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POINCARÉ RECURRENCE THEOREM IN IMPULSIVE SYSTEMS

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Abstract. In this article, we generalize the Poincaré recurrence theorem to impulsive dynamical systems in $\mathbb{R}^n$. For a measure preserving system, we present some sufficient conditions to establish an impulsive system that is also measure preserving. Then, two recurrence theorems are proved. Finally, we use two examples to illustrate our results.

1. Introduction

Consider the differential equation $\dot{x} = f(x)$ on $\mathbb{R}^n$, where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field. Let the vector field define a dynamical system or flow $\varphi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. For a subset $\mathcal{D} \subset \mathbb{R}^n$, the flow φ is volume-preserving on $\mathcal{D}$ if for every measurable set $A \subset \mathcal{D}$ and every $t \in \mathbb{R}$ the set $\varphi^t(A) = \varphi(A \times \{t\})$ is measurable and $\mu(\varphi^t(A)) = \mu(A)$, where μ is a measure. For example, a Hamiltonian flow is volume-preserving (Liouville's Theorem). One of the most significant consequences of volume preservation is the following result.

Theorem 1.1 (Poincaré Recurrence Theorem). *If φ is a volume-preserving flow on an invariant bounded subset $\mathcal{D}$ of $\mathbb{R}^n$, then each point in $\mathcal{D}$ is nonwandering.*

This celebrated theorem has many generalizations, for example, see [1] and [8]–[12]. Our goal in this article is to generalize the Poincaré Recurrence Theorem

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to the impulsive systems. Since an impulsive system admits abrupt perturbations, its dynamical behavior is much richer than that of the corresponding system. Now, the theory of impulsive systems is an important and flourishing area of investigation, see [2]–[7]. In the next section, we establish some sufficient conditions to guarantee that an impulsive system is also volume-preserving. Then, in Section 3, we generalize the Poincaré Recurrence Theorem to impulsive dynamical systems. Finally, two examples are presented in Section 4 to illustrate the recurrence theorems.

2. Impulsive system

We first recall the definition and basic properties of a system with impulse action, and the reader may consult [2]–[7] for instance. Let φ be a flow on $\mathbb{R}^n$ defined by the vector field f, i.e. $\varphi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous such that $\varphi(x,0) = x$ for all $x \in \mathbb{R}^n$, and $\varphi(\varphi(x,t),s) = \varphi(x,t+s)$ for all $x \in \mathbb{R}^n$, $t,s \in \mathbb{R}$. Note that $(\partial/\partial t)\varphi(x,t) = f(\varphi(x,t))$. The mappings $\varphi_x\colon \mathbb{R} \to \mathbb{R}^n$ $(t \mapsto \varphi(x,t))$ for $x \in \mathbb{R}^n$ are called the *motions* of the flow, and the mappings $\varphi^t\colon \mathbb{R}^n \to \mathbb{R}^n$ $(x \mapsto \varphi(x,t))$ for $t \in \mathbb{R}$ are called the *transitions*. If $A \subset \mathbb{R}^n$ and $J \subset \mathbb{R}$, then we write $\varphi(A \times J) = A \cdot J$, in particular, $\varphi(x,t) = x \cdot t$. If $x \in \mathbb{R}^n$, the *orbit* of x is the set $\gamma(x) = x \cdot \mathbb{R}$, or $\varphi_x(\mathbb{R})$. The *positive and negative semi-orbits* of x are the sets $\gamma^+(x) = x \cdot \mathbb{R}^+$ and $\gamma^-(x) = x \cdot \mathbb{R}^-$, respectively.

A set $M \subset \mathbb{R}^n$ is called a *smooth submanifold* of codimension one or a surface with dimension $n-1$ if it can be written as $M = \{x \in U : g(x) = 0\}$, where $U \subset \mathbb{R}^n$ is open, $g\colon U \to \mathbb{R}$ is a smooth function, and $\partial g/\partial x \neq 0$ for all $x \in U$. The submanifold M is said to be *transversal* to the vector field f if the dot product $\partial g/\partial x \cdot f(x) \neq 0$ for all $x \in M$, it is also called a cross section (see [10]).

Let M be a smooth $(n-1)$-dimensional surface in $\mathbb{R}^n$, and denote $\Omega = \mathbb{R}^n \backslash M$. Let $I\colon M \to \Omega$ be a diffeomorphism, then $N = I(M)$ is also a smooth $(n-1)$-dimensional surface. If $x \in M$, we shall denote $I(x)$ by x^+ and say that x *jumps* to x^+. Meanwhile, M is said to be an *impulsive set* and I is called an *impulsive function*. For each $x \in \Omega$, by $M^+(x)$ we mean the set $\gamma^+(x) \cap M$. We can define a function $\psi\colon \Omega \to \mathbb{R}^+ \cup \{+\infty\}$ (the space of extended positive reals) by

$$\psi(x) = \begin{cases} s & \text{if } x \cdot s \in M \text{ and } x \cdot t \notin M \text{ for } t \in [0,s), \\ +\infty & \text{if } M^+(x) = \emptyset. \end{cases}$$

In general, $\psi\colon \Omega \to \mathbb{R}^+ \cup \{+\infty\}$ is not continuous. Ciesielski [2] has established some easy conditions to guarantee the continuity of ψ. In this paper, we always assume that ψ is continuous on Ω.

We define an impulsive system $(\Omega, \widetilde{\varphi})$ by portraying the trajectory of each point in Ω. Let $x \in \Omega$, the *impulsive trajectory* of x is an Ω-valued function $\widetilde{\varphi}_x$ defined on a subset of $\mathbb{R}^+$. If $M^+(x) = \emptyset$, then $\psi(x) = +\infty$, and we set $\widetilde{\varphi}_x(t) =$