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RELATIVE INDEX THEORIES AND APPLICATIONS

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Abstract. We develop some relative index theories for abstract operator equations. As applications, we prove a new Galerkin approximation formula and a new saddle point reduction formula for the P-index. We apply these new formulas to the minimal periodic problem for P-symmetric periodic solutions of nonlinear Hamiltonian systems.

1. Introduction

Many problems can be displayed as a self-adjoint operator equation

$$Au = F'(u), \quad u \in D(A) \subset H, \tag{O.E.}$$

where H is an infinite-dimensional separable Hilbert space, A is a self-adjoint operator on H with domain $D(A)$, F is a nonlinear functional on H. For example, the Dirichlet problem for Laplace's equation on bounded domain, periodic problem for periodic solutions of Hamiltonian systems, Schrödinger equations, periodic problem for periodic solutions of wave equations and so on. By the variational method, we know that the solutions of (O.E.) correspond to the critical points of a functional on a Hilbert space. For any critical point of a functional, one can define its so-called Morse index (may be infinite). In many cases with

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the help of Morse theory the relationship between the global and local behavior of the functional can be established. However in the case of the so-called strongly indefinite functionals, such as the functionals related to periodic solutions of first order Hamiltonian systems, Schrödinger equations, wave equations, etc., their Morse indices are infinite. Hence one needs to define some relative Morse indices which could replace the classical Morse index.

For example, basing on the analytic approach, by using a Galerkin approximation sequence, one can define a kind of relative Morse index, which can be used in place of the Morse index when dealing with variational problems, see e.g. [8], [18], [24], [35], [37], [44], etc. Similarly, by using the so-called saddle point reduction method (a kind of the Lyapunov–Schmidt procedure, see e.g. [1], [2] and [7]), one can define a kind of relative Morse index, which in many cases coincides with the relative Morse index defined via the Galerkin approximation method (cf. [35] for the case of symplectic paths related to the periodic solutions of Hamiltonian systems). In the case of convex Hamiltonian systems, due to the dual variational method and convex analysis theory (see e.g. [4], [14], [17]) one can define a Morse index for any critical point of the corresponding dual functionals (cf. [13]–[16]). In [42], Wang and the author developed an index theory for linear self-adjoint operator equation where the operator A in (O.E.) may contain a nonempty essential spectrum.

Basing on the algebraic approach, for a linear Hamiltonian system its fundamental solution is a path in a symplectic group starting from the identity. Here the symplectic group is defined as $\mathrm{Sp}(2n)=\{M\in\mathcal{L}(2n) : M^TJM=J\}$, $J=\left(\begin{smallmatrix}0 & -I_n\\ I_n & 0\end{smallmatrix}\right)$, I_n is the $n\times n$ identity matrix. The set of symplectic paths starting from the identity is denoted by $\mathcal{P}_\tau(2n)=\{\gamma:\gamma\in C([0,\tau],\mathrm{Sp}(2n)),\gamma(0)=I_{2n}\}$. We say that a symplectic path $\gamma\in\mathcal{P}_\tau(2n)$ is non-degenerate if $\dim\ker_{\mathbf{C}}(\gamma(\tau)-I)=0$. In 1984, Conley and Zehnder in [9] developed an index theory for the non-degenerate symplectic paths with $n\geq 2$. In 1990, Long and Zehnder in [36] generalized it to the non-degenerate case with $n=1$. Long in [31], [32] and Viterbo in [41] extended this Maslov-type index theory to the degenerate case, they assigned a pair of integers $(i_1(\gamma),\nu_1(\gamma))\in\mathbf{Z}\times\{0,1,\dots,2n\}$ to $\gamma\in\mathcal{P}_\tau(2n)$. In [33], the index pair $(i_1(\gamma),\nu_1(\gamma))$ was further extended to an index function $(i_\omega(\gamma),\nu_\omega(\gamma))\in\mathbf{Z}\times\{0,1,\dots,2n\}$ with $\omega\in\mathbf{U}=\{z\in\mathbf{C}:|z|=1\}$. It was proved that this index pair in fact coincides with the relative index pair defined via the Galerkin approximation method [35] and the relative index pair defined via the saddle point reduction method [35]. So in this case, the relative Morse index introduced via the analytic approach is the same as the index introduced via the algebraic approach.

For any $P\in\mathrm{Sp}(2n)$, the author in [24] and Dong in [10] independently and with different methods defined the so-called P-index pair $(i^P(\gamma),\nu^P(\gamma))\in$