

## ON EXTREME VALUES OF NEHARI MANIFOLD METHOD VIA NONLINEAR RAYLEIGH'S QUOTIENT

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**ABSTRACT.** We study applicability conditions of the Nehari manifold method to the equation of the form  $D_u T(u) - \lambda D_u F(u) = 0$  in a Banach space  $W$ , where  $\lambda$  is a real parameter. Our study is based on the development of the Rayleigh quotient theory for nonlinear problems. It turns out that the extreme values of parameter  $\lambda$  which define intervals of applicability of the Nehari manifold method can be found through the critical values of the corresponding nonlinear generalized Rayleigh quotient. In the main part of this paper, we provide general results on this relationship. Theoretical results are illustrated by considering several examples of nonlinear boundary value problems. Furthermore, we demonstrate that the introduced tool of nonlinear generalized Rayleigh quotient can also be applied to prove new results on the existence of multiple solutions for nonlinear elliptic equations.

### 1. Introduction

The Nehari manifold method (NM-method), which was introduced in [28] and [29], by now is a well-established and useful tool in finding solutions of equations in variational form. Let us briefly describe it. Assume  $W$  is a real

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Banach space,  $\Phi_\lambda: W \rightarrow \mathbb{R}$  is a Fréchet-differentiable functional with derivative  $D_u\Phi_\lambda$  and  $\lambda \in \mathbb{R}$  is a parameter. Consider the equation in variational form

$$(1.1) \quad D_u\Phi_\lambda(u) = 0, \quad u \in W.$$

The Nehari manifold associated with (1.1) is defined as

$$\mathcal{N}_\lambda := \{u \in W \setminus 0 : D_u\Phi_\lambda(u)(u) = 0\}.$$

Since any solution of (1.1) belongs to  $\mathcal{N}_\lambda$ , a natural idea to solve (1.1) is to consider the constrained minimization problem

$$\Phi_\lambda(u) \rightarrow \min, \quad u \in \mathcal{N}_\lambda.$$

Suppose that there exists a local minimizer  $u$  of this problem and  $\Phi_\lambda \in C^2(U, \mathbb{R})$  for some neighbourhood  $U \subset W$  of  $u$ . Then by the Lagrange multiplier rule one has  $\mu_0 D_u\Phi_\lambda(u) + \mu_1 (D_u\Phi_\lambda(u) + D_{uu}\Phi_\lambda(u)(u, \cdot)) = 0$  for some  $\mu_0, \mu_1$  such that  $|\mu_0| + |\mu_1| \neq 0$ . Testing this equality by  $u$  we obtain  $\mu_1 D_{uu}\Phi_\lambda(u)(u, u) = 0$ . Hence, if  $D_{uu}\Phi_\lambda(u)(u, u) \neq 0$ , then we have successively  $\mu_1 = 0$ ,  $\mu_0 \neq 0$  and therefore  $D_u\Phi_\lambda(u) = 0$ . Thus, one has the following sufficient condition for the applicability of the NM-method:

$$(1.2) \quad D_{uu}\Phi_\lambda(u)(u, u) \neq 0 \quad \text{for any } u \in \mathcal{N}_\lambda.$$

The feasibility of this condition often depends on the parameter  $\lambda$ . Thus, we may expect that there exists the set of *extreme values of the NM-method*  $\sigma_\mathcal{N} := \{\lambda_{\min, i}, \lambda_{\max, i}\}_{i=1}^\infty$  such that the sufficient condition (1.2) is satisfied only for  $\lambda \in \bigcup_{i=1}^\infty (\lambda_{\min, i}, \lambda_{\max, i})$ . This brings up the question of *how to find these extreme values*.

In general, this question is related to finding bifurcations for critical points of family fibering functions  $\phi_{\lambda, v}(s) := \Phi_\lambda(sv)$ ,  $s \in \mathbb{R}^+$ , where  $v \in S := \{v \in W : \|v\|_W = 1\}$  and  $\lambda \in \mathbb{R}$ . Indeed, if  $u_\lambda = s_\lambda v_\lambda$  satisfies (1.1), then  $d\phi_{\lambda, v_\lambda}(s_\lambda)/ds = 0$  and hence  $s_\lambda v_\lambda \in \mathcal{N}_\lambda$ , whereas condition (1.2) is equivalent to  $d^2\phi_{\lambda, v_\lambda}(s_\lambda)/ds^2 \neq 0$ . Thus, in general, an extreme value  $\lambda^*$  of the NM-method may occur only: (1) as a bifurcation at zero or at infinity, when  $s_\lambda \rightarrow 0$  or  $s_\lambda \rightarrow +\infty$  as  $\lambda \rightarrow \lambda^*$ , respectively; (2) as a bifurcation at a point  $(s^*, \lambda^*, v^*)$ , where  $d^2\phi_{\lambda^*, v^*}(s^*)/ds^2 = 0$  and  $(s_\lambda^1, v_\lambda^1) \rightarrow (s^*, v^*)$ ,  $(s_\lambda^2, v_\lambda^2) \rightarrow (s^*, v^*)$  as  $\lambda \rightarrow \lambda^*$ , for some branches of critical points  $s_\lambda^1$  and  $s_\lambda^2$  of  $\phi_{\lambda, v}(s)$ . In fact, when for each  $v \in S$  the function  $\phi_{\lambda, v}(s)$  may possess at most one critical point in  $\mathbb{R}^+$  the extreme values of the NM-method either do not exist or can be found directly. Essential dependence of equation (1.1) on the parameter  $\lambda$  and the necessity of finding the extreme values of the NM-method take place when  $\phi_{\lambda, v}(s)$  may have more than one critical point of various types. Nonlinear partial differential equations with such property have been studied in a number of papers dealing with the multiplicity of