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RESONANT ROBIN PROBLEMS WITH INDEFINITE AND UNBOUNDED POTENTIAL

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Abstract. We study a semilinear Robin problem with an indefinite and unbounded potential and a reaction term which asymptotically at $\pm\infty$ is resonant with respect to any nonprincipal nonnegative eigenvalue. We prove two multiplicity theorems producing three and four nontrivial solutions respectively. Our approach uses variational methods based on the critical point theory, truncation and perturbation techniques, and Morse theory (critical groups).

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following Robin problem:

$$(1.1)\qquad -\Delta u(z) + \xi(z)u(z) = f(z, u(z)) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} + \beta(z)u = 0 \quad \text{on } \partial\Omega.$$

Here $\xi \in L^s(\Omega)$ with $s > N$ and in general it is sign-changing (indefinite potential) and unbounded from below. The aim of this work is to prove multiplicity theorems for problem (2.1) when the reaction f asymptotically as the second argument tends to $\pm\infty$ interacts with the spectrum of the differential operator $-\Delta u + \xi(z)u$ with Robin boundary conditions (resonant problems). In particular, we show that if $f(z,\,\cdot\,)$ at $\pm\infty$ is resonant with respect to any nonnegative and nonprincipal eigenvalue of the differential operator, then problem (2.1)

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admits at least three nontrivial solutions. Subsequently, by strengthening the conditions on $f(z,\,\cdot\,)$, we show the existence of at least four nontrivial solutions.

Multiplicity theorems for resonant Dirichlet problems with no potential term (that is, $\xi = 0$), were proved by Bartsch and Wang [3], Castro, Cossio and Velez [4], Hofer [8], Liu and Li [10]. For resonant Neumann problems also with zero potential term, we mention the works of Gasinski and Papageorgiou [7], Motreanu, Motreanu and Papageorgiou [12] and Tang and Wu [19]. Recently there have been some multiplicity theorems for elliptic problems with an indefnite and unbounded potential. We mention the works of Kyritsi and Papageorgiou [9], Papageorgiou and Papalini [15] (Dirichelt problems) and Papageorgiou and Smyrlis [17] (Neumann problems). For the Robin problem, there is only the recent work of Papageorgiou and Radulescu [16], who examine a class of parametric coercive equations.

Our approach combines variational methods based on the critical point theory together with truncation and perturbation techniques and Morse theory (critical groups).

2. Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle\,\cdot\,,\,\cdot\,\rangle$ we denote the duality brackets for the pair (X^*,X). Given $\varphi \in C^1(X,\mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following is true:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1}$ is bounded and

$$(1+||u_n||)\varphi'(u_n) \to 0 \quad \text{in } X^* \text{ as } n\to\infty,$$

admits a strongly convergent subsequence".

This is a compactness-type condition on the functional φ, it is more general than the usual "Palais–Smale condition" and it suffices to prove a deformation theorem and from it to derive the minimax theory of the critical values of φ. A basic result in this theory is the well-known "mountain pass theorem", which we state below in a slightly more general form (see, for example, Gasinski and Papageorgiou [6, p. 648]).

Theorem 2.1. *If $\varphi \in C^1(X)$ satisfies the C-condition, $u_0,u_1 \in X$, $r>0$ are such that $||u_0-u_1||>r$ and*

$$\max\{\varphi(u_0),\varphi(u_1)\} < \inf\,[\varphi(u) : ||u-u_0|| = r] = m_r,$$

$$c = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1}\varphi(\gamma(t)), \quad \text{where } \Gamma = \{\gamma \in C([0,1],X) : \gamma(0)=u_0,\ \gamma(1)=u_1\},$$

then $c \geq m_r$ and c is a critical value of φ.

For problem (2.1) the relevant function spaces are the Sobolev space $H^1(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue spaces $L^p(\partial\Omega)$ $(1\leq p\leq\infty)$.