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CONVEX HULL DEVIATION AND CONTRACTIBILITY

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Abstract. We study the Hausdorff distance between a set and its convex hull. Let X be a Banach space, define the CHD-constant of the space X as the supremum of this distance over all subsets of the unit ball in X. In the case of finite dimensional Banach spaces we obtain the exact upper bound of the CHD-constant depending on the dimension of the space. We give an upper bound for the CHD-constant in L_p spaces. We prove that the CHD-constant is not greater than the maximum of Lipschitz constants of metric projection operators onto hyperplanes. This implies that for a Hilbert space the CHD-constant equals 1. We prove a characterization of Hilbert spaces and study the contractibility of proximally smooth sets in a uniformly convex and uniformly smooth Banach space.

1. Introduction

Let X be a Banach space. For a set $A \subset X$, we denote by $\partial A, \operatorname{int} A$ and $\operatorname{co} A$ the boundary, interior and convex hull of A, respectively. We use $\langle p, x \rangle$ to denote the value of the functional $p \in X^*$ at the vector $x \in X$. For $R > 0$ and $c \in X$ we denote by $B_R(c)$ a closed ball with center c and radius R. We denote the origin by 0.

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By $\rho(x, A)$ we denote the distance between a point $x \in X$ and a set A. We define the deviation from a set A to a set B as follows:

$$h^+(A, B) = \sup_{x \in A} \rho(x, B). \tag{1.1}$$

For the case $B \subset A$, which takes place below, the deviation $h^+(A, B)$ coincides with the Hausdorff distance between the sets A and B.

Given $D \subset X$, the deviation $h^+(\operatorname{co} D, D)$ is called the *convex hull deviation* (CHD) of D. We define the *CHD-constant* ζ_X of X as

$$\zeta_X = \sup_{D \subset B_1(0)} h^+(\operatorname{co} D, D).$$

Remark 1.1. Directly from our definition it follows that for any normed linear space X we have $1 \leq \zeta_X \leq 2$.

We denote by ℓ_p^n the n-dimensional real vector space endowed with p-norm.

This article presents estimates for the CHD-constant for different spaces and some of its geometrical applications. In particular, for finite-dimensional spaces we obtain the exact upper bound of the CHD-constant depending on the dimension of the space:

Theorem 1.2. *Let X_n be a normed linear space,* $\dim X_n = n \geq 2$, *then $\zeta_{X_n} \leq 2(n-1)/n$. If $X_n = \ell_1^n$ or $X_n = \ell_\infty^n$, then this bound is tight.*

Let the sets P and Q be the intersections of the unit ball with two parallel affine hyperplanes of dimension k, where P is a central section. In Corollary 2.3 we obtain the exact upper bound of the homothety coefficient, that provides covering of Q by P.

The next theorem gives an estimate for the CHD-constant in the L_p spaces, $1 \leq p \leq +\infty$:

Theorem 1.3. *For any $p \in [1, +\infty]$*

$$\zeta_{L_p} \leq 2^{|1/p - 1/p'|}, \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1. \tag{1.2}$$

Theorem 3.3 shows that the CHD-constant is not greater than the maximum of the Lipschitz constants of metric projection operators onto hyperplanes. This implies that for a Hilbert space the CHD-constant equals 1. Besides that, we prove a characterization of a Hilbert space in terms of the CHD-constant. The idea of the proof is analogous to the idea used by A.L. Garkavi in [9].

Theorem 1.4. *The equation $\zeta_X = 1$ holds for a Banach space X if and only if X is a Euclidian space or* $\dim X = 2$.

In addition we study the contractibility of a covering of a convex set with balls.