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ISOMORPHIC EXTENSIONS AND APPLICATIONS

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Abstract. If $\pi\colon (X,T) \to (Z,S)$ is a topological factor map between uniquely ergodic topological dynamical systems, then (X,T) is called an isomorphic extension of (Z,S) if π is also a measure-theoretic isomorphism. We consider the case when the systems are minimal and we pay special attention to equicontinuous (Z,S). We first establish a characterization of this type of isomorphic extensions in terms of mean equicontinuity, and then show that an isomorphic extension need not be almost one-to-one, answering questions of Li, Tu and Ye.

1. Introduction

Throughout, by a topological dynamical system (denoted (X,T), or similarly) we will always mean the action of a homeomorphism T on an infinite compact metric space X. Although many of our results apply to the noninvertible case, for simplicity, we focus on invertible systems only.

By a *topological model* of an (invertible) ergodic measure-preserving transformation (*m.p.t.* for short) $(\Omega,\mathcal{F},\nu,S)$, we will mean any uniquely ergodic topological system (X,T) (with the unique invariant measure μ), such that $(X,\mathsf{Borel}(X),\mu,T)$ and $(\Omega,\mathcal{F},\nu,S)$ are measure-theoretically isomorphic.

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The celebrated Jewett–Krieger theorem asserts that every invertible ergodic m.p.t. has a strictly ergodic (i.e. minimal and uniquely ergodic) topological model. In this work, we are interested in the relations between various strictly ergodic models of the same ergodic system. More specifically we will focus on the situation when one model is a topological extension of another. Also, of particular interest to us is the case when the underlying m.p.t. belongs to the class of the simplest ergodic systems, namely those with discrete spectrum. Recall that a "standard model" of a discrete spectrum ergodic system has the form of a rotation, by a topological generator, on a compact monothetic group or, equivalently, is a minimal equicontinuous system (Halmos–von Neumann Theorem, see e.g. [9, Chapter 1, Section 2]).

We will provide a natural classification of all the strictly ergodic topological models of an ergodic system with discrete spectrum which topologically extend the standard model. Before we formulate our results, we need to establish some basic terminology. We will assume that the reader is familiar with the textbook notions of a factor (in particular, maximal equicontinuous factor), extension, isomorphism, ergodicity, minimality, discrete spectrum, etc. as well as with the definition of lower and upper Banach density of a subset of integers and some related notions. When dealing with an m.p.t. which arises from a topological dynamical system equipped with an invariant probability measure, we will always assume that the sigma-algebra in question is the Borel sigma-algebra completed with respect to that measure (and skip it in the notation of the system). The reader is referred to Furstenberg's classical monograph [5], or to [9] for more details and background.

Most of the time we will consider a pair of uniquely ergodic topological dynamical systems (X,T) and (Z,S). In such a case, μ and ν will always denote the unique invariant measures on X and Z, respectively. Notice that a topological factor of a uniquely ergodic system is uniquely ergodic as well.

Definition 1.1. We say that (X,T) is an *isomorphic extension* of (Z,S) if (X,T) is uniquely ergodic and there exists a topological factor map $\pi\colon X\to Z$ which is, at the same time, a measure-theoretic isomorphism between (X,μ,T) and (Z,ν,S).

It is clear that an isomorphic extension of a topological model of some (ergodic) m.p.t. is another topological model of the same m.p.t. Notice that the requirement in the definition is stronger than just assuming that (X,T) is a topological extension of (Z,S) and that the systems (X,μ,T) and (Z,ν,S) are measure-theoretically isomorphic. The measure-theoretic isomorphism must be realized by the same map which establishes the topological factor relation. Isomorphic extensions are important because they carry over many "hybrid properties" of the base system (Z,S) to the extended system (X,T). The adjective