alpha}|v|^{\beta}\, dx$ and $\dot{W} := (W_0^{1,p}(\Omega) \setminus 0) \times (W_0^{1,q}(\Omega) \setminus 0)$.

The corresponding vector fibering Rayleigh quotient is given as follows:

$$r_{(u,v)}(t,s) := R(tu,sv) = \frac{t^p \displaystyle\int |\nabla u|^p\,dx + s^q \int |\nabla v|^q\,dx - t^\alpha s^\beta (\alpha+\beta) F(u,v)}{t^p \displaystyle\int |u|^p\,dx + s^q \int |v|^q\,dx}$$

for $t,s \in \dot{\mathbb{R}}^+$ and $(u,v) \in \dot{W}$. Evidently conditions (A_1)–(A_3) are satisfied.

Consider the Hessian matrix

$$\mathcal{H}(\Phi_\lambda)(u,v) = \begin{pmatrix} (p-1)P_\lambda(u) - \alpha(\alpha-1)F(u,v) & -\alpha\beta F(u,v) \\ -\alpha\beta F(u,v) & (q-1)Q_\lambda(v) - \beta(\beta-1)F(u,v) \end{pmatrix}.$$

Here we denote

$$P_\lambda(u) := \int |\nabla u|^p dx - \lambda \int |u|^p\,dx, \qquad Q_\lambda(v) := \int |\nabla v|^q dx - \lambda \int |v|^q\,dx.$$

Then, for $(u,v) \in \mathcal{N}_\lambda$ we have

$$\mathcal{H}(\Phi_\lambda)(u,v) = \begin{pmatrix} \alpha(p-\alpha)F(u,v) & -\alpha\beta F(u,v) \\ -\alpha\beta F(u,v) & \beta(q-\beta)F(u,v) \end{pmatrix}.$$

Proposition 4.1. *$r_{(u,v)}(t,s)$ satisfies* (R) *and* (S0).

Proof. Observe that $\det \mathcal{H}(\Phi_\lambda)(u,v) = \alpha\beta(pq - p\beta - q\alpha)F^2(u,v)$ for $(u,v) \in \mathcal{N}_\lambda$. By (4.2), $pq - p\beta - q\alpha \neq 0$. Hence $\det \mathcal{H}(\Phi_\lambda)(u,v) = 0$ for $(u,v) \in \mathcal{N}_\lambda$ if and only if $F(u,v) = 0$. However, $\mathcal{H}(\Phi_\lambda)(u,v)1_2 = 0_2$ if $F(u,v) = 0$. Thus, condition (R) holds.

Observe, for $(u,v) \in \dot{W}$, $t > 0$, $s > 0$ we have

$$\frac{\partial}{\partial t} r_{(u,v)}(t,s) = \frac{1}{t\Big(t^p \displaystyle\int |u|^p\,dx + s^q \int |v|^q\,dx\Big)}(pP_\lambda(tu) - \alpha(\alpha+\beta)F(tu,sv)),$$

$$\frac{\partial}{\partial s} r_{(u,v)}(t,s) = \frac{1}{s\Big(t^p \displaystyle\int |u|^p\,dx + s^q \int |v|^q\,dx\Big)}(qQ_\lambda(sv) - \beta(\alpha+\beta)F(tu,sv)).$$

Thus, if $(t_0u, s_0v) \in \mathcal{N}_{r_{(u,v)}(t_0,s_0)}$ and $\partial r_{(u,v)}(t_0,s_0)/\partial t = 0$, $\partial r_{(u,v)}(t_0,s_0)/\partial s = 0$ for some $t_0 > 0$, $s_0 > 0$, then $P_\lambda(t_0u) = 0$, $Q_\lambda(s_0v) = 0$ and $F(t_0u, s_0v) = 0$. Hence we have successively $P_\lambda(u) = 0$, $Q_\lambda(v) = 0$, $F(u,v) = 0$ and $\partial r_{(u,v)}(t,s)/\partial t \equiv 0$, $\partial r_{(u,v)}(t,s)/\partial s \equiv 0$ for all $t > 0$, $s > 0$. Thus, condition (S0) holds. $\square$

Let us prove

Lemma 4.2. *The extreme value* $\lambda^*_{\max}$ *of Nehari manifold* (4.4) *is expressed by the following explicit variational form:*

$$\lambda^*_{\max} = \inf_{(u,v)\in \dot{W}} \left\{ \max\left\{ \frac{\displaystyle\int |\nabla u|^p\,dx}{\displaystyle\int |u|^p\,dx}, \frac{\displaystyle\int |\nabla v|^q\,dx}{\displaystyle\int |v|^q\,dx} \right\} : F(u,v) \geq 0 \right\}. \tag{4.5}$$

Proof. We claim that

$$\Lambda(u,v) = \sup_{t,s\in\dot{\mathbb{R}}^+} r_{(u,v)}(t,s) = \begin{cases} \max\left\{ \dfrac{\int |\nabla u|^p dx}{\int |u|^p dx}, \dfrac{\int |\nabla v|^q dx}{\int |v|^q dx} \right\} & \text{if } F(u,v) \geq 0, \\ +\infty & \text{if } F(u,v) < 0, \end{cases}$$

and

$$\lambda(u,v) = \inf_{t,s\in\dot{\mathbb{R}}^+} r_{(u,v)}(t,s) = \begin{cases} -\infty & \text{if } F(u,v) > 0, \\ \min\left\{ \dfrac{\int |\nabla u|^p dx}{\int |u|^p\,dx}, \dfrac{\int |\nabla v|^q\,dx}{\int |v|^q\,dx} \right\} & \text{if } F(u,v) \leq 0. \end{cases}$$

Let us show, as an example, the first equality. Assume $F(u,v) < 0$. Then setting $t = \sigma^q$, $s = \sigma^p$ we obtain

$$\frac{\displaystyle\int |\nabla u|^p\,dx + \int |\nabla v|^q\,dx - \sigma^{pq(\alpha/p+\beta/q-1)} F(u,v)}{\displaystyle\int |u|^p\,dx + \int |v|^q\,dx} \to +\infty$$

as $\sigma \to +\infty$, since $\alpha/p + \beta/q > 1$. Consider now the case $F(u,v) \geq 0$. Without loss of generality, we can suppose that

$$\frac{\displaystyle\int |\nabla u|^p\,dx}{\displaystyle\int |u|^p\,dx} \geq \frac{\displaystyle\int |\nabla v|^q\,dx}{\displaystyle\int |v|^q\,dx}.$$

This implies that

$$\frac{\displaystyle\int |\nabla u|^p\,dx + \tau \int |\nabla v|^q\,dx}{\displaystyle\int |u|^p\,dx + \tau \int |v|^q\,dx} \leq \frac{\displaystyle\int |\nabla u|^p\,dx}{\displaystyle\int |u|^p\,dx}$$

for any $\tau \geq 0$. Since $F(u,v) \geq 0$,

$$r_{(u,v)}(t,s) = \frac{\displaystyle\int |\nabla u|^p\,dx + s^q t^{-p} \int |\nabla v|^q\,dx - t^{\alpha-p} s^{\beta} (\alpha+\beta) F(u,v)}{\displaystyle\int |u|^p\,dx + s^q t^{-p} \int |v|^q\,dx}$$

$$\leq \frac{\displaystyle\int |\nabla u|^p\,dx}{\displaystyle\int |u|^p\,dx}.$$

for any $s \geq 0$ and $t > 0$. Taking into account that this inequality becomes equality if $s = 0$, we get the proof of the assertion and the lemma. $\square$

Observe, that $\lambda^*_{\min} = \sup_{(u,v)\in \dot{W}} \lambda(u,v) = +\infty$ if the set $\{x \in \Omega : f(x) \leq 0\}$ contains an open domain up to a subset of Lebesgue measure zero. Consider $\lambda_{\min} = \inf_{(u,v)\in \dot{W}} \lambda(u,v)$. Simple analysis shows that $\mathcal{N}_\lambda \neq \emptyset$ as $\lambda \in (\lambda_{\min}, \lambda^*_{\max})$ (see e.g. [9]). Let us prove

Lemma 4.3. *Assume* (4.2), (f1) *are satisfied. Then* $\lambda_{\min} < \lambda^*_{\max}$ *and for* $\lambda \in (\lambda_{\min}, \lambda^*_{\max})$*, the vector NM-method* (4.3) *is applicable to* (4.1) *so that* (4.4) *is a* C^1*-manifold of codimension* 2 *and any solution of* (4.3) *satisfies* (4.1).

Proof. Consider $\lambda^l_1 := \min\{\lambda_{1,p}, \lambda_{1,q}\}$, $\lambda^u_1 := \max\{\lambda_{1,p}, \lambda_{1,q}\}$. Clearly,

$$\lambda^l_1 = \inf_{(u,v)\in \dot{W}} \min \left\{ \frac{\displaystyle\int |\nabla u|^p \, dx}{\displaystyle\int |u|^p \, dx}, \frac{\displaystyle\int |\nabla v|^q \, dx}{\displaystyle\int |v|^q \, dx} \right\},$$

$$\lambda^u_1 = \inf_{(u,v)\in \dot{W}} \max \left\{ \frac{\displaystyle\int |\nabla u|^p \, dx}{\displaystyle\int |u|^p \, dx}, \frac{\displaystyle\int |\nabla v|^q \, dx}{\displaystyle\int |v|^q \, dx} \right\}.$$

Hence $\lambda^*_{\max} \geq \lambda^u_1 \geq \lambda^l_1$. Observe that

$$\lambda_{\min} = \begin{cases} -\infty & \text{if there exists } (u,v) \in \dot{W} \text{ such that } F(u,v) > 0, \\ \lambda^l_1 & \text{if for all } (u,v) \in \dot{W}, F(u,v) \leq 0. \end{cases}$$

Now taking into account that $\lambda^*_{\max} = +\infty$ if $F(u,v) \leq 0$, for all $(u,v) \in \dot{W}$, we get $\lambda_{\min} < \lambda^*_{\max}$. By Proposition 4.1, conditions (R), (S) are satisfied. Thus, the proof of the lemma follows from Theorem 3.11. $\square$

Note that if $f > 0$ almost everywhere in Ω, then $-\infty = \lambda^*_{\min} < \lambda_{\max} = +\infty$. Thus, in this case, we can apply Theorem 3.11 with the extreme values $\lambda^*_{\min}, \lambda_{\max}$ that is (4.3) is applicable to (4.1) for any $\lambda \in \mathbb{R}$.

The existence of the solution of (4.3) for $\lambda \in (\lambda_{\min}, \lambda^l_1) \cup (\lambda^u_1, \lambda^*_{\max})$ follows from [9], [10]. Herein, we did not study (4.1) using the scalar NM-method. However, by (3.13), (3.14) we have $(\lambda^s_{\min}, \lambda^{s,*}_{\max}) \subseteq (\lambda_{\min}, \lambda^*_{\max})$. Thus, the scalar NM-method is not expected to provide better results than the vector NM-method (4.3).

The extreme values like (4.5) are not believed to be obtained directly as for Ouyang's extreme value (3.12) or applying the spectral analysis by the fibering procedure [17], [18], [20] as for (3.8). We would like to draw the reader's attention to the fact that the extreme value (4.5) has been presented in our paper already [9]. However, in that case it was found by applying the same approach as in the proof of Lemma 4.2.

5. Multiplicity result

Nehari manifold method is often used to prove the existence of multiple solutions, see e.g. [15], [18], [30]. In this section, we show how to obtain such type of results in terms of categories of NG-Rayleigh quotient. First we prove a result on existence of multiple solutions for an abstract system of equations and as a consequence therefrom we obtain a new result on existence of multiple sign-constant solutions for a boundary value problem with a general convex-concave type nonlinearity and p-Laplacian.

We will study (2.1) using the scalar NM-method. In what follows, we always assume that $D_{\overline{u}} G(\overline{u})(\overline{u}) > 0$ for all $\overline{u} \in W \setminus 0_n$ so that (A_1) is satisfied. Denote $S = \{\overline{v} \in W : \|\overline{v}\|_W = 1\}$. We will suppose $r_{\overline{u}}(s)$ satisfies the following conditions:

(1) *For all $\overline{u} \in W \setminus 0_n$, $r_{\overline{u}}(s)$ has a unique critical point $s_{\max}(\overline{u}) \in \dot{\mathbb{R}}^+$ which is a global maximum point.*
(2) *There exists $\delta_0 > 0$ such that $s_{\max}(\overline{v}) > \delta_0$ for any $\overline{v} \in S$.*
(3) *If $(\overline{v}_m) \subset S$ is weakly separated from $0_n \in W$, then the set of functions $(r_{\overline{v}_m}(t))_{m=1}^{\infty}$ is bounded in $C^1[\sigma, T]$ for any $\sigma, T \in (0, \infty)$.*
(4) *For any $\lambda \in \mathbb{R}$, $0 < \sigma < T < +\infty$, the set $\mathcal{N}^s_\lambda \cap \{\sigma < \|u\|_W < T\}$ is weakly separated from $0_n \in W$.*

Typical graph of the function $r_{\overline{u}}(s)$ satisfying (1) is presented in Figure 2. The condition of type (2) is common in study of variational problems, where fibering functions are used (see eg. [11], [34], [36]). Roughly speaking, this condition, as well as conditions (3) and (4), ensures that the solutions of problem (2.3) are separated from zero and from each others. In the example below, we will see that conditions (3) and (4) will be verified by standard methods as a consequence of Sobolev's and Holder's inequalities and the Rellich–Kondrachov theorem.

Evidently, (1) yields (R), (S) (see (3.2) and Remark 3.8) and that it satisfies $dr_{\overline{u}}(s_{\max}(\overline{u}))/ds = 0$,

$$\frac{d}{ds} r_{\overline{u}}(s) > 0 \;\Leftrightarrow\; 0 < s < s_{\max}(\overline{u}) \quad \text{and} \quad \frac{d}{ds} r_{\overline{u}}(s) < 0 \;\Leftrightarrow\; s > s_{\max}(\overline{u}). \tag{5.1}$$

Furthermore, it follows that for all $\overline{u} \in W \setminus 0_n$ there exist limits: $r_{\overline{u}}(s) \to \widehat{r}_{\overline{u}}(0)$ as $s \to 0$ and $r_{\overline{u}}(s) \to \widehat{r}_{\overline{u}}(\infty)$ as $s \to +\infty$, where $-\infty \le \widehat{r}_{\overline{u}}(0), \widehat{r}_{\overline{u}}(\infty) < +\infty$. Introduce

$$\lambda^\partial_{\min} = \sup_{\overline{u} \in W \setminus 0_n} \max\{\widehat{r}_{\overline{u}}(0), \widehat{r}_{\overline{u}}(\infty)\}.$$

Consider $\lambda^*_{\max} = \inf_{\overline{u}\in W\setminus 0_n} \sup_{s>0} r_{\overline{u}}(s)$. Observe, that (1) entails

$$\lambda^*_{\min} = \sup_{\overline{u}\in W\setminus 0_n} \inf_{s>0} r_{\overline{u}}(s) \le \lambda^{\partial}_{\min}. \tag{5.2}$$

Let us introduce the following sets:

$$\mathcal{N}^{s,1}_\lambda := \mathcal{N}^s_\lambda \cap \left\{ \overline{u} \in W\setminus 0_n : \frac{d}{ds} r_{\overline{u}}(s)\bigg|_{s=1} < 0 \right\}, \tag{5.3}$$

$$\mathcal{N}^{s,2}_\lambda := \mathcal{N}^s_\lambda \cap \left\{ \overline{u} \in W\setminus 0_n : \frac{d}{ds} r_{\overline{u}}(s)\bigg|_{s=1} > 0 \right\}. \tag{5.4}$$

Obviously, $\mathcal{N}^{s,1}_\lambda \cap \mathcal{N}^{s,2}_\lambda = \emptyset$ and $\mathcal{N}^{s,1}_\lambda \cup \mathcal{N}^{s,2}_\lambda = \mathcal{N}^s_\lambda$ if $\lambda < \lambda^*_{\max}$. In view of Lemma 3.7, $\mathcal{N}^{s,1}_\lambda \ne \emptyset$, $\mathcal{N}^{s,2}_\lambda \ne \emptyset$ if $\lambda \in (\lambda^{\partial}_{\min}, \lambda^*_{\max})$. Thus, for $\lambda \in (\lambda^{\partial}_{\min}, \lambda^*_{\max})$, one may split minimization problem (2.3) into

$$\widehat{\Phi}^1_\lambda := \min\{\Phi_\lambda(\overline{u}) : \overline{u} \in \mathcal{N}^{s,1}_\lambda\}, \tag{5.5}$$

$$\widehat{\Phi}^2_\lambda := \min\{\Phi_\lambda(\overline{u}) : \overline{u} \in \mathcal{N}^{s,2}_\lambda\}. \tag{5.6}$$

Theorem 5.1. *Suppose W is a reflexive Banach space, $\Phi_\lambda \in C^1(W\setminus 0_n, \mathbb{R})$, $dT(s\,\overline{u})/ds$, $dG(s\overline{u})/ds$ are maps of class C^1 on $\dot{\mathbb{R}}^+ \times (W\setminus 0_n)$, $D_{\overline{u}}G(\overline{u})(\overline{u}) > 0$, for all $\overline{u} \in W\setminus 0_n$,* (1)–(4) *hold and the following conditions are fulfilled*:

- (a) *for all $\lambda\in\mathbb{R}$, $\Phi_\lambda(\overline{u}) \to +\infty$ as $\|\overline{u}\|_W \to \infty$, $\overline{u}\in\mathcal{N}^s_\lambda$,*
- (b) *$\Phi_\lambda(\overline{u})$, for all $\lambda\in\mathbb{R}$ and $R(\overline{u})$ are sequentially weakly lower semi-continuous functionals on W.*

Assume $\lambda^{\partial}_{\min} < \lambda^*_{\max}$. Then for every $\lambda \in (\lambda^{\partial}_{\min}, \lambda^*_{\max})$ system of equations (2.1) *has two distinct solutions $\overline{u}^1_\lambda, \overline{u}^2_\lambda \in W\setminus 0_n$ such that*

$$d^2\Phi_\lambda(s\overline{u}^1_\lambda)/ds^2|_{s=1} < 0, \quad d^2\Phi_\lambda(s\overline{u}^2_\lambda)/ds^2|_{s=1} > 0, \quad \Phi_\lambda(\overline{u}^2_\lambda) \equiv \widehat{\Phi}^2_\lambda < \Phi_\lambda(0).$$

Furthermore, for $\lambda \in (\lambda^{\partial}_{\min}, \lambda^*_{\max})$, $\overline{u}^2_\lambda$ is a ground state of (2.1) *and $\mathcal{N}^{s,1}_\lambda$, $\mathcal{N}^{s,2}_\lambda$ are C^1-manifolds of codimension* 1.

Proof. Since $\mathcal{N}^{s,1}_\lambda \cap \mathcal{N}^{s,2}_\lambda = \emptyset$ and $\mathcal{N}^{s,1}_\lambda \cup \mathcal{N}^{s,2}_\lambda = \mathcal{N}^s_\lambda$ for $\lambda \in (\lambda^{\partial}_{\min}, \lambda^*_{\max})$, any solution of (5.5) or (5.6) is a critical point (local minimizer) of Φ_λ in $\mathcal{N}^s_\lambda$. Thus, in view of Theorem 3.9, to prove the existence of two distinct solutions of (2.1), it is sufficient to show that (5.5) and (5.6) for $\lambda \in (\lambda^{\partial}_{\min}, \lambda^*_{\max})$ possess minimizers $\overline{u}^1_\lambda \in \mathcal{N}^{s,1}_\lambda$, and $\overline{u}^2_\lambda \in \mathcal{N}^{s,2}_\lambda$, respectively.

Let $\lambda \in (\lambda^{\partial}_{\min}, \lambda^*_{\max})$ and $(\overline{u}^i_m)$, $i = 1,2$, be minimizing sequences of (5.5) and (5.6), respectively.

Proposition 5.2. *For $i = 1,2$, the minimizing sequence $(\overline{u}^i_m)$ has a non-zero limit point $\overline{u}^i_0 \in W$.*

Proof. Let $i = 1,2$. Observe, (a) implies that $(\overline{u}^i_m)$ is bounded in W. Write $\overline{u}^i_m = s_m\overline{v}_m$, where $s^i_m = \|\overline{u}^i_m\|_W$, $\overline{v}^i_m \in S$, $m = 1,2,\ldots$ Then (s^i_m) is bounded above and we may assume that $s^i_m \to s^i_0$, $\overline{v}^i_m \rightharpoonup \overline{v}^i_0$ weakly in W as $m \to \infty$ for some $s^i_0 \geq 0$ and $\overline{v}^i_0 \in W$.

Let us show that $\overline{u}^i_0 := s^i_0 \overline{v}^i_0 \neq 0_n$, $i = 1,2$. Consider first minimizing problem (5.5). In view of (5.1), we have $s^1_m > s_{\max}(\overline{v}_m)$. Then (2) entails $\inf_m s^1_m > \delta_0 > 0$ for any $m = 1,2,\ldots$ and thus $s^1_0 \neq 0$. Consider now minimizing problem (5.6). Observe, $\widehat{\Phi}^2_\lambda < 0$ for $\lambda \in (\lambda^\partial_{\min}, \lambda^*_{\max})$. Indeed, let $\overline{u} \in \mathcal{N}^{s,2}_\lambda$, then (1) entails $r_{\overline{u}}(s) < \lambda$ for every $s \in (0,1)$. Consequently by Proposition 3.3, $d\Phi_\lambda(s\overline{u})/ds < 0$ for all $s \in (0,1)$ and therefore $0 = \Phi_\lambda(0\overline{u}) > \Phi_\lambda(\overline{u}) \geq \widehat{\Phi}^2_\lambda$. Assume $s^2_m \to 0$. Then $\|\overline{u}^2_m\|_W \to 0$ and $\Phi_\lambda(\overline{u}^2_m) \to 0$. However, this contradicts to $\widehat{\Phi}^2_\lambda < 0$ and therefore $s^2_0 \neq 0$.

Thus $\delta < \|\overline{u}^i_m\|_W < K < +\infty$, $m = 1,2,\ldots$, $i = 1,2$, with some $\delta, K \in (0,\infty)$, and assumption (4) entails that $\overline{u}^i_0 \neq 0_n$, $i = 1,2$. □

PROPOSITION 5.3.

$$\frac{d}{ds}\, r_{\overline{u}^1_0}(s)\bigg|_{s=1} < 0, \tag{5.7}$$

$$\frac{d}{ds}\, r_{\overline{u}^2_0}(s)\bigg|_{s=1} > 0. \tag{5.8}$$

PROOF. Let $i = 1,2$. Since $(\overline{v}^i_m)$ is weakly separated from $0_n \in W$, assumption (3) yields that the set of functions $(r^i_{\overline{v}_m}(t))^\infty_{m=1}$ is bounded in $C^1[\sigma,T]$ for any $\sigma, T \in (0,+\infty)$. Consequently by the Arzelá–Ascoli compactness criterion we can assume that

$$r^i_{\overline{v}_m}(t) \to \psi^i(t) \quad \text{in } C[\sigma,T], \text{ as } m \to \infty \text{ for all } \sigma,\ T \in (0,+\infty), \tag{5.9}$$

for some limit function $\psi^i \in C(0,+\infty)$. Since $s^i_0 > 0$,

$$r_{\overline{u}^i_m}(t) = R(ts^i_m \overline{v}^i_m) \to \psi^i(ts^i_0) =: \widehat{\psi}^i(t) \quad \text{as } m \to \infty \tag{5.10}$$

for all $t \in (0,+\infty)$. Observe that by the weak lower semi-continuity of R

$$r_{\overline{u}^i_0}(s) \equiv R(s\overline{u}^i_0) \leq \liminf_{m\to\infty} R(s\overline{u}^i_m), \quad \text{for all } s > 0. \tag{5.11}$$

This and (5.10) yield that for $s \geq 0$

$$r_{\overline{u}^i_0}(s) \leq \widehat{\psi}^i(s). \tag{5.12}$$

Let us show (5.7). Suppose, contrary to our claim, that $dr_{\overline{u}^1_0}(s)/ds|_{s=1} \geq 0$. Then (5.1) entails $s_{\max}(\overline{u}^1_0) \geq 1$. Since $r_{\overline{u}^1_m}(s) \leq \lambda$ for $s \in [1,\infty)$, $m = 1,2,\ldots$, (5.10) implies $\widehat{\psi}^1(s) \leq \lambda$ for $s \in [1,\infty)$ and consequently by (5.12), $r_{\overline{u}^1_0}(s) \leq \lambda$ for $s \in [1,\infty)$. Hence $\max_{s>0} r_{\overline{u}^1_0}(s) = r_{\overline{u}^1_0}(s_{\max}(\overline{u}^1_0)) \leq \lambda$. However, by the assumption $\lambda < \lambda^*_{\max} \leq \max_{s>0} r_{\overline{u}^1_0}(s)$. Thus we get a contradiction.

Assertion (5.8) can be handled in a similar way. Indeed, if $dr_{\overline{u}^2_0}(s)/ds|_{s=1} \leq 0$, then (5.1) entails $s_{\max}(\overline{u}^2_0) \leq 1$. Since $r_{\overline{u}^2_m}(s) \leq \lambda$ for $s \in (0,1]$, $m = 1,2\ldots$, (5.10), (5.12) yield $r_{\overline{u}_0^2}(s) \le \lambda$ for any $s \in (0,1]$. Hence $\lambda < \lambda^*_{\max} \le \max_{s>0} r_{\overline{u}_0^2}(s) = r_{\overline{u}_0^2}(s_{\max}(\overline{u}_0^2)) \le \lambda$, which is a contradiction. □

Proposition 5.4. *$\overline{u}_0^1$ and $\overline{u}_0^2$ are minimizers of* (5.5) *and* (5.6)*, respectively.*

Proof. By the weak lower semi-continuity of Φ_λ and R we have

$$-\infty < \Phi_\lambda(\overline{u}_0^i) \le \liminf_{m\to\infty} \Phi_\lambda(\overline{u}_m^i) = \widehat{\Phi}_\lambda^i, \quad i = 1,2, \tag{5.13}$$

$$-\infty < R(\overline{u}_0^i) \le \liminf_{m\to\infty} R(\overline{u}_m^i) = \lambda, \qquad i = 1,2. \tag{5.14}$$

Note that if $R(\overline{u}_0^i) = \lambda$, $i = 1,2$, then (5.7), (5.8) imply that $\overline{u}_0^1 \in \mathcal{N}_\lambda^{s,1}$, $\overline{u}_0^2 \in \mathcal{N}_\lambda^{s,2}$. Consequently, in (5.13) are possible only equalities $\Phi_\lambda(\overline{u}_0^i) = \widehat{\Phi}_\lambda^i$, $i = 1,2$, that yield the proof of the proposition.

Consider (5.5). Suppose that $R(\overline{u}_0^1) < \lambda$. Since (5.7), assumptions $\lambda < \lambda^*_{\max}$ and (1) entail that there is $s_1 < 1$ such that $R(s_1\overline{u}_0^1) = \lambda$ and $dR(s\overline{u}_0^1)/ds|_{s=s_1} < 0$. Since $R(s_1\overline{u}_m^1) \ge R(s_1\overline{u}_0^1) = \lambda$ and $R(\overline{u}_m^1) = \lambda$ for $m = 1,2,\ldots$, assumption (1) entails $R(s\overline{u}_m^1) > \lambda$, $m = 1,2,\ldots$, for all $s \in (s_1,1]$. Then by Proposition 3.3, $d\Phi_\lambda(s\overline{u}_m^1)/ds > 0$, $m = 1,2,\ldots$, for all $s \in (s_1,1]$. Consequently, $\Phi_\lambda(s_1\overline{u}_m^1) < \Phi_\lambda(\overline{u}_m^1)$ and by the weak lower semi-continuity of Φ_λ we have

$$\Phi_\lambda(s_1\overline{u}_0^1) \le \liminf_{m\to\infty} \Phi_\lambda(s_1\overline{u}_m^1) \le \liminf_{m\to\infty} \Phi_\lambda(\overline{u}_m^1) = \widehat{\Phi}_\lambda^1.$$

Notice that $s_1\overline{u}_0^1 \in \mathcal{N}_\lambda^{s,1}$. Hence if $\Phi_\lambda(s_1\overline{u}_0^1) = \widehat{\Phi}_\lambda^1$, then $s_1\overline{u}_0^1$ is a minimizer of (5.5) and we are done. Otherwise, if $\Phi_\lambda(s_1\overline{u}_0^1) < \widehat{\Phi}_\lambda^1$, then we obtain a contradiction and thus $R(\overline{u}_0^1) = \lambda$.

Consider (5.6). Suppose contrary to our claim that $R(\overline{u}_0^2) < \lambda$. Then $R(\overline{u}_0^2) < \lambda < \lambda^*_{\max} \le R(s_{\max}(\overline{u}_0^2)\,\overline{u}_0^2)$. By (5.8), we have $dR(s\overline{u}_0^2)/ds|_{s=1} > 0$. Hence, assumption (1) yields that there exists $s_1 \in (1, s_{\max}(\overline{u}_0^2))$ such that $R(s_1\overline{u}_0^2) = \lambda$ and $dR(s\overline{u}_0^2)/ds|_{s=s_1} > 0$, i.e. $s_1\overline{u}_0^2 \in \mathcal{N}_\lambda^{s,2}$. On the other hand, by Proposition 3.3 the inequality $R(s\overline{u}_0^2) < \lambda$, $s \in [1,s_1)$, implies $d\Phi_\lambda(s\overline{u}_0^2)/ds < 0$ for $s \in [1,s_1)$. Consequently, by (5.13) we have

$$\Phi_\lambda(s_1\overline{u}_0^2) < \Phi_\lambda(\overline{u}_0^2) \le \widehat{\Phi}_\lambda^2.$$

But for $s_1\overline{u}_0^2 \in \mathcal{N}_\lambda^{s,2}$, this is impossible. This completes the proof of the proposition. □

Now let us conclude the proof of the theorem. Since (5.2), Proposition 5.4 and Theorem 3.9 yield $\overline{u}_\lambda^1 = \overline{u}_0^1$ and $\overline{u}_\lambda^2 = \overline{u}_0^2$ satisfy (2.1). Since (5.7), (5.8), Proposition 3.4 implies that $d^2\Phi_\lambda(s\overline{u}_\lambda^1)/ds^2|_{s=1} < 0$, $d^2\Phi_\lambda(s\overline{u}_\lambda^2)/ds^2|_{s=1} > 0$.

Thus, it remains to show that that $\overline{u}_\lambda^2$ is a ground state of (2.1). Assumption (a) and $\lambda \in (\lambda^\partial_{\min}, \lambda^*_{\max})$ yield that the equation $d\Phi_\lambda(\tau\overline{u}_\lambda^1)/d\tau = 0$ has precisely two solutions $\tau_{\min} < 1$ and $\tau_{\max} = 1$ such that

$$d^2\Phi_\lambda(\tau\overline{u}_\lambda^1)/d\tau^2|_{\tau=1} < 0 \quad \text{and} \quad d^2\Phi_\lambda(\tau\overline{u}_\lambda^1)/d\tau^2|_{\tau=\tau_{\min}} > 0.$$

This and Proposition 3.4 yield that $\tau_{\min}\overline{u}^1_\lambda \in \mathcal{N}^{s,2}_\lambda$. Hence

$$\Phi_\lambda(\overline{u}^1_\lambda) > \Phi_\lambda(\tau_{\min}\overline{u}^1_\lambda) \geq \widehat{\Phi}^2_\lambda \equiv \Phi_\lambda(\overline{u}^2_\lambda).$$

Since $\mathcal{N}^{s,1}_\lambda \cup \mathcal{N}^{s,2}_\lambda = \mathcal{N}^s_\lambda$ for $\lambda \in (\lambda^\partial_{\min}, \lambda^*_{\max})$, this implies that $\overline{u}^2_\lambda$ is a minimizer of the problem $\widehat{\Phi}_\lambda := \min\{\Phi_\lambda(u) : u \in \mathcal{N}^s_\lambda\}$, i.e. $\overline{u}^2_\lambda$ is a ground state of (2.1) .□

We emphasize that the value $\lambda^\partial_{\min}$ has been used above only in order to allocate the values λ in $(\lambda^\partial_{\min}, \lambda^*_{\max})$ for which $\mathcal{N}^{s,1}_\lambda \neq \emptyset$, $\mathcal{N}^{s,2}_\lambda \neq \emptyset$. In fact, the above proof of Theorem 5.1 can be easily adapted to other assumptions on the behaviour of $r_v(s)$ at $s \to 0$ and $s \to \infty$. In particular, let us assume that for all $v \in S$ there holds

$$r_{\overline{v}}(s) \to 0 \quad \text{as } s \to 0 \quad \text{and} \quad r_{\overline{v}}(s) \to -\infty \quad \text{as } s \to \infty. \tag{5.15}$$

Then $\lambda^\partial_{\min} = 0$, $\lambda^*_{\min} = -\infty$, and now $\mathcal{N}^{s,1}_\lambda \neq \emptyset$ for all $\lambda < \lambda^*_{\max}$. It is easily seen that the above proof of the existence of the minimizer $\overline{u}^1_\lambda$ of (5.5) remains valid for all $\lambda < \lambda^*_{\max}$, provided (5.15) is satisfied. Thus we have

Corollary 5.5. *Suppose the assumptions of Theorem* 5.1 *and* (5.15) *hold. Then for every $\lambda < \lambda^*_{\max}$ there exists a minimizer $\overline{u}^1_\lambda$ of* (5.5) *which satisfies* (2.1). *Furthermore, $d^2\Phi_\lambda(s\overline{u}^1_\lambda)/ds^2|_{s=1} < 0$ and $\overline{u}^1_\lambda$ is the ground state of* (2.1) *for $\lambda \leq 0$.*

5.1. Multiplicity nonnegative solutions for problems with a general convex-concave type nonlinearity. In this subsection, using Theorem 5.1 we obtain a result on the existence of multiple sign-constant solutions for problems with a general convex-concave type nonlinearity and p-Laplacian.

Consider the following system of the equations:

$$\begin{cases} -\Delta_p u_1 = \lambda|\overline{u}|^{q-2}u_1 + f_1(x,\overline{u}) & \text{in } \Omega, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ -\Delta_p u_n = \lambda|\overline{u}|^{q-2}u_n + f_n(x,\overline{u}) & \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0 & i = 1,\ldots,n, \end{cases} \tag{5.16}$$

where $1 < q < p < +\infty$, $f_i\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^+$, $i = 1,\ldots,n$, are Carathéodory functions such that $f_i(x, 0_n) = 0$, $f_i(x,\cdot\,) \in C^1(\mathbb{R}^n,\mathbb{R})$ for almost all $x \in \Omega$, with primitive $F(x,\overline{u})$ so that $f_i(x,\overline{u}) = \partial F(x,\overline{u})/\partial u_i$ for almost all $x \in \Omega$, $\overline{u} \in \mathbb{R}^n$, $i = 1,\ldots,n$.

We will suppose that $\overline{f} := (f_1,\ldots,f_n)$ satisfies the following conditions:

(F_1) There exist $\gamma_1, \gamma_2 \in (p, p^*)$, $\gamma_1 \leq \gamma_2$ and exist $g_j \in L^{\beta_j}(\Omega)$, $g_j \geq 0$, $j = 1, 2$, so that: for $i = 1,\ldots,n$, for all $\overline{u} \in \mathbb{R}^n \setminus 0_n$, for all $s \in \mathbb{R} \setminus 0$,

$$0 < s\,\frac{\partial}{\partial s}\, f_i(x, s\,\overline{u}) \leq g_1(x)|s\,\overline{u}|^{\gamma_1 - 1} + g_2(x)|s\,\overline{u}|^{\gamma_2 - 1} \quad \text{a.e. in } \Omega,$$

where $\beta_j > p^*/(p^* - \gamma_j)$, if $N > p$ and $\beta_j > 1$, if $N \leq p$, $j = 1, 2$.

(F_2) There exist $\theta > p$, $K_1 > 0$ such that

$$0 < \theta F(x,\overline{u}) \le \sum_{i=1}^{n} f_i(x,\overline{u})u_i, \quad \text{a.e. in } \Omega,\ |\overline{u}| \ge K_1.$$

(F_3) For all $\overline{u} \in \mathbb{R}^n \setminus 0_n$ and for almost all $x \in \Omega$,

$$\rho(s) := \sum_{i=1}^{n} \frac{\partial}{\partial s} \frac{s^{1-q} f_i(x, s\,\overline{u})u_i}{s^{p-q-1}}$$

is a monotone function such that $\rho(s) \to +\infty$ as $s \to +\infty$.

By a solution of (5.16) we shall mean a weak solution $\overline{u} \in W := (W_0^{1,p}(\Omega))^n$. Problem (5.16) has a variational form with the Euler–Lagrange functional

$$\Phi_\lambda(\overline{u}) = \frac{1}{p}\int |\nabla \overline{u}|^p\,dx - \lambda\frac{1}{q}\int |\overline{u}|^q\,dx - \int F(x,\overline{u})\,dx. \tag{5.17}$$

Here $\nabla\overline{u} := (\nabla u_1, \ldots, \nabla u_n)$ and $|\nabla\overline{u}|^p = \sum_{i=1}^n |\nabla u_i|^p$. Consider the NG-Rayleigh quotient

$$R(\overline{u}) = \frac{\displaystyle\int |\nabla\overline{u}|^p\,dx - \sum_{i=1}^{n}\int f_i(x,\overline{u})u_i\,dx}{\displaystyle\int |\overline{u}|^q\,dx}, \quad \overline{u} \in W \setminus 0_n \tag{5.18}$$

and the corresponding fibering map

$$r_{\overline{u}}(s) = \frac{\displaystyle s^{p-q}\int |\nabla\overline{u}|^p\,dx - s^{1-q}\sum_{i=1}^{n}\int f_i(x,s\,\overline{u})u_i\,dx}{\displaystyle\int |\overline{u}|^q dx} \tag{5.19}$$

for $\overline{u} \in W \setminus 0_n$, $s > 0$. Note that (F_1) implies that, for all $\overline{u} \in \mathbb{R}^n \setminus 0_n$,

$$0 < f_i(x,\overline{u}) \le g_1'(x)|\overline{u}|^{\gamma_1-1} + g_2'(x)|\overline{u}|^{\gamma_2-1} \quad \text{for a.a. } x\in\Omega,\ i = 1,\ldots,n. \tag{5.20}$$

Here $g_j' = g_j/(\gamma_j - 1)$, $j = 1,2$. For $\overline{u} \in W$ and $j = 1,2$, by Sobolev's and Holder's inequalities one has

$$\left|\int g_j'(x)|\overline{u}|^{\gamma_j}\,dx\right| \le C\|\overline{u}\|^{\gamma_j/p^*}_{(L^{p^*})^n}\left(\int |g_j'(x)|^{p^*/(p^*-\gamma_i)}\,dx\right)^{(p^*-\gamma_j)/p^*}, \tag{5.21}$$

where $C < +\infty$. Here and in what follows we denote $(L^d)^n = (L^d(\Omega))^n$, $1 < d < \infty$. This implies that Φ_λ and r are well defined on W and $W \setminus 0_n$, respectively.

Consider the extreme value

$$\lambda^*_{\max} = \inf_{\overline{u}\in W\setminus 0_n}\sup_{s>0}\frac{\displaystyle s^{p-q}\int |\nabla\overline{u}|^p\,dx - s^{1-q}\sum_{i=1}^{n}\int f_i(x,s\,\overline{u})u_i\,dx}{\displaystyle\int |\overline{u}|^q\,dx}. \tag{5.22}$$

We prove

Theorem 5.6. *Assume* $1 < q < p < +\infty$, $f_i\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^+$, $i = 1, \ldots, n$, *are Carathéodory functions such that* $f_i(x, 0_n) = 0$, $f_i(x, \cdot\,) \in C^1(\mathbb{R}^n, \mathbb{R})$ *for almost all* $x \in \Omega$ *and* (F$_1$)–(F$_3$) *hold. Then* $0 < \lambda^*_{\max}$ *and for any* $\lambda < \lambda^*_{\max}$, *problem* (5.16) *admits a weak solution* $\overline{u}^1_\lambda \neq 0$. *Furthermore, when* $\lambda \in (0, \lambda^*_{\max})$ *problem* (5.16) *has a second weak solution* $\overline{u}^2_\lambda \neq 0$. *Moreover,*

(a) $d^2\Phi_\lambda(s\,\overline{u}^1_\lambda)/ds^2|_{s=1} < 0$, $d^2\Phi_\lambda(s\,\overline{u}^2_\lambda)/ds^2|_{s=1} > 0$, $\Phi_\lambda(\overline{u}^2_\lambda) < 0$;
(b) *if* $\lambda \in (-\infty, 0]$, *then* $\overline{u}^1_\lambda$ *is a ground state of* (5.16);
(c) *if* $\lambda \in (0, \lambda^*_{\max})$, *then* $\overline{u}^2_\lambda$ *is a ground state of* (5.16).

Proof. We will obtain the proof by applying Theorem 5.1 and Corollary 5.5. First we verify conditions (1)–(4) of Theorem 5.1.

Claim. (1) holds. Compute

(5.23)
$$\frac{d}{ds}\, r_{\overline{u}}(s) = \frac{(p-q)s^{p-q-1}\displaystyle\int |\nabla \overline{u}|^p\,dx - \int \frac{\partial}{\partial s}\bigg(s^{1-q}\sum_{i=1}^n f_i(x, s\,\overline{u})u_i\bigg)\,dx}{\displaystyle\int |\overline{u}|^q\,dx}.$$

Clearly, (F$_1$), (5.20) (5.21) yield

$$\int \frac{\partial}{\partial s}\bigg(s^{1-q}\sum_{i=1}^n f_i(x, s\,\overline{u})u_i\bigg)\,dx/s^{p-q-1} \to 0$$

as $s \to 0$ for any $\overline{u} \in W$. Hence by (F$_3$) the equation

$$\int |\nabla \overline{u}|^p dx - \int \frac{\partial}{\partial s}\bigg(s^{1-q}\sum_{i=1}^n f_i(x, s\,\overline{u})u_i\bigg)\,dx/s^{p-q-1} = 0$$

has a unique solution $s_{\max}(\overline{u}) \in \dot{\mathbb{R}}^+$ which is a global maximum point of $r_{\overline{u}}(s)$. Thus we get (1).

Claim. (2) holds. Suppose by contradiction that there is a sequence $(\overline{v}_m) \subset S$ such that $s_m := s_{\max}(\overline{v}_m) \to 0$ as $m \to \infty$. In view of (5.23), we have

$$(p-q)s_m^{p-q-1} - \int \frac{\partial}{\partial s}\bigg(s^{1-q}\sum_{i=1}^n f_i(x, s\,\overline{v}_m)v_{m,i}\bigg)\,dx = 0.$$

Now using (F$_1$), (5.20), (5.21) we obtain

$$(p-q)s_m^{p-q-1} - c_1 s_m^{\gamma_2-q-1} - c_2 s_m^{\gamma_1-q-1} \leq 0,$$

where c_1, c_2 do not depend on $s > 0$ and $m = 1, 2, \ldots$ However, since $q < p < \gamma_2 \leq \gamma_1$, we get a contradiction as $s_m \to 0$. Thus, we get (2).

Claim. (3) holds. Assume that $(\overline{v}_m) \subset S$ is weakly separated from $0_n \in W$. Since $(\overline{v}_m)$ is bounded in W and W is the reflexive Banach space, we may assume that $\overline{v}_m \rightharpoonup \overline{v}_0$ weakly in W for some $\overline{v}_0 \in W$. Furthermore, by the Rellich–Kondrachov theorem $\|\overline{v}_m\|_{L^d} < C_1 < +\infty$ for $m = 1, 2, \ldots$, $1 \leq d \leq p^*$, and $\overline{v}_m \to \overline{v}_0$ in $L^d(\Omega)$ for $d < p^*$. Since $(\overline{v}_m) \subset S$ is weakly separated from $0_n \in W$, $\overline{v}_0 \neq 0$ and consequently there exists $\delta_0 > 0$ such that $\int |\overline{v}_m|^q\,dx > \delta_0$ for all $m = 1, 2, \ldots$ Hence by (F_1), (5.20) and (5.21) we have: for any $s > 0$,

$$|R(s\,\overline{v}_m)| \leq \delta_0^{-1}\bigg(s^{p-q} + s^{1-q}\sum_{i=1}^n \int |f_i(x, s\,\overline{v}_m)v_{m,i}|\,dx\bigg) \leq C_2 s^{p-q} + C_3 s^{\gamma_2 - q},$$

and

$$\begin{aligned}\bigg|\frac{d}{ds}\,R(s\,\overline{v}_m)\bigg| &\leq \delta_0^{-1}\bigg((p-q)s^{p-q-1} + (q-1)s^{-q}\sum_{i=1}^n \int |f_i(x, s\,\overline{v}_m)v_{m,i}|\,dx \\ &\qquad + s^{1-q}\sum_{i=1}^n \int \bigg|\frac{\partial}{\partial s} f_i(x, s\,\overline{v}_m)v_{m,i}\bigg|\,dx\bigg) \\ &\leq C_4 s^{p-q-1} + C_5 s^{\gamma_2 - q - 1},\end{aligned}$$

where $C_2, \ldots, C_5$ do not depend on $s > 0$ and $m = 1, 2, \ldots$ Thus we get (2).

Claim. (4) holds. Observe that (5.20) and (5.21) imply

$$r_{\overline{u}}(s) \geq \frac{s^{p-q}\|\overline{u}\|_W^p - C_1' s^{\gamma_2 - q}\|\overline{u}\|^{\gamma_2}_{(L^{\gamma_2})^n} - C_2' s^{\gamma_1 - q}\|\overline{u}\|^{\gamma_1}_{(L^{\gamma_1})^n}}{\|\overline{u}\|^q_{(L^q)^n}} \tag{5.24}$$

for $s > 0$, $\overline{u} \in W \setminus 0$, where C_1', C_2' do not depend on $s > 0$. Suppose by contradiction that there exists $(s_m \overline{v}_m) \subset \mathcal{N}_\lambda^s$, $\lambda \in \mathbb{R}$ such that $(\overline{v}_m) \subset S$, $\sigma < s_m < T$, $m = 1, 2, \ldots$, for some $\sigma, T \in (0, +\infty)$ and $\overline{v}_m \rightharpoonup 0$ weakly in W. Then, we may assume that $\overline{v}_m \to 0$ in $(L^q)^n$, $(L^{\gamma_1})^n$ and $(L^{\gamma_2})^n$. However, by (5.24) we have

$$s_m^{\gamma_1 - q} \geq \frac{\sigma^{p-q} - \lambda\|\overline{v}_m\|^q_{(L^q)^n}}{C_1' T^{\gamma_2 - \gamma_1}\|\overline{v}_m\|^{\gamma_2}_{(L^{\gamma_2})^n} + C_2'\|\overline{v}_m\|^{\gamma_1}_{(L^{\gamma_1})^n}} \to \infty \quad \text{as } m \to \infty$$

that contradicts to the assumption $s_m < T$, $m = 1, 2, \ldots$ Thus (4) also holds.

It is readily seen, (F_1), (5.20), (5.21) imply that (5.19) satisfies condition (5.15) of Corollary 5.5.

Let us show that conditions (a), (b) of Theorem 5.1 are satisfied. For $\overline{u} \in \mathcal{N}_\lambda^s$ we have

$$\Phi_\lambda(\overline{u}) = \frac{(\theta - p)}{p}\int |\nabla \overline{u}|^p\,dx - \lambda\,\frac{(\theta - q)}{q}\int |\overline{u}|^q\,dx - \int\bigg(\theta F(x, \overline{u}) - \sum_{i=1}^n f_i(x, \overline{u})u_i\bigg)\,dx.$$

Hence (F_2) and Sobolev's inequalities yield

$$\Phi_\lambda(\overline{u}) \geq \frac{(\theta - p)}{p}\,\|\overline{u}\|_W^p - \frac{\lambda(\theta - q)}{q}\,\|\overline{u}\|_W^q$$

for $\|\overline{u}\|_W > K_1$. Since $q < p$, this implies $\Phi_\lambda(\overline{u}) \to +\infty$ as $\|\overline{u}\|_W \to +\infty$ and thus condition (a) of Theorem 5.1 is satisfied.

Clearly, the functionals $\int F(x,\overline{u})\,dx$, $\int f_i(x,\overline{u})u_i\,dx$, $\int |\overline{u}|^q\,dx$ are weakly continuous on W and $\int |\nabla \overline{u}|^p\,dx$ is a weakly lower semi-continuous functional on W. This yields that (5.17) and (5.18) satisfy (b) of Theorem 5.1.

Note that (5.24), (F_1) and Sobolev's inequalities yield

$$\begin{aligned}\lambda^*_{\max} &\geq \inf_{\|\overline{v}\|=1}\sup_{s>0}\frac{s^{p-q}\|\overline{v}\|^p_W - \widetilde{c}_1 s^{\gamma_2-q}\|\overline{v}\|^{\gamma_2}_W - \widetilde{c}_2 s^{\gamma_1-q}\|\overline{v}\|^{\gamma_1}_W}{\widetilde{c}_3\|\overline{v}\|^q_W}\\ &= \max_{s>0}\{s^{p-q} - \widetilde{c}_1 s^{\gamma_2-q} - \widetilde{c}_2 s^{\gamma_1-q}\}/\widetilde{c}_3 > 0\end{aligned}$$

for some $\widetilde{c}_1, \widetilde{c}_2, \widetilde{c}_3 > 0$. Hence $\lambda^*_{\max} > 0$. Thus, all assumptions of Theorem 5.1 and Corollary 5.5 are satisfied. □

Denote $|\overline{u}| = (|u_1|,\ldots,|u_n|)$. The next corollary on the existence of sign-constant solutions follows in the standard way.

Corollary 5.7. *Suppose the assumptions of Theorem* 5.6 *are satisfied and* $F(x,\overline{u}) = F(x,|\overline{u}|)$ *almost everywhere in* Ω, *for any* $\overline{u}\in\mathbb{R}^n$. *Then, for* $\lambda < \lambda^*_{\max}$, *system of equations* (5.16) *admits a pair of non-trivial weak solutions* $\overline{u}^{1,+}_\lambda \geq 0_n \geq \overline{u}^{1,-}_\lambda$ *and for* $\lambda\in(0,\lambda^*_{\max})$, *system of equations* (5.16) *has a second pair of non-trivial weak solutions* $\overline{u}^{2,+}_\lambda \geq 0_n \geq \overline{u}^{2,-}_\lambda$. *Furthermore, assertions* (a)–(c) *of Theorem* 5.6 *are satisfied for* $\overline{u}^{1,\pm}_\lambda$, $\overline{u}^{2,\pm}_\lambda$.

In the scalar version of (5.16), this result can be strengthened. Let us consider

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + f(x,u) & \text{in } \Omega,\\ u|_{\partial\Omega} = 0.\end{cases} \tag{5.25}$$

Consider the extreme value

$$\lambda^*_{\max} = \inf_{v\in W\setminus 0}\sup_{s>0}\frac{s^{p-q}\displaystyle\int |\nabla u|^p\,dx - s^{1-q}\int f(x,su)u\,dx}{\displaystyle\int |u|^q\,dx}. \tag{5.26}$$

Theorem 5.8. *Assume* $1<q<p<+\infty$, $f\colon\Omega\times\mathbb{R}\to\mathbb{R}$ *is a Carathéodory function such that* $f(x,\cdot)\in C^1(\mathbb{R},\mathbb{R})$, $f(x,0)=0$, $\partial f(x,s)/\partial s|_{s=0}=0$ *for almost all* $x\in\Omega$ *and* (F_1)–(F_3) (*with* $n=1$) *hold. Then* $0<\lambda^*_{\max}$ *and for any* $\lambda<\lambda^*_{\max}$, *problem* (5.25) *admits a pair of non-trivial weak solutions* $u^{1,+}_\lambda \geq 0 \geq u^{1,-}_\lambda$. *Furthermore, when* $\lambda\in(0,\lambda^*_{\max})$ *equation* (5.25) *has a second pair of non-trivial weak solutions* $u^{2,+}_\lambda \geq 0 \geq u^{2,-}_\lambda$. *Moreover,*

(a) $d^2\Phi_\lambda(su^{1,\pm}_\lambda)/ds^2|_{s=1}<0$, $d^2\Phi_\lambda(su^{2,\pm}_\lambda)/ds^2|_{s=1}>0$, $\Phi_\lambda(u^{2,\pm}_\lambda)<0$;

(b) *if* $\lambda\in(-\infty,0]$, *then one of the solutions* $u^{1,+}_\lambda$ *or* $u^{1,-}_\lambda$ *is a ground state of* (5.25);

(c) *if* $\lambda\in(0,\lambda^*_{\max})$, *then one of the solutions* $u^{2,+}_\lambda$ *or* $u^{2,-}_\lambda$ *is a ground state of* (5.25).

Remark 5.9. Similar result on the existence of multiple sign-constant solutions for scalar problems with a general convex-concave type nonlinearity has been obtained in [2], [3], [26]. However, our assumptions on the function $f(x,u)$ are different from that of made in [2], [3], [26]. In particular, functions g_i, $i=1,2$, may not be bounded from above, which may result in difficulties in applying the super-sub solution method (cf. [2], [3], [26]). Furthermore, the presence of p-Laplacian with $p \neq 2$ in (5.25) can complicate the application of mountain pass theorem in order intervals (cf. [26]).

Proof. In order to obtain sign-constant solutions $u_\lambda^{1,+} \geq 0 \geq u_\lambda^{1,-}$ and $u_\lambda^{2,+} \geq 0 \geq u_\lambda^{2,-}$, we truncate and reflect $f(x,u)$ as follows:

$$f^{\pm}(x,u) = \begin{cases} f(x,u) & \text{if } \pm u \geq 0, \\ -f(x,-u) & \text{if } \pm u < 0. \end{cases} \tag{5.27}$$

Let $F^{\pm}(x,u)$ denote the primitive of $f^{\pm}(x,u)$ and consider

$$\Phi_\lambda^{\pm}(u) = \frac{1}{p}\int |\nabla u|^p\,dx - \lambda\,\frac{1}{q}\int |u|^q\,dx - \int F^{\pm}(x,u)\,dx. \tag{5.28}$$

Clearly, $\Phi_\lambda^{\pm}(u) \in C^1(W\backslash 0,\mathbb{R})$. Furthermore, since $f(x,0)=0$, $\partial f(x,s)/\partial s|_{s=0} = 0$, for almost all $x \in \Omega$, $\frac{\partial}{\partial s}\int F^{\pm}(x,su)\,dx$ is a map of class C^1 on $\dot{\mathbb{R}}^+ \times (W \setminus 0)$. As above in the proof of Theorem 5.6, it can be shown that all the other assumptions of Theorem 5.1 and Corollary 5.5 are also satisfied. Thus there exist weak solutions $u_\lambda^{1,\pm}, u_\lambda^{2,\pm} \in W_0^{1,p}(\Omega)$ of

$$-\Delta_p u = \lambda |u|^{q-2}u + f^{\pm}(x,u)$$

for $\lambda < \lambda^*_{\max}$ and $\lambda \in (0,\lambda^*_{\max})$, respectively. Since $\Phi_\lambda^{\pm}(|u|) = \Phi_\lambda^{\pm}(u)$ we may assume that the minimizers $u_\lambda^{1,+}, u_\lambda^{2,+}$ of $\widehat{\Phi}_\lambda^{j,+} := \min\{\Phi_\lambda^{+}(u) : u \in \mathcal{N}_\lambda^{s,j}\}$, $j=1,2$, respectively, are non-negative, whereas the minimizers $u_\lambda^{1,-}, u_\lambda^{2,-}$ of $\widehat{\Phi}_\lambda^{j,-} := \min\{\Phi_\lambda^{-}(u) : u \in \mathcal{N}_\lambda^{s,j}\}$, $j=1,2$, respectively, are non-positive. Now taking into account (5.27) we get that the functions $u_\lambda^{1,\pm}, u_\lambda^{2,\pm}$ in fact are weak solutions of the original problem (5.25). Finally, assertions (a)–(c) of Theorem 5.8 follow from Theorem 5.1 and Corollary 5.5. □

References

[1] S. Alama and G. Tarantello, *On semilinear elliptic problems with indefinite nonlinearities*, Cal. Var. Partial Differential Equations **1** (1993), 439–475.

[2] A. Ambrosetti, J.G. Azorero and I. Peral, *Multiplicity results for some nonlinear elliptic equations*, J. Funct. Anal. **137** (1996), no. 1, 219–242.

[3] A. Ambrosetti, H. Brezis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122** (1994), no. 2, 519–543.

[4] A. Ambrosetti and A. Malchiodi, *Nonlinear Analysis and Semilinear Elliptic Problems*, Cambridge Studies in Advanced Mathematics, Vol. 104, Cambridge University Press, Cambridge, 2007.

[5] A. Anane, *Etude des valeurs propres et de la résonance pour l'opérateur p-Laplacien*, Thèse de doctorat, Université Libre de Bruxelles, Brussels, 1988.

[6] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, *Variational methods for indefinite superlinear homogeneous elliptic problems*, NoDEA **2** (1995), 553–572.

[7] H. Berestycki and P.L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), no. 4, 313–345.

[8] V. Bobkov, *Least energy nodal solutions for elliptic equations with indefinite nonlinearity*, Electron. J. Qual. Theory Differ. Equ. **56** (2014), 1–15.

[9] V. Bobkov and Y. Il'yasov, *Asymptotic behaviour of branches for ground states of elliptic systems*, EJDE **212** (2013), 1–21.

[10] ______, *Maximal existence domains of positive solutions for two-parametric systems of elliptic equations*, Complex Var. Elliptic Equ. **5** (2015), 1–21.

[11] K.J. Brown and T.F. Wu, *A fibering map approach to a potential operator equation and its applications*, Differential Integral Equations **22** (2009), no. 11/12, 1097–1114.

[12] L. Cherfils and Y. Il'yasov, *On the stationary solutions of generalized reaction diffusion equations with p&q-Laplacian*, Comm. Pure Appl. Math. **4** (2005), no. 1, 9–22.

[13] J.I. Díaz, J. Hernández and Y. Il'yasov, *On the existence of positive solutions and solutions with compact support for a spectral nonlinear elliptic problem with strong absorption*, Nonlinear Anal. **119** (2015), 484–500.

[14] P. Drábek and J. Milota, *Methods of Nonlinear Analysis. Applications to Differential Equations*, Springer, Basel, 2013.

[15] P. Drábek and S.I. Pohozaev, *Positive solution for the p-Laplacian: application of the fibering method*, Proc. Roy. Soc. Edinburg Sect. A. **127** (1997), 703–726.

[16] Y.V. Egorov and Y. Il'yasov, *On conformal invariants for elliptic systems with multiple critical exponents*, Ann. Global Anal. Geom. **32** (2007), no. 1, 39–66.

[17] Y. Il'yasov, *On a procedure of projective fibering of functionals on Banach spaces*, Proc. Steklov Inst. Math. **232** (2001), 150–156.

[18] ______, *Non-local investigation of bifurcations of solutions of non-linear elliptic equations*, Izv. Ross. Acad. Nauk Ser. Mat. **66** (2002), no. 6, 1103–1130.

[19] ______, *On nonlocal existence results for elliptic equations with convex-concave nonlinearities*, Nonlinear Anal. **61** (2005), no. 1–2, 211–236.

[20] ______, *On calculation of the bifurcations by the fibering approach*, Harmonic, Wavelet and P-adic Analysis (N.M. Chuong et al., eds.), World Scientific, Hackensack, 2007, pp. 141–155.

[21] Y. Il'yasov and Y. Egorov, *Hopf boundary maximum principle violation for semilinear elliptic equations*, Nonlinear Anal. **72** (2010), no. 7–8, 3346–3355.

[22] Y. Il'yasov and T. Runst, *Positive solutions for indefinite inhomogeneous Neumann elliptic problems*, Electron. J. Differential Equations **57** (2003), 1–21.

[23] ______, *On nonlocal calculation for inhomogeneous indefinite Neumann boundary value problems*, Calc. Var. Partial Differential Equations **22** (2005), no. 1, 101–127.

[24] Y. Il'yasov, T. Runst and A. Youssfi, *On the existence of pair positive-negative solutions for resonance problems*, Nonlinear Anal. **70** (2009), no. 10, 3461–3471.

[25] Y.S. Il'yasov and P. Takáč, *Optimal-regularity, Pohozhaev's identity, and nonexistence of weak solutions to some quasilinear elliptic equations*, J. Differential Equations **252** (2012), no. 3, 2792–2822.

[26] S. Li and Z.Q. Wang, *Mountain pass theorem in order intervals and multiple solutions for semilinear elliptic Dirichlet problems*, J. Anal. Math. **81** (2000), no. 1, 373–396.

[27] P. Lindqvist, *On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\lambda|u|^{p-2}u=0$, Proc. Amer. Math. Soc. **109** (1990), no. 1, 157–164.

[28] Z. Nehari, *On a class of nonlinear second-order differential equations*, Trans. Amer. Math. Soc. (1960), 101–123.

[29] ______, *Characteristic values associated with a class of nonlinear second-order differential equations*, Acta Math. **105** (1961), 141–175.

[30] T. Ouyang, *On the positive solutions of semilinear equations* $\Delta u+\lambda u+hu^p=0$ *on the compact manifolds*, Part II, Indiana Univ. Math. J. **40** (1991), no. 3, 1083–1141.

[31] A. Pankov, *Periodic nonlinear Schrödinger equation with application to photonic crystals*, Milan J. Math **73** (2005), no. 1, 259–287.

[32] S.I. Pohozaev, *On an approach to Nonlinear equations*, Doklady Acad. Sci. USSR. **247** (1979), 1327–1331.

[33] ______, *The fibering method in nonlinear variational problems*, In: Topological and Variational Methods for Nonlinear Boundary Value Problems, Pitman Res. Notes Math. Ser., Vol. 365, Longman, Harlow, 1997, pp. 35–88.

[34] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conf. Series Math., Vol. 65. American Mathematical Society, Providence, 1986.

[35] W.A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), no. 2, 149–162.

[36] A. Szulkin and T. Weth, *The method of Nehari manifold*, In: Handbook of Nonconvex Analysis and Applications, International Press, Somerville, 2010, pp. 597–632.

[37] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and their Applications, Vol. 24, Springer, Boston, 1997.

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Yavdat Ilyasov
Institute of Mathematics
Ufa Scientific Center
Russian Academy of Sciences
Chernyshevsky 112
Ufa, RUSSIA

E-mail address: ilyasov02@gmail.com