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ISOLATED SETS, CATENARY LYAPUNOV FUNCTIONS AND EXPANSIVE SYSTEMS

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Abstract. It is a paper about models for isolated sets and the construction of special hyperbolic Lyapunov functions. We prove that after a suitable surgery every isolated set is the intersection of an attractor and a repeller. We give linear models for attractors and repellers. With these tools we construct hyperbolic Lyapunov functions and metrics around an isolated set whose values along the orbits are catenary curves. Applications are given to expansive flows and homeomorphisms, obtaining, among other things, a hyperbolic metric on local cross sections for an arbitrary expansive flow on a compact metric space.

1. Introduction

A hanging chain describes a curve that is called *catenary*. Galileo's first approximation to this curve was a *parabola* but, after the development of the infinitesimal calculus, this curve was shown to be related with hyperbolic cosines and it is not parabolic. As shown in [11] hyperbolic cosines also appear in the expression of the catenary, even if gravity is not assumed to be constant but associated with a varying potential $-1/r$, which is a more realistic model of gravity.

In the present paper we consider dynamical systems and the purpose is to construct Lyapunov functions whose values along the orbits have the harmony of a hanging chain. They will be called *catenary functions* and as we will see they

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are hyperbolic Lyapunov functions. We will show that every isolated set admits a catenary function defined on an isolating neighbourhood. The construction of these functions is based on two results. First, in Theorem 2.16 we show that after a cut and paste procedure every isolated set is the intersection of an attractor with a repeller. Second, we prove in Theorem 3.4 that attractors and repellers have linear models. Precise definitions and statements are given in the corresponding sections. The applications to expansive systems, given in Sections 5 and 6, are natural if an isolated set is found. As we will see, the difficulty of this task depends on the form of expansivity that we consider and if we are dealing with flows or homeomorphisms.

Let us give an example illustrating the main concepts of the paper. Consider the following differential equations in the plane:

$$\begin{cases} \dot{x} = x, \\ \dot{y} = -y. \end{cases} \tag{1.1}$$

This system determines a hyperbolic equilibrium point of saddle type at the origin. Its solutions are given by $\phi_t(x, y) = (xe^t, ye^{-t})$. Consider the norm $\mathcal{L}(x, y) = |x| + |y|$. We have that

$$\mathcal{L}(\phi_t(x, y)) = |x|e^t + |y|e^{-t} \quad \text{and} \quad \dot{\mathcal{L}}(x, y) = -|x| + |y|.$$

Consequently $\ddot{\mathcal{L}} = \mathcal{L}$. As usual, the dots indicate time derivatives. When a function satisfies $\ddot{\mathcal{L}} = \mathcal{L}$ we call it a *catenary function* for the flow ϕ. If d is the distance induced by the norm $\mathcal{L}$, we have that $\ddot{\mathrm{d}} = \mathrm{d}$, and we call it a *catenary metric*. In this example $\Lambda = \{(0, 0)\}$ is an isolated set because there is a compact neighbourhood $N = [-1, 1] \times [-1, 1]$ of Λ such that the whole orbit of a point is contained in N if and only if the point is in Λ. In Theorem 4.2 we will show that every isolated point admits a catenary metric and that every isolated set admits a catenary pseudo-metric vanishing on pairs of points of the isolated set. This result will be proved for partial flows on metric spaces. In Theorem 4.4 we show that every isolated set admits a catenary function $\mathcal{L}$.

The construction of Lyapunov functions is a classical tool for proving the asymptotic stability of an equilibrium point of a differential equation. In [21], Massera considered the converse problem in the setting of autonomous or periodic differential equations in $\mathbb{R}^n$. He showed that every asymptotically stable singular point admits a positive and decreasing Lyapunov function of class C^1. From a topological viewpoint, i.e. Lyapunov functions of class C^0, simpler constructions can be made even on metric spaces, see for example [3], [4], [7], [15], [29]. In Section 3 we will show that every attractor admits a positive and decreasing Lyapunov function $\mathcal{L}$ satisfying $\dot{\mathcal{L}} = -\mathcal{L}$, which is a key step in the construction of a catenary function for an isolated set. In Theorem 4.14 we apply Massera's theorem to construct a differentiable Lyapunov function $\mathcal{L}$ for