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EXISTENCE OF SOLUTIONS TO A SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEM WITH AUGMENTED MORSE INDEX BIGGER THAN TWO

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Abstract. Building on the construction of least energy sign-changing solutions to variational semilinear elliptic boundary value problems introduced in [5], we prove the existence of a solution with *augmented Morse index* at least three when a sublevel of the corresponding action functional has nontrivial topology. We provide examples where the set of least energy sign changing solutions is disconnected, hence has nontrivial topology.

1. Introduction

We consider the existence of solutions to the equation

$$\begin{cases} -\Delta u = f(u) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded subset of $\mathbb{R}^n$, its boundary $\partial\Omega$ is Lipschitizian, and f is a differentiable function.

The solvability of (1.1) has motivated fundamental developments in critical point theory in the last fifty years. The *mountain pass lemma* was developed in [2]

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by A. Ambrosetti and P.H. Rabinowitz in order the prove the existence of positive solutions to (1.1). The *saddle point principle* proved by P.H. Rabinowitz in [14] was motivated by the solvability of (1.1) in the presence of resonance. In [16], Z.-Q. Wang studied connections between mountain passes in order to establish the existence of solutions to (1.1) given by critical points with *augmented Morse* index greater that or equal to two, see Definition 1.1. Refinements of the arguments in [16] led to the existence of solutions to (1.1) that change sign exactly once and have Morse index 2, see [5]. This paper builds on the constructions in [5] obtaining solutions with *augmented Morse index* greater than two, see Theorem 1.3.

We assume that there exist $A > 0$ and $p \in [1, (N+2)/(N-2))$ such that

$$|f'(u)| \leq A(|u|^{p-1} + 1) \quad \text{for } u \text{ in } \mathbb{R}. \tag{1.2}$$

Let $\lambda_1 < \lambda_2 \leq \ldots \to +\infty$ denote the eigenvalues of $-\Delta$ with Dirichlet boundary condition in Ω. We also assume the following hypotheses:

- (h_1) $f(0) = 0$, $f'(0) < \lambda_1$.
- (h_2) $\lim_{|u|\to\infty} f(u)/u = \infty$, i.e. f is *superlinear*.
- (h_3) $f'(u) > f(u)/u$ for all $u \neq 0$.
- (h_4) There exist $m \in (0,1)$ and $\rho > 0$ such that $(m/2)uf(u) - F(u) \geq 0$ for $|u| > \rho$, where $F(u) = \int_0^u f(s)\,ds$.

From these hypotheses it follows that there exists a positive constant K such that

$$\alpha t f(\alpha t) \geq K\alpha^{2/m} t f(t) \quad \text{for } \alpha \geq 1 \text{ and } |t| > \rho. \tag{1.3}$$

Let $\mathbb{H}(\Omega) := \mathbb{H}$ denote the Sobolev space of functions vanishing in $\partial\Omega$ and having square integrable first order partial derivatives. The solutions to (1.1) are the critical points of the functional $J\colon \mathbb{H} \to \mathbb{R}$,

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(u)\,dx, \tag{1.4}$$

where $F(t) = \int_0^t f(s)\,ds$. The functional J is of class C^2. Its gradient is given by

$$\langle \nabla J(u), v\rangle = \int_\Omega (\nabla u \cdot \nabla v - f(u)v)\,dx, \tag{1.5}$$

for all $u, v \in \mathbb{H}$, and its Hessian is given by

$$\langle D^2 J(u)v, w\rangle = \int_\Omega (\nabla v \cdot \nabla w - f'(u)vw)\,dx, \tag{1.6}$$

for all $u, v, w \in \mathbb{H}$.

Definition 1.1. If u is a critical point of J, we will say that u has *Morse index k* if $D^2J(u)$ has exactly k negative eigenvalues, counting multiplicity; and