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# ON GLOBAL INVERSE THEOREMS

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Abstract. Since the Hadamard Theorem, several metric and topological conditions have emerged in the literature to date, yielding global inverse theorems for functions in different settings. Relevant examples are the mappings between infinite-dimensional Banach–Finsler manifolds, which are the focus of this work. Emphasis is given to the nonlinear Fredholm operators of nonnegative index between Banach spaces. The results are based on good local behavior of $f$ at every $x$, namely, $f$ is a local homeomorphism or $f$ is locally equivalent to a projection. The general structure includes a condition that ensures a global property for the fibres of $f$, ideally expecting to conclude that $f$ is a global diffeomorphism or equivalent to a global projection. A review of these results and some relationships between different criteria are shown. Also, a global version of the Graves Theorem is obtained for a suitable submersion $f$ with image in a Banach space: given $r > 0$ and $x_0$ in the domain of $f$ we give a radius $\varrho(r) > 0$, closely related to the hypothesis of the Hadamard Theorem, such that $B_\varrho(f(x_0)) \subset f(B_r(x_0))$.

## 1. Introduction

Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable mapping. Consider the nonlinear system $f(x) = y$. In his seminal article [23] of 1906, Hadamard establishes an existence and unicity condition for a nonlinear system $f(x) = y$ in terms of

$$\mu(x) = \min_{v\neq 0} \frac{\|\mathrm{J}f(x)v\|_2}{\|v\|_2},$$

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where $\mathrm{J}f(x)$ is the Jacobian matrix of $f$ at $x$ and $\|\cdot\|_2$ is the Euclidean norm in $\mathbb{R}^n$. If $n=1$, Hadamard points out that if the first derivative is positive at every point $x \in \mathbb{R}^n$ then the nonlinear system has at most one solution, but its existence cannot always be assured unless $\int_{-\infty}^{a} f'(x)\,dx = \infty$ and $\int_{a}^{\infty} f'(x)\,dx = \infty$. For $n > 1$, Hadamard asserts that it is not enough to replace the derivative $f'(x)$ in the above integrals by the Jacobian determinant at $x$, but the appropriate condition for global inversion is

$$\int_0^\infty \min_{\|x\|=\rho} \mu(x)\,d\rho = \infty,$$

referred to in this paper as the *Hadamard integral condition*, provided that $\mu(x) > 0$ for all $x \in \mathbb{R}^n$. Hadamard also conjectures the "properness criterion" established in the mid thirties by Banach and Mazur [6] and by Cacciopoli [11], which asserts that a map between Banach spaces is a proper local homeomorphism if and only if $f$ is a global one. The properness condition has been widely reported and extended in the literature in different frameworks. The same can be said of the Hadamard integral condition and similar metric criteria in settings where a correct generalization of $\mu(x)$ can be established, sometimes devoted to special cases. The properness criterion was relaxed to closedness by Browder [10] in the context of topological spaces. Besides, the Hadamard Theorem was extended to the infinite-dimensional setting by Lévy [35], who considered the case of smooth mappings between Hilbert spaces. In the late sixties, John [29] also obtained an extension of the Hadamard integral condition for nonsmooth mappings $f$ between Banach spaces in terms of the lower scalar Dini derivative of $f$. For the proof, he used the prolongation of local inverses of $f$ along lines. Soon after, Plastock [47] introduced a limiting property for lines, called the *condition* $L$, analogous to the continuation property used by Rheinboldt [54] in a more abstract context. Plastock proved that a local homeomorphism $f$ satisfies the condition $L$ if and only if it is a global homeomorphism. He also showed that the properness, closedness, and the Hadamard integral condition all imply the condition $L$. Since then, the condition $L$ has proved to be quite useful in global inversion theorems. In the same vein, Ioffe [26] extended the Hadamard Theorem in terms of the so-called surjection constant of the mapping $f$ making use of the condition $L$. More recently, this approach was used to give necessary and sufficient conditions for a map $f$ to be a global homeomorphism for a large class of metric spaces with nice local structure, which includes Banach–Finsler manifolds [22]. As a consequence, an extension of the Hadamard Theorem is obtained in terms of a metric version of $\mu(x)$, a kind of lower scalar Dini derivate. In [19] an estimation of the domain of invertibility around a point is provided for a local homeomorphism between length metric space, inspired by the aforementioned work of John [29]. In finite-dimensional case, analogous results were