Topological Methods in Nonlinear Analysis
Volume 49, No. 2, 2017, 511–527
DOI: 10.12775/TMNA.2016.085

STRONGLY RESONANT ROBIN PROBLEMS WITH IDEFINITE AND UNBOUNDED POTENTIAL

Nikolaos S. Papageorgiou — George Smyrlis

Abstract. We consider a Robin boundary value problem driven by the Laplacian plus an indefinite and unbounded potential. We assume that the reaction term of the equation is resonant with respect to the principal eigenvalue and the resonance is strong. Using primarily variational tools we prove two multiplicity theorems producing respectively two and three nontrivial smooth solutions.

1. Introduction

In a recent paper Papageorgiou–Smyrlis [23] studied semilinear resonant Robin problems driven by the Laplacian plus an indefinite and unbounded potential. In [23] the resonance occurs asymptotically at $\pm\infty$ with respect to any nonprincipal, nonnegative eigenvalue of the differential operator $u \mapsto -\Delta u + \xi(z)u$ for all $u \in H^1(\Omega)$ with Robin boundary condition and with $\xi(\,\cdot\,)$ being the indefinite and unbounded potential function.

In the present paper we examine what happens when resonance occurs with respect to the principal eigenvalue $\widehat{\lambda}_1$. More precisely, we investigate the more interesting and more difficult case of "strong resonance"' with respect to $\widehat{\lambda}_1$.

2010 *Mathematics Subject Classification.* 35J20, 35J60, 58E05.
Key words and phrases. Indefinite and unbounded potential; Robin boundary condition; strong resonance; multiple nontrivial solutions; critical groups.

So, let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We study the following semilinear Robin problem:

$$(1.1)\qquad \begin{cases} -\Delta u(z) + \xi(z)u(z) = \widehat{\lambda}_1 u(z) + g(z, u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this problem $\xi \in L^s(\Omega)$ with $s > N$ and so it is in general unbounded. Also $\xi(\,\cdot\,)$ is in general indefinite, that is, $\xi(\,\cdot\,)$ is sign changing. In the reaction term (right-hand side of the equation), $\widehat{\lambda}_1 \in \mathbb{R}$ is the first eigenvalue of the differential operator $u \mapsto -\Delta u + \xi(z)u$, $u \in H^1(\Omega)$, with Robin boundary condition. In this reaction term the perturbation $g(\,\cdot\,,\,\cdot\,)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto g(z,x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto g(z,x)$ is continuous) such that

$$\frac{g(z,x)}{x} \to 0 \quad \text{as } x \to \pm\infty \text{ uniformly for a.a. } z \in \Omega.$$

In the Robin boundary condition, $\beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$. When $\beta \equiv 0$ we recover the Neumann boundary value problem. This makes problem (1.1) resonant with respect to the principal eigenvalue $\widehat{\lambda}_1 \in \mathbb{R}$. In fact, we assume that the resonance is "strong" in the sense that the perturbation $g(z,\,\cdot\,)$ has a smaller rate of increase as $x \to \pm\infty$. More precisely, if $G(z,x) = \int_0^x g(z,s)\,ds$, then there exist functions $G_\pm \in L^1(\Omega)$ such that

$$g(z,x) \to 0 \quad \text{and} \quad G(z,x) \to G_\pm(z) \quad \text{as } x \to \pm\infty \text{ uniformly for a.a. } z \in \Omega.$$

In the terminology introduced by Landesman and Lazer [12], such problems are called "strongly resonant" and are the most interesting class of resonant problems, since as we will see in the sequel, exhibit a partial lack of compactness, that is the energy (Euler) functional of the problem does not satisfy the C-condition (the compactness condition) at all levels.

In the boundary condition $\frac{\partial u}{\partial n}$ denotes the usual normal derivative of $u(\,\cdot\,)$, hence $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ for all $u \in H^1(\Omega)$ with $n(\,\cdot\,)$ being the outward unit normal on $\partial\Omega$. Also the boundary coefficient $\beta(\,\cdot\,) \in W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$. When $\beta \equiv 0$ we recover the Neumann problem. So, our framework here incorporates as a special case Neumann problems.

In the past resonant problems were studied primarily in the context of Dirichlet problems with zero potential (that is, $\xi \equiv 0$) and not for strongly resonant equations. We mention the works of Bartsch and Wang [5], Castro, Cossio and Velez [6], Hofer [10], Liu and Li [13]. More recently, the study was extended to resonant Neumann problems again with zero potential. In this direction we mention the works of Gasinski and Papageorgiou [9], Motreanu, Motreanu and Papageorgiou [15]. Equations with indefinite and unbounded potential were