

**GLOBAL PHASE PORTRAITS
OF KUKLES DIFFERENTIAL SYSTEMS
WITH HOMOGENOUS POLYNOMIAL NONLINEARITIES
OF DEGREE 5 HAVING A CENTER**

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(Submitted by J. Llibre)

ABSTRACT. We provide 22 different global phase portraits in the Poincaré disk of all centers of the so-called Kukles polynomial differential systems of the form $\dot{x} = -y$, $\dot{y} = x + Q_5(x, y)$, where Q_5 is a real homogeneous polynomial of degree 5 defined in \mathbb{R}^2 .

1. Introduction and statement of the main result

Consider a system of the form

$$(1.1) \quad \begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + Q_n(x, y), \end{aligned}$$

where the dot denotes the derivative with respect to the independent variable t and Q_n is a real homogeneous polynomial of degree n . A system of this form was

2010 *Mathematics Subject Classification*. Primary: 34C05.

Key words and phrases. Centers; Kukles; polynomial vector fields; phase portrait; Poincaré disk.

The final version of this paper will be submitted for publication elsewhere.

The first author was supported in part by the MINECO/FEDER grant MTM2008-03437, MINECO grant MTM2013-40998-P, the AGAUR grant 2014SGR-568, the ICREA Academia, grants FP7-PEOPLE-2012-IRSES 318999 and 316338, the MINECO/FEDER grant UNAB13-4E-1604, and the CAPES grant 88881.030454/2013-01 from the program CSF-PVE.

The second author was supported by CAPES.

called a *Kukles polynomial differential system* by Giné in [4]. In [11], Volokitin and Ivanov raised the following open question about such differential systems:

Open Question. *Is it true that a differential system of the form (1.1) with polynomial nonlinearity Q_n of degree higher than 2 has a center at the origin if and only if its vector field is symmetric with respect to one of the coordinate axes?*

For degrees $n = 2, 3$ the authors of the open problem knew that the answer is positive. A positive answer to this question for degrees $n = 4, 5$ was given in [4]. In [5], Giné, Llibre and Valls proved that this conjecture holds for $n \geq 5$ odd, and in [6] they proved the conjecture for $n \geq 6$ even.

More precisely, in [5] the authors proved that for a planar differential system of the form (1.1), the origin is a center if and only if its vector field is symmetric with respect to one of the coordinate axes.

In this work we use this information to classify topologically the global phase portraits of differential systems

$$(1.2) \quad \begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + ax^5 + bx^3y^2 + cxy^4, \end{aligned}$$

in the Poincaré disk, where a, b, c are real parameters. Note that systems (1.2) are invariant under the change of coordinates $(t, x, y) \mapsto (-t, -x, y)$. That is, the phase portrait is symmetric with respect to the y -axis.

In [1], [8], [12]–[14], there were classified the global phase portraits of linear systems with homogeneous nonlinearities of degree 3, so in particular the phase portraits of the Kukles systems (1.1) of degree 3. Moreover, in [2] and [7], there have been classified the global phase portraits of the Kukles systems (1.1) of degree 4.

Our main result is the following one.

THEOREM 1.1. *The Poincaré compactification of system (1.2) is topologically equivalent to the Poincaré compactification of one of the following systems:*

- (i) $\dot{x} = -y, \dot{y} = x,$
- (ii) $\dot{x} = -y, \dot{y} = x + Axy^4,$
- (iii) $\dot{x} = -y, \dot{y} = x + Ax^3y^2,$
- (iv) $\dot{x} = -y, \dot{y} = x + Ax^3y^2 + Cxy^4,$
- (v) $\dot{x} = -y, \dot{y} = x + Ax^5,$
- (vi) $\dot{x} = -y, \dot{y} = x + Ax^5 + Cxy^4,$
- (vii) $\dot{x} = -y, \dot{y} = x + Ax^5 + Bx^3y^2,$
- (viii) $\dot{x} = -y, \dot{y} = x + Ax^5 + Bx^3y^2 + Cxy^4,$

for an appropriate choice of $A \in \{-1, 1\}$, $B, C \in \mathbb{R} \setminus \{0\}$. The phase portraits in the Poincaré disk of systems (i)–(viii) are topologically equivalent to one of 22

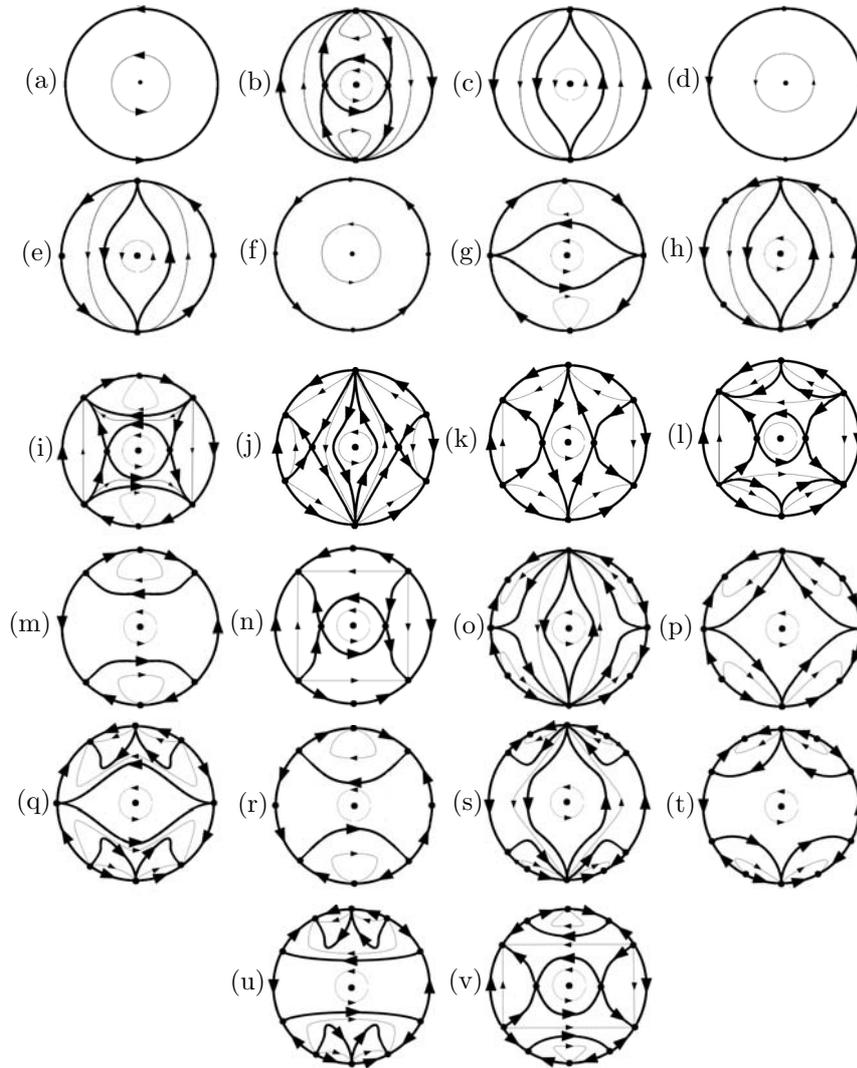


FIGURE 1. Phase portrait of systems (1.2) in the Poincaré disk.

phase portraits presented in Figure 1. The possibilities for the phase portraits of systems (1.2) and the corresponding systems (i)–(viii) are provided in Table 1.

With Program P4 we have checked that all possibilities hold.

In the phase portraits of our figures the separatrices are thicker than orbits of the canonical regions, and the thickest lines denote lines of singularities. For more details about Program P4 see Chapters 9 and 10 of [3]. We present the relation between the parameters of system (1.2) and the parameters of its corresponding normal form in Table 2.

Phase Portrait of Figure 1	System (1.2)	Normal Form (i)–(viii)
(a)	$a = b = c = 0$	(i)
(b)	$a < 0, b = c = 0$	(v), $A = -1$
	$a < 0, b > 0, c < 0, b^2 - 4ac < 0$	(viii), $A = -1, B > 0, C < 0, B^2 + 4C < 0$
	$a < 0, b < 0, c < 0$	(viii), $A = -1, B < 0, C < 0$
	$a < 0, b < 0, c = 0$	(vii), $A = -1, B < 0$
	$a < 0, b = 0, c < 0$	(vi), $A = -1, C < 0$
(c)	$a > 0, b < 0, c > 0, b^2 - 4ac < 0$	(viii), $A = 1, B < 0, C > 0, B^2 - 4C < 0$
	$a > 0, b > 0, c > 0$	(viii), $A = 1, B > 0, C > 0$
	$a > 0, b = 0, c > 0$	(vi), $A = 1, C > 0$
(d)	$a > 0, b = c = 0$	(v), $A = 1$
	$a > 0, b > 0, c = 0$	(vii), $A = 1, B > 0$
(e)	$a = b = 0, c > 0$	(ii), $A = 1$
	$a = 0, b > 0, c > 0$	(iv), $A = 1, C > 0$
(f)	$a = 0, b > 0, c = 0$	(iii), $A = 1$
(g)	$a = b = 0, c < 0$	(ii), $A = -1$
	$a = 0, b < 0, c = 0$	(iii), $A = -1$
	$a = 0, b < 0, c < 0$	(iv), $A = -1, C < 0$
(h)	$a > 0, b < 0, c > 0, b^2 - 4ac = 0$	(h), $A = 1, B < 0, C > 0, B^2 - 4C = 0$
(i)	$a < 0, b > 0, c < 0, b^2 - 4ac = 0$	(h), $A = -1, B > 0, C < 0, B^2 + 4C = 0$
(j)–(l)	$a < 0, b > 0, c > 0$	(viii), $A = -1, B > 0, C > 0$
	$a < 0, b < 0, c > 0$	(viii), $A = -1, B < 0, C > 0$
	$a < 0, b = 0, c > 0$	(vi), $A = -1, C > 0$
(m)	$a > 0, b \neq 0, c < 0$	(viii), $A = 1, C < 0$
	$a > 0, b < 0, c = 0$	(vii), $A = 1, B < 0$
	$a > 0, b = 0, c < 0$	(vi), $A = 1, C < 0$
(n)	$a < 0, b > 0, c = 0$	(vii), $A = -1, B > 0$
(o)–(q)	$a = 0, b < 0, c > 0$	(iv), $A = -1, C > 0$
(r)	$a = 0, b > 0, c < 0$	(iv), $A = 1, C < 0$
(s)–(u)	$a > 0, b < 0, c > 0, b^2 - 4ac > 0$	(viii), $A = 1, B < 0, C > 0, B^2 - 4C > 0$
(v)	$a < 0, b > 0, c < 0, b^2 - 4ac > 0$	(viii), $A = -1, B > 0, C < 0, B^2 + 4C > 0$

TABLE 1. Classification of phase portraits and normal forms of systems (1.2).

This work is organized in the following way. In Section 2, we introduce basic definitions and results. Since we use the Poincaré compactification to study the orbits near the infinity, we present a summary of this technique in Section 3. The Markus–Neumann–Peixoto Theorem implies that to determine the phase portrait of a given planar vector field, it is sufficient to determine the behavior of a completed separatrix skeleton. We present the ingredients of this result in Section 4. We study the finite and infinite singular points of system (1.2) in Sections 5 and 6, respectively. Finally, we prove Theorem 1.1 in Section 7.

2. Definitions and useful results

Let $X: U \rightarrow \mathbb{R}^2$ be a vector field defined on the open subset of \mathbb{R}^2 . Suppose that $(x_0, y_0) \in U$ is a singular point of X . We say that (x_0, y_0) is a *hyperbolic singular point* of X when the real parts of both eigenvalues of $DX(x_0, y_0)$ are different from zero. If exactly one of the eigenvalues of $DX(x_0, y_0)$ is different from zero then (x_0, y_0) is a *semi-hyperbolic singular point* of X . If (x_0, y_0) is a hyperbolic or a semi-hyperbolic singular point of X then it is called an *elementary singularity of X* else, (x_0, y_0) is a *non-elementary singular point of X* .

Here we use definitions of (stable and unstable) node, saddle point, elliptic, hyperbolic and parabolic sector (attracting or repelling) as in [3].

In this work, the topological behavior of the flow near a hyperbolic singular point of X is described by Theorem 2.15 of [3]. To determine the behavior of the flow near a semi-hyperbolic singular point we use Theorem 2.19 of [3].

If a non-elementary singular point (x_0, y_0) of X is such that both eigenvalues of $DX(x_0, y_0)$ are zero, but $DX(x_0, y_0)$ is not zero, then (x_0, y_0) is called a *nilpotent singularity*. If $DX(x_0, y_0)$ is the null matrix then (x_0, y_0) is a *linearly zero singularity*. The study of nilpotent singular points is performed by the change of coordinates of the form $x \mapsto x, y \mapsto xy$, and $x \mapsto xy, y \mapsto y$, called the *homogeneous blow-up* or, simply *blow-up*, and it is summarized in Theorem 3.5 of [3].

But, all nonelementary singularities of system (1.2) are linearly zero. To study these singular points we will use the *quasihomogeneous blow-up*, that is, a change of variables of the form

$$\begin{aligned} (x, y) &\mapsto (x^\alpha, x^\beta y), && \text{positive } x\text{-direction,} \\ (x, y) &\mapsto (-x^\alpha, x^\beta y), && \text{negative } x\text{-direction,} \\ (x, y) &\mapsto (xy^\alpha, y^\beta), && \text{positive } y\text{-direction,} \\ (x, y) &\mapsto (xy^\alpha, -y^\beta), && \text{negative } y\text{-direction,} \end{aligned}$$

where α, β are positive integers. For quasihomogeneous blow-ups in the x -direction (respectively y -direction), when α (respectively β) is odd, the information obtained in the positive x -direction (respectively y -direction) is useful for the negative x -direction (respectively y -direction). To avoid successive blow-ups we choose α and β using the Newton diagram (see p. 104 of [3]).

3. Poincaré compactification

In this section we present a technique useful for studying the behavior of trajectories of polynomial vector fields near the infinity. See p. 149 of [3] for more details.

Consider the polynomial vector field X defined on \mathbb{R}^2 by the system

$$\begin{aligned}\dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y),\end{aligned}$$

where P and Q are real polynomial functions defined on \mathbb{R}^2 . We define the *degree* d of X as the maximum between the degrees of P and Q .

We consider the notation $\mathbb{S}^2 = \{(z_1, z_2, z_3) \in \mathbb{R}^3; z_1^2 + z_2^2 + z_3^2 = 1\}$, $\mathbb{S}^1 = \{(z_1, z_2, z_3) \in \mathbb{S}^2; z_3 = 0\}$ and identify \mathbb{R}^2 with the tangent plane π of \mathbb{S}^2 at the north pole $(0, 0, 1)$. If we consider the central projection of π in \mathbb{S}^2 , we obtain a tangent vector field defined on $\mathbb{S}^2 \setminus \mathbb{S}^1$ such that the infinity points of π are projected in \mathbb{S}^1 .

This vector field is symmetric about the center of \mathbb{S}^2 and, in general, is unbounded near \mathbb{S}^1 . After a multiplication by an appropriate factor, the resultant vector field admits an analytical extension to \mathbb{S}^2 . Due to the symmetry, it is sufficient to consider the extended vector field defined only in the closed northern hemisphere H of \mathbb{S}^2 . The orthogonal projection of H into the disk $\{(z_1, z_2, z_3) \in \mathbb{R}^3; z_1^2 + z_2^2 \leq 1, z_3 = 0\}$ is called the *Poincaré disk*.

More precisely, we denote the northern and southern hemisphere of \mathbb{S}^2 by $H^+ = \{(z_1, z_2, z_3) \in \mathbb{S}^2; z_3 > 0\}$ and $H^- = \{(z_1, z_2, z_3) \in \mathbb{S}^2; z_3 < 0\}$, respectively. The central projections of $\pi := \{(z_1, z_2, z_3) \in \mathbb{R}^3; z_3 = 1\}$ in H^+ and H^- are defined, respectively, by

$$\begin{aligned}f^+ : \pi &\rightarrow H^+, & z &\mapsto \frac{1}{\Delta(z)}(z_1, z_2, 1), \\ f^- : \pi &\rightarrow H^-, & z &\mapsto \frac{1}{\Delta(z)}(-z_1, -z_2, -1),\end{aligned}$$

where $\Delta(z) = \sqrt{z_1^2 + z_2^2 + 1}$, $z \in \pi$. Define the vector field \overline{X} on $\mathbb{S}^2 \setminus \mathbb{S}^1$ by

$$\overline{X}(w) = \begin{cases} Df^+(z)X(z) & \text{if } w = f^+(z), \\ Df^-(z)X(z) & \text{if } w = f^-(z). \end{cases}$$

In general \overline{X} is unbounded in a neighbourhood of \mathbb{S}^1 , then there is no extension of \overline{X} to \mathbb{S}^2 . But we can prove that if we multiply \overline{X} by w_3^d then the resultant vector field has a unique analytical extension to \mathbb{S}^2 , called the *Poincaré compactification* of X and denoted by $p(X)$.

Notice that, in general, in each hemisphere we have that $p(X)$ is C^ω -equivalent, but not C^ω -conjugated, to X . So we can study the behavior of trajectories of X near a singularity using the correspondent singularities of $p(X)$. But it may be that $p(X)$ has singularities in \mathbb{S}^1 . A singular point of $p(X)$ which belongs to $\mathbb{S}^2 \setminus \mathbb{S}^1$ (respectively \mathbb{S}^1) is called a *finite* (respectively *infinite*) *singular point* of X . We can prove that \mathbb{S}^1 is invariant under the flow of $p(X)$.

From the definition of f^+ and f^- it follows that $p(X)$ is symmetric with respect to the origin, so to study the trajectories of X it is sufficient to study $p(X)$ in the closed northern hemisphere $H^+ \cup \mathbb{S}^1$.

To obtain expressions of $p(X)$ in local coordinates, we consider the charts of the sphere \mathbb{S}^2 . For $j = 1, 2, 3$ define $U_j = \{(z_1, z_2, z_3) \in \mathbb{S}^2; z_j > 0\}$, $V_j = \{(z_1, z_2, z_3) \in \mathbb{S}^2; z_j < 0\}$ and $\varphi_j: U_j \rightarrow \mathbb{R}^2, \psi_j: V_j \rightarrow \mathbb{R}^2$ given by

$$\varphi_1(z) = -\psi_1(z) = \frac{(z_2, z_3)}{z_1}, \quad \varphi_2(z) = -\psi_2(z) = \frac{(z_1, z_3)}{z_2}, \quad \varphi_3(z) = \frac{(z_1, z_2)}{z_3}.$$

If we denote by (u, v) the value of φ_j or ψ_j at the point z we can prove that the expression of $p(X)$ in the chart (U_1, φ_1) is given by

$$\dot{u} = v^d \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right) \right], \quad \dot{v} = -v^{d+1}P\left(\frac{1}{v}, \frac{u}{v}\right).$$

The expression of $p(X)$ in the chart (U_2, φ_2) is

$$\dot{u} = v^d \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \quad \dot{v} = -v^{d+1}Q\left(\frac{u}{v}, \frac{1}{v}\right),$$

and the expression of $p(X)$ in the chart (U_3, φ_3) is

$$\dot{u} = P(u, v), \quad \dot{v} = Q(u, v).$$

Finally, for each $j = 1, 2, 3$, the expression of $p(X)$ in the chart (V_j, ψ_j) is the expression of $p(X)$ in the chart (U_j, φ_j) multiplied by the factor $(-1)^{d-1}$.

Using this notation we observe that $(u, v) \in U_j$ is a infinite singular point of X if and only if the expression of $p(X)$ in the chart (U_j, φ_j) vanishes in (u, v) and $v = 0$.

Notice that if z is a infinite singular point of X then $-z$ is also a infinite singular point of X . In this case, from the expressions of $p(X)$ in local coordinates it follows that the behavior of the flow near $-z$ can be determined by the behavior of the flow near z , because the flow near $-z$ differs from the flow near z by the factor $(-1)^{d-1}$. Then the study of $p(X)$ in the charts (V_j, ψ_j) , $j = 1, 2, 3$, is superfluous.

Moreover, notice that if z is an infinite singular point of X with $z \in U_2$, $z \neq (0, 1, 0)$, then $z \in U_1 \cup V_1$. It follows that to study all infinite singular points of X , it is sufficient to study the singularities of $p(X)$ in U_1 and the origin of U_2 .

4. Markus–Neumann–Peixoto Theorem

In this section we present the Markus–Neumann–Peixoto Theorem which reduces the work of determining the phase portrait of a given planar vector fields to the determination of their separatrices (see definition below) and a finite number of special orbits. More details in [9], [10] or p. 33 of [3]. It is sufficient to consider C^1 -vector fields.

Let X and Y be C^1 -vector fields defined on the open sets U and V of \mathbb{R}^2 , respectively. Using the notation of Markus and Neumann, we denote by (U, Φ) and (V, Ψ) the flow of X and Y , respectively. We say that (U, Φ) and (V, Ψ) are *topologically equivalent* if there exists a homeomorphism of U in V which carries the orbits of X in orbits of Y , preseving or reversing the orientation of all orbits. If (U, Φ) and (V, Ψ) are topologically equivalent then we also say that their phase portraits are *topologically equivalent*.

Suppose that $U = \mathbb{R}^2$. We say that the flow (\mathbb{R}^2, Φ) is *parallel* if it is topologically equivalent of the flow (V, Ψ) defined by one of the following vector fields:

- Y defined on \mathbb{R}^2 by $Y(x, y) = (1, 0)$, for all $(x, y) \in \mathbb{R}^2$,
- Y defined on $\mathbb{R}^2 \setminus \{(0, 0)\}$ such that, in polar coordinates Y is given by $\dot{r} = 0, \dot{\theta} = 1$,
- Y defined on $\mathbb{R}^2 \setminus \{(0, 0)\}$ such that, in polar coordinates Y is given by $\dot{r} = r, \dot{\theta} = 0$.

The flows of the above three vector fields are called *strip flow*, *annulus flow* and *nodal flow*, respectively.

Given $p \in U$, we denote the orbit, α -limit and ω -limit of p by $\gamma(p)$, $\alpha(p)$ and $\omega(p)$, respectively. We say that $\gamma(p)$ is a *separatrix* if

- $\gamma(p)$ is a singular point, or
- $\gamma(p)$ is a periodic orbit and there is no neighbourhood of $\gamma(p)$ consisting of periodic orbits, or
- $\gamma(p)$ is homeomorphic to \mathbb{R} and there is no neighbourhood W of $\gamma(p)$ with the following two properties:
 - $q \in W \Rightarrow \alpha(q) = \alpha(p)$ and $\omega(q) = \omega(p)$,
 - the boundary of W is composed by $\alpha(p), \omega(p)$ and by two another orbits $\gamma(p_1), \gamma(p_2)$ such that $\alpha(p_1) = \alpha(p_2) = \alpha(p)$ and $\omega(p_1) = \omega(p_2) = \omega(p)$.

The union of all separatrices of a given flow (U, Φ) is called an *extended separatrix skeleton*, which is a closed invariant subset of U and is denoted by Σ . Each connected component of $U \setminus \Sigma$ is an open invariant set, called a *canonical region*. The next result says that there are only three possibilities for the flow in each canonical region.

PROPOSITION 4.1. *In each canonical region the flow is parallel.*

The *completed separatrix skeleton* is the union of the extended separatrix skeletons with one orbit in each canonical region. Given the flows (\mathbb{R}^2, Φ) and (\mathbb{R}^2, Ψ) , consider their extended separatrix skeletons C_1 and C_2 , respectively. We say that C_1 and C_2 are *topologically equivalent* if there exists a homeomorphism of \mathbb{R}^2 in \mathbb{R}^2 which maps orbits of C_1 to orbits of C_2 preseving or reversing the orientation.

Finally we recall the Markus–Newmann–Peixoto Theorem. This result implies that, to draw the phase portrait of a given planar vector field it is sufficient to determine its separatrices and the behavior of one orbit in each canonical region.

THEOREM 4.2 (Markus–Neumann–Peixoto). *Suppose that (\mathbb{R}^2, Φ) and (\mathbb{R}^2, Ψ) are continuous flows with only isolated singular points. Then (\mathbb{R}^2, Φ) and (\mathbb{R}^2, Ψ) are topologically equivalent if and only if their completed separatrices skeletons are topologically equivalent.*

5. Symmetries and finite singular points of system (1.2)

From now on X is the vector field defined by system (1.2). Additionally to the symmetry of X with respect to the y -axis, X is invariant under the change of coordinates $(x, y) \mapsto (-x, -y)$. Now to draw the phase portrait of X it is sufficient to study the quadrant given by $x, y \geq 0$ in the Poincaré disk.

We study the finite singular points of system (1.2) using Theorem 2.15 of [3] in the following remark.

REMARK 5.1. If $a \geq 0$ then $(0, 0)$ is the unique singularity of (1.2). If $a < 0$ then there exist another two singularities, $(\pm 1/\sqrt[4]{-a}, 0)$. In this case

$$DX\left(\pm \frac{1}{\sqrt[4]{-a}}, 0\right) = \begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix},$$

and there are hyperbolic saddle points. Moreover, $[1, -2]$ and $[1, 2]$ are eigenvectors associated to the eigenvalues 2 and -2 , respectively.

6. Infinite singular points of system (1.2)

To study the infinity singularities of X , we observe that the Poincaré compactification of X in the charts (U_1, φ_1) and (U_2, φ_2) is given by

$$(6.1) \quad \begin{aligned} \dot{u} &= (1 + u^2)v^4 + p(u), \\ \dot{v} &= uv^5, \end{aligned}$$

and

$$(6.2) \quad \begin{aligned} \dot{u} &= -(1 + u^2)v^4 - u^2q(u), \\ \dot{v} &= -uv^5 - uvq(u), \end{aligned}$$

respectively, where $p(u) = cu^4 + bu^2 + a$ and $q(u) = au^4 + bu^2 + c$.

We will perform the study of infinite singular points in U_1 and the origin of U_2 in Subsections 6.1 and 6.2, respectively.

6.1. Infinity singular points in U_1 . In this section we study the local phase portrait of infinite singular points of (1.2) which belong to U_1 . Note that

REMARK 6.1. $(u_0, 0)$ is a singular point of (6.1) if and only if $p(u_0) = 0$.

Suppose that $(u_0, 0)$ is a singularity of system (6.1). From both symmetries of system (6.1) it follows that it is sufficient to study the infinity singular points in U_1 of the form $(u_0, 0)$ with $u_0 \geq 0$.

The Jacobian matrix of the vector field defined by system (6.1) evaluated at $(u_0, 0)$ is

$$(6.3) \quad \begin{pmatrix} p'(u_0) & 0 \\ 0 & 0 \end{pmatrix},$$

so $(u_0, 0)$ is a semi-hyperbolic or a linearly zero singular point of system (6.1).

6.1.1. Semi-hyperbolic singular points. The semi-hyperbolic singular points in U_1 are treated in the next result.

LEMMA 6.2. *Suppose that $p(u_0) = 0$ and $p'(u_0) \neq 0$. Then $u_0 \neq 0$, moreover, if $u_0 > 0$ and $p'(u_0) > 0$ then $(u_0, 0)$ is a semi-hyperbolic unstable node of system (6.1). If $u_0 > 0$ and $p'(u_0) < 0$ then $(u_0, 0)$ is a semi-hyperbolic saddle point of system (6.1).*

PROOF. From the definition of p it follows that $u_0 \neq 0$. After the translation $u \mapsto u - u_0$ system (6.1) becomes

$$(6.4) \quad \begin{aligned} \dot{u} &= (1 + (u + u_0)^2)v^4 + p(u + u_0), \\ \dot{v} &= (u + u_0)v^5. \end{aligned}$$

Define

$$\begin{aligned} A(u, v) &= (1 + (u + u_0)^2)v^4 + p(u + u_0) - p'(u_0)u, \\ B(u, v) &= (u + u_0)v^5, \end{aligned}$$

then we can write system (6.4) in the form

$$(6.5) \quad \begin{aligned} \dot{u} &= p'(u_0)u + A(u, v), \\ \dot{v} &= B(u, v). \end{aligned}$$

Let $f = f(v)$ be the solution of the equation $p'(u_0)f(v) + A(f(v), v) = 0$ in a neighbourhood of $(0, 0)$ and define $g(v) = B(f(v), v)$. Then

$$g^{(j)}(0) = 0, \quad j = 0, 1, 2, 3, 4, \quad g^{(5)}(0) = 5!u_0,$$

and Lemma 6.2 follows from Theorem 2.19 of [3]. \square

6.1.2. *Nonelementary singular points.* Suppose that $p(u_0) = p'(u_0) = 0$, then $(u_0, 0)$ is a linearly zero singular point of system (6.1) and we use quasihomogeneous blow-up to study this singular point. The choice of the exponents α, β of the quasihomogeneous blow-up is given by the Newton diagram and depends of the order of vanishing of p at u_0 .

LEMMA 6.3. *Assume that $p(u_0) = p'(u_0) = 0$. Then we can write p in the form $p(u) = (u - u_0)^N \tilde{p}(u - u_0)$, for a suitable choice of $N = 2$ or $N = 4$. Moreover,*

$$(6.6) \quad N = 2 \quad \Rightarrow \quad \tilde{p}(u) = cu^2 + 4cu_0u + 6cu_0^2 + b, \quad \tilde{p}(0) = \frac{p''(u_0)}{2},$$

and the case $N = 4$ corresponds to $a = b = 0, c \neq 0, \tilde{p} \equiv c$.

PROOF. It is sufficient to show that the case $N = 3$ is impossible. It is true when $c = 0$. Suppose that $c \neq 0$ and $p'''(u_0) = 24cu_0 \neq 0$, then $u_0 \neq 0$. Since p is an even function, $p(-u_0) = 0$.

On the other hand, if we take $u = -u_0$ in the Taylor formula $p(u) = (u - u_0)^3(4cu_0 + c(u - u_0))$ we obtain $p(-u_0) = -16cu_0^4 \neq 0$. \square

In Lemmas 6.4 and 6.5 we consider cases $N = 2$ and $N = 4$, respectively.

LEMMA 6.4. *Suppose that $p(u_0) = p'(u_0) = 0$. If $u_0 \geq 0$ and $p''(u_0) > 0$ then the sectorial decomposition of system (6.1) at $(u_0, 0)$ is composed by two hyperbolic sectors (Figure 3). If $u_0 = 0$ and $p''(0) < 0$ then the sectorial decomposition of system (6.1) at $(u_0, 0)$ is composed by six hyperbolic sectors (Figure 5). If $u_0 > 0$ and $p''(u_0) < 0$ then the sectorial decomposition of system (6.1) at $(u_0, 0)$ is composed by two hyperbolic sectors and one parabolic repelling sector, that is, it is a saddle-node (Figure 7).*

PROOF. Suppose that $u_0 \geq 0$. After the translation $u \mapsto u - u_0$ and using notation of Lemma 6.3, system (6.1) becomes

$$(6.7) \quad \begin{aligned} \dot{u} &= (1 + (u + u_0)^2)v^4 + u^2\tilde{p}(u), \\ \dot{v} &= (u + u_0)v^5. \end{aligned}$$

Notice that for system (6.7) we have

$$(6.8) \quad u = 0, v \neq 0 \quad \Rightarrow \quad \dot{u} > 0.$$

We take the quasihomogeneous blow-up in the positive u -direction defined by $u = \bar{u}^2, v = \bar{u} \bar{v}$ which carries system (6.7) into

$$(6.9) \quad \begin{aligned} \dot{\bar{u}} &= \frac{1}{2} \bar{u}^3 [(1 + (\bar{u}^2 + u_0)^2)\bar{v}^4 + \tilde{p}(\bar{u}^2)], \\ \dot{\bar{v}} &= (\bar{u}^2 + u_0)\bar{u}^4\bar{v}^5 - \frac{1}{2} \bar{u}^2\bar{v} [(1 + (\bar{u}^2 + u_0)^2)\bar{v}^4 + \tilde{p}(\bar{u}^2)]. \end{aligned}$$

Rescaling the independent variable by the rule $dt = \bar{u}^2 ds$, system (6.9) becomes

$$(6.10) \quad \begin{aligned} \bar{u}' &= \frac{1}{2} \bar{u} [(1 + (\bar{u}^2 + u_0)^2) \bar{v}^4 + \tilde{p}(\bar{u}^2)], \\ \bar{v}' &= (\bar{u}^2 + u_0) \bar{u}^2 \bar{v}^5 - \frac{1}{2} \bar{v} [(1 + (\bar{u}^2 + u_0)^2) \bar{v}^4 + \tilde{p}(\bar{u}^2)], \end{aligned}$$

where the prime denotes derivative with respect to s .

The singularities of system (6.10) with $\bar{u} = 0$ are isolated and are given by $(0, \bar{v}_0)$, where $\bar{v}_0 = 0$ or \bar{v}_0 is a solution of the equation

$$(6.11) \quad \bar{v}_0^4 = -\tilde{p}(0)/(1 + u_0^2).$$

The Jacobian matrix of the vector-field defined by system (6.10) evaluated at $(0, \bar{v}_0)$ is

$$\begin{pmatrix} [(1 + u_0^2) \bar{v}_0^4 + \tilde{p}(0)]/2 & 0 \\ 0 & -[5(1 + u_0^2) \bar{v}_0^4 + \tilde{p}(0)]/2 \end{pmatrix}.$$

If $\tilde{p}(0) > 0$ then the origin is the unique singular point of system (6.10) with $\bar{u} = 0$ and it is a hyperbolic saddle point.

If $\tilde{p}(0) < 0$, then there are three singular points of system (6.10) with $\bar{u} = 0$, namely: $(0, 0)$ which is a hyperbolic saddle point, and $(0, \bar{v}_0)$ with \bar{v}_0 given by the two real solutions of the equation (6.11), which are semi-hyperbolic saddle points.

Now, we take the quasihomogeneous blow-up in the negative u -direction defined by $u = -\bar{u}^2$, $v = \bar{u} \bar{v}$, which carries system (6.7) into

$$(6.12) \quad \begin{aligned} \dot{\bar{u}} &= -\frac{1}{2} \bar{u}^3 [(1 + (-\bar{u}^2 + u_0)^2) \bar{v}^4 + \tilde{p}(-\bar{u}^2)], \\ \dot{\bar{v}} &= (-\bar{u}^2 + u_0) \bar{u}^4 \bar{v}^5 + \frac{1}{2} \bar{u}^2 \bar{v} [(1 + (-\bar{u}^2 + u_0)^2) \bar{v}^4 + \tilde{p}(-\bar{u}^2)]. \end{aligned}$$

Rescaling the independent variable by the rule $dt = \bar{u}^2 ds$, system (6.12) becomes

$$(6.13) \quad \begin{aligned} \bar{u}' &= -\frac{1}{2} \bar{u} [(1 + (-\bar{u}^2 + u_0)^2) \bar{v}^4 + \tilde{p}(-\bar{u}^2)], \\ \bar{v}' &= (-\bar{u}^2 + u_0) \bar{u}^2 \bar{v}^5 + \frac{1}{2} \bar{v} [(1 + (-\bar{u}^2 + u_0)^2) \bar{v}^4 + \tilde{p}(-\bar{u}^2)]. \end{aligned}$$

The singular points of system (6.13) with $\bar{u} = 0$ are isolated and given by $(0, \bar{v}_0)$, where $\bar{v}_0 = 0$ or \bar{v}_0 is defined by the real solutions of (6.11). The Jacobian matrix of the vector-field defined by system (6.13) evaluated at $(0, \bar{v}_0)$ is

$$\begin{pmatrix} -[(1 + u_0^2) \bar{v}_0^4 + \tilde{p}(0)]/2 & 0 \\ 0 & [5(1 + u_0^2) \bar{v}_0^4 + \tilde{p}(0)]/2 \end{pmatrix}.$$

If $\tilde{p}(0) > 0$ then the origin is the unique singular point of system (6.13) with $\bar{u} = 0$ and it is a hyperbolic saddle point.

If $\tilde{p}(0) < 0$ then there are three singular points of system (6.13) with $\bar{u} = 0$, namely: $(0, 0)$ which is a hyperbolic saddle point, and $(0, \bar{v}_0)$ with \bar{v}_0 given by the two real solutions of equation (6.11), which are semi-hyperbolic singular points.

Moreover, if $u_0 = 0$ then $(0, \bar{v}_0)$ is a semi-hyperbolic saddle point and if $u_0 > 0$ then $(0, \bar{v}_0)$ is a semi-hyperbolic unstable node.

For the case $u_0 \geq 0$, $p(u_0) = p'(u_0) = 0$ and $p''(u_0) > 0$ we summarize the blow-ups in Figure 2. From (6.8) it follows that the sectorial decomposition of system (6.1) at $(u_0, 0)$ is as given in Figure 3.

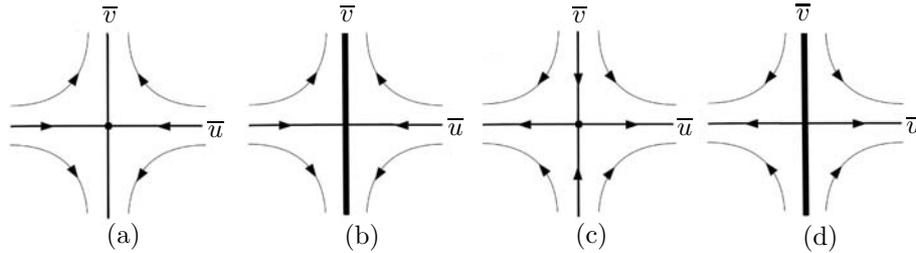


FIGURE 2. Case $u_0 \geq 0$, $p(u_0) = p'(u_0) = 0$ and $p''(u_0) > 0$. (a) Blow-up in the negative u -direction, system (6.13), (b) system (6.12), (c) blow-up in the positive u -direction, system (6.10), (d) system (6.9).

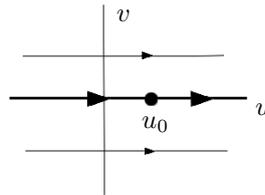


FIGURE 3. Sectorial decomposition of system (6.1) at $(u_0, 0)$ when $u_0 \geq 0$, $p(u_0) = p'(u_0) = 0$ and $p''(u_0) > 0$.

For the case $u_0 = 0$, $p(0) = p'(0) = 0$ and $p''(0) < 0$ we summarize the blow-ups in Figure 4. From (6.8) it follows that the sectorial decomposition of system (6.1) at the origin of U_1 is as given in Figure 5.

For the case $u_0 > 0$, $p(u_0) = p'(u_0) = 0$ and $p''(u_0) < 0$ we summarize the blow-ups in Figure 6. From (6.8) it follows that the sectorial decomposition of system (6.1) at $(u_0, 0)$ is as given in Figure 7. \square

LEMMA 6.5. *If $a = b = 0$ and $c > 0$ then the sectorial decomposition of system (6.1) at the origin is composed by two hyperbolic sectors (Figure 8(c)). If $a = b = 0$ and $c < 0$ then the sectorial decomposition of system (6.1) at the origin is composed by six hyperbolic sectors (Figure 10).*

PROOF. Suppose that $a = b = 0$ and $c \neq 0$. In this case for system (6.1) we have

$$(6.14) \quad u = 0, v \neq 0 \Rightarrow \dot{u} > 0$$

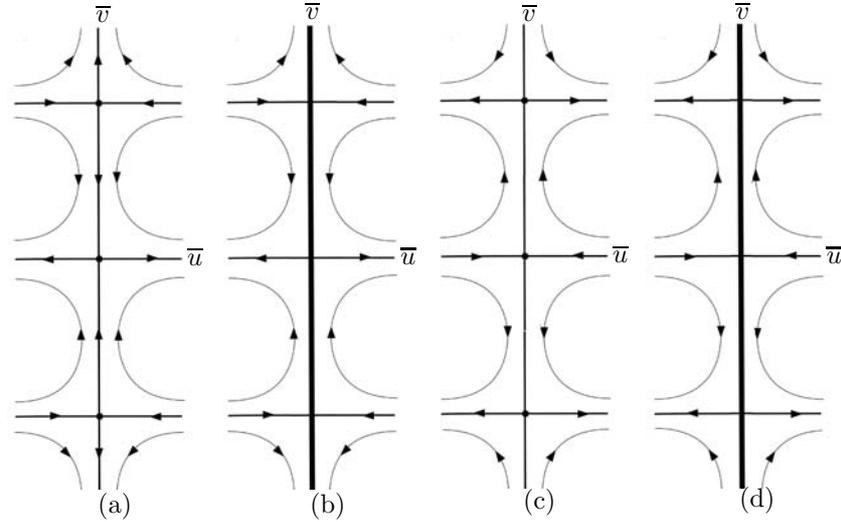


FIGURE 4. Case $u_0 = 0, p(0) = p'(0) = 0$ and $p''(0) < 0$. (a) Blow-up in the negative u -direction, system (6.13), (b) system (6.12), (c) blow-up in the positive u -direction, system (6.10), (d) system (6.9).

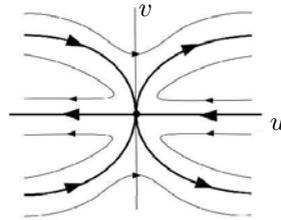


FIGURE 5. Sectorial decomposition of system (6.1) at the origin when $u_0=0, p(0) = p'(0) = 0$ and $p''(0) < 0$.

and

$$(6.15) \quad v = 0 \quad \Rightarrow \quad \dot{u} = cu^4.$$

For the case $a = b = 0$ and $c > 0$ we take the blow-up $u = \bar{u}, v = \bar{u}\bar{v}$ which carries system (6.1) into

$$(6.16) \quad \begin{aligned} \dot{\bar{u}} &= (1 + \bar{u}^2)\bar{u}^4\bar{v}^4 + c\bar{u}^4, \\ \dot{\bar{v}} &= \bar{u}^5\bar{v}^5 - \bar{u}^3\bar{v}^5(1 + \bar{u}^2) - c\bar{u}^3\bar{v}. \end{aligned}$$

Rescaling the independent variable by the rule $dt = \bar{u}^3 ds$, system (6.16) becomes

$$(6.17) \quad \begin{aligned} \bar{u}' &= (1 + \bar{u}^2)\bar{u}\bar{v}^4 + c\bar{u}, \\ \bar{v}' &= \bar{u}^2\bar{v}^5 - \bar{v}^5(1 + \bar{u}^2) - c\bar{v}. \end{aligned}$$

Then the origin is a hyperbolic saddle point of system (6.17).

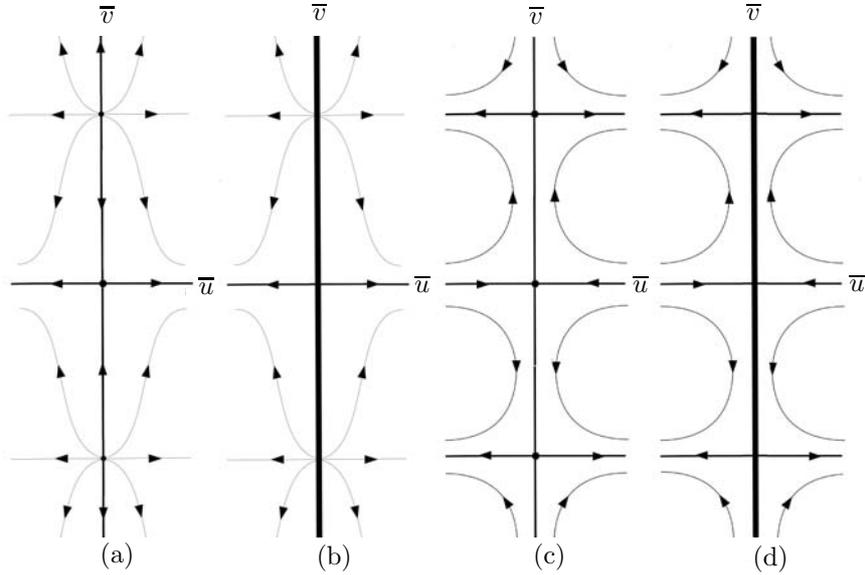


FIGURE 6. Case $u_0 > 0$, $p(u_0) = p'(u_0) = 0$ and $p''(u_0) < 0$. (a) Blow-up in the negative u -direction, system (6.13), (b) system (6.12), (c) blow-up in the positive u -direction, system (6.10), (d) system (6.9).

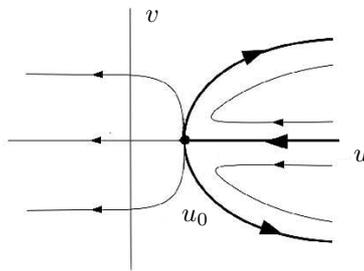


FIGURE 7. Sectorial decomposition of system (6.1) at $(u_0, 0)$ when $u_0 > 0$, $p(u_0) = p'(u_0) = 0$ and $p''(u_0) < 0$.

When $c > 0$ the blow-up is presented in Figures 8 (a)–(b). From (6.14) it follows that the sectorial decomposition of system (6.1) at the origin is given by Figure 8 (c).

For the case $a = b = 0$ and $c < 0$ we take the blow-up $u = \bar{u}\bar{v}$, $v = \bar{v}$ which carries system (6.1) into

$$(6.18) \quad \begin{aligned} \dot{\bar{u}} &= (1 + c\bar{u}^4)\bar{v}^3, \\ \dot{\bar{v}} &= \bar{u}\bar{v}^6. \end{aligned}$$

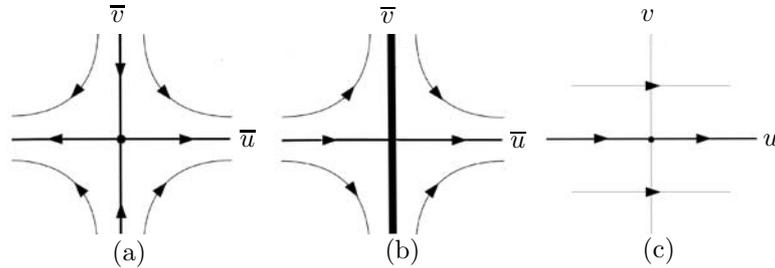


FIGURE 8. Case $a = b = 0$ and $c > 0$. (a) Blow-up, system (6.17), (b) system (6.16), (c) sectorial decomposition of system (6.1) at the origin.

Rescaling the independent variable by the rule $dt = \bar{v}^3 ds$, system (6.18) becomes

$$(6.19) \quad \begin{aligned} \bar{u}' &= 1 + c\bar{u}^4, \\ \bar{v}' &= \bar{u} \bar{v}^3. \end{aligned}$$

Then the singularities with $\bar{v} = 0$ are $(\pm 1/\sqrt[4]{-c}, 0)$ which are semi-hyperbolic saddle points.

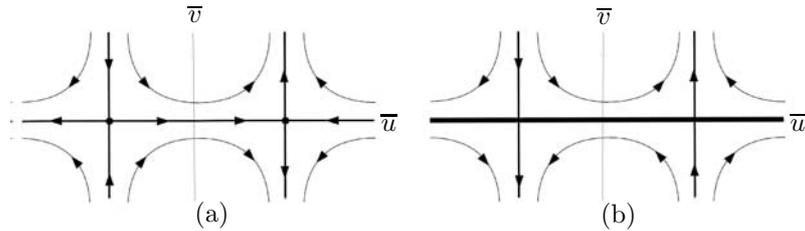


FIGURE 9. Case $a = b = 0$ and $c < 0$. (a) Blow-up, system (6.19), (b) system (6.18).

When $c < 0$ the blow-up is presented in Figures 9(a)–(b). From (6.14) and (6.15) it follows that the sectorial decomposition of system (6.1) at the origin is given by Figure 10. □

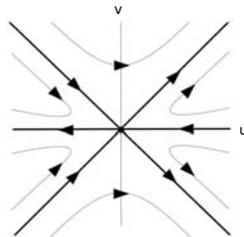


FIGURE 10. Case $a = b = 0$, $c > 0$. Sectorial decomposition of system (6.1) at the origin.

6.2. The origin of U_2 . In this section we study the local phase portrait of the origin of U_2 .

REMARK 6.6. For system (6.2) we have that $\dot{u} < 0$ when $u = 0$, $v \neq 0$. Moreover, this system is invariant under the change of coordinates $(t, u, v) \mapsto (-t, -u, v)$.

As in the previous section the choice of exponents α and β of the quasi-homogeneous blow-up depends of the order of the vanishing of q at the origin. So we will divide the study of the origin of U_2 in the cases $c \neq 0$; $c = 0$, $b \neq 0$ and $c = b = 0$, $a \neq 0$ which will be treated in Lemmas 6.7–6.9, respectively.

In the next section we will prove that all possibilities for the sectorial decomposition of system (6.2) at the origin presented in Lemmas 6.7 and 6.8 are realized for an appropriate choice of parameters a, b and c . On the other hand, for the case $a < 0$ and $b = c = 0$, only the second possibility for the sectorial decomposition of system (6.2) at the origin presented by Lemma 6.9 is realized.

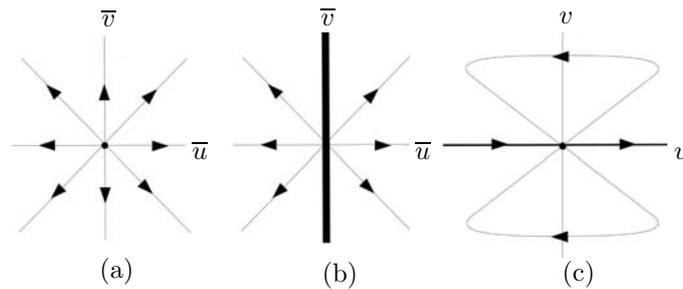


FIGURE 11. Case $c < 0$. (a) Blow-up in the positive u -direction, system (6.21), (b) system (6.20), (c) sectorial decomposition of system (6.2) at the origin.

LEMMA 6.7. *If $c < 0$ then there are two possibilities for the sectorial decomposition of system (6.2) at the origin: either two elliptic sectors, or two elliptic sectors, one parabolic repelling sector and one parabolic attracting sector (Figure 11 (c)). If $c > 0$ then the sectorial decomposition of system (6.2) at the origin is composed by two hyperbolic sectors, one parabolic repelling sector and one parabolic attracting sector (Figure 12 (c)).*

PROOF. Suppose $c \neq 0$. After the blow-up in the positive u -direction defined by $u = \bar{u}^2$, $v = \bar{u}\bar{v}$, system (6.2) becomes

$$(6.20) \quad \begin{aligned} \dot{\bar{u}} &= -\frac{1}{2}\bar{u}^3[(1 + \bar{u}^4)\bar{v}^4 + q(\bar{u}^2)], \\ \dot{\bar{v}} &= \frac{1}{2}\bar{u}^2\bar{v}[(1 + \bar{u}^4)\bar{v}^4 - q(\bar{u}^2)]. \end{aligned}$$

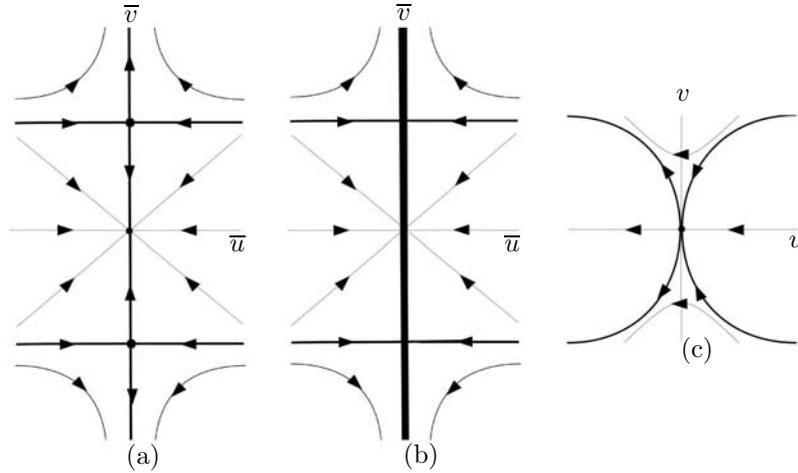


FIGURE 12. Case $c > 0$. (a) Blow-up in the positive u -direction, system (6.21), (b) system (6.20), (c) sectorial decomposition of system (6.2) at the origin.

Rescaling the independent variable by the rule $dt = \bar{u}^2 ds$, system (6.20) writes

$$(6.21) \quad \begin{aligned} \bar{u}' &= -\frac{1}{2}\bar{u}[(1 + \bar{u}^4)\bar{v}^4 + q(\bar{u}^2)], \\ \bar{v}' &= \frac{1}{2}\bar{v}[(1 + \bar{u}^4)\bar{v}^4 - q(\bar{u}^2)]. \end{aligned}$$

The singular points of system (6.21) with $\bar{u} = 0$ are of the form $(0, \bar{v}_0)$ where $\bar{v}_0 = 0$ or \bar{v}_0 is a solution of the equation $\bar{v}_0^4 = c$, then $(0, \bar{v}_0)$ is an isolated singular point.

The Jacobian matrix of the vector field defined by system (6.21) evaluated at $(0, \bar{v}_0)$ is

$$\begin{pmatrix} -(\bar{v}_0^4 + c)/2 & 0 \\ 0 & (5\bar{v}_0^4 - c)/2 \end{pmatrix}.$$

If $c < 0$ then the origin is the unique singular point of (6.21) with $\bar{u} = 0$, moreover, this is a hyperbolic unstable node. The blow-up in the positive u -direction is presented in Figures 11 (a)–(b). From Remark 6.6 it follows that the sectorial decomposition of system (6.2) at the origin is given by Figure 11 (c).

If $c > 0$ then there are three singular points of (6.21) with $\bar{u} = 0$, namely: $(0, 0)$ is a hyperbolic stable node and $(0, \pm \sqrt[4]{c})$ are hyperbolic saddle points. The blow-up in the positive u -direction is presented in Figures 12 (a)–(b). From Remark 6.6 it follows that the sectorial decomposition of system (6.2) at the origin is given by Figure 12 (c). \square

LEMMA 6.8. *If $c = 0$ and $b < 0$ then there are two possibilities for the sectorial decomposition of system (6.2) at the origin: either two elliptic sectors, or two elliptic sectors, one parabolic repelling sector and one parabolic attracting*

sector (Figure 13(c)). If $c = 0$ and $b > 0$ then the sectorial decomposition of system (6.2) at the origin is composed by two hyperbolic sectors (Figure 14(c)).

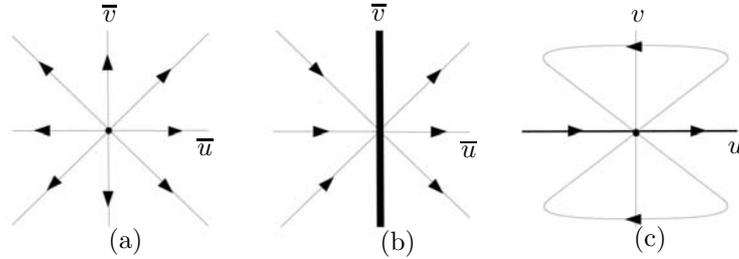


FIGURE 13. Case $c = 0$ and $b < 0$. (a) Blow-up, system (6.23), (b) system (6.22), (c) sectorial decomposition of system (6.2) at the origin.

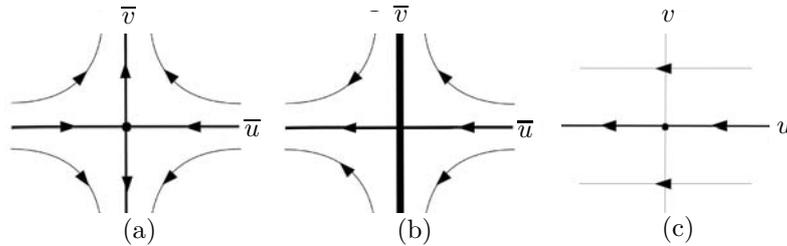


FIGURE 14. Case $c = 0$ and $b > 0$. (a) Blow-up, system (6.23), (b) system (6.22), (c) sectorial decomposition of system (6.2) at the origin.

PROOF. Suppose that $c = 0, b \neq 0$. We take the blow-up defined by $u = \bar{u}, v = \bar{u}\bar{v}$ which carries system (6.2) into

$$(6.22) \quad \begin{aligned} \dot{\bar{u}} &= -(1 + \bar{u}^2)\bar{u}^4\bar{v}^4 - \bar{u}^4(a\bar{u}^2 + b), \\ \dot{\bar{v}} &= \bar{u}^3\bar{v}^5. \end{aligned}$$

Rescaling the independent variable by the rule $dt = \bar{u}^3 ds$, system (6.22) becomes

$$(6.23) \quad \begin{aligned} \bar{u}' &= -(1 + \bar{u}^2)\bar{u}\bar{v}^4 - \bar{u}(a\bar{u}^2 + b), \\ \bar{v}' &= \bar{v}^5. \end{aligned}$$

The singular point with $\bar{u} = 0$ is $(0, 0)$ and this is an isolated singularity. The Jacobian matrix of the vector-field defined by system (6.23) evaluated at $(0, 0)$ is

$$\begin{pmatrix} -b & 0 \\ 0 & 0 \end{pmatrix}.$$

If $b < 0$ then $(0, 0)$ is a semi-hyperbolic unstable node of system (6.23). The blow-up is presented in Figures 13(a)–(b). From Remark 6.6 it follows that the sectorial decomposition of system (6.2) at the origin is given by Figure 13(c).

If $b > 0$ then $(0, 0)$ is a semi-hyperbolic saddle point of system (6.23). The blow-up is presented in Figures 14 (a)–(b). Using Remark 6.6 it follows that the sectorial decomposition of system (6.2) at the origin is given by Figure 14 (c). \square

LEMMA 6.9. *If $a < 0$ and $b = c = 0$ then there are two possibilities for the sectorial decomposition of system (6.2) at the origin: either two elliptic sectors, or two elliptic sectors, one parabolic repelling sector and one parabolic attracting sector (Figure 15 (c)). If $a > 0$ and $b = c = 0$ then the sectorial decomposition of system (6.2) at the origin is composed by two hyperbolic sectors (Figure 16 (c)).*

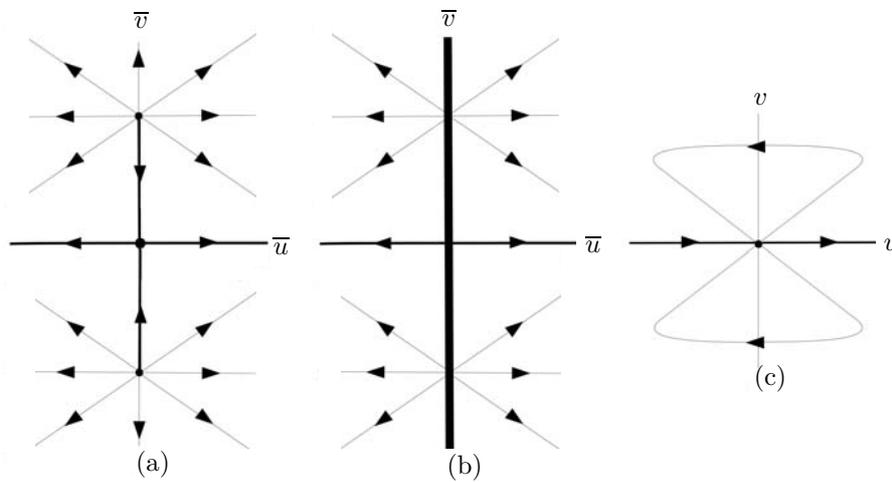


FIGURE 15. Case $a < 0$ and $b = c = 0$. (a) Blow-up in the positive u -direction, system (6.25), (b) system (6.24), (c) sectorial decomposition of system (6.2) at the origin.

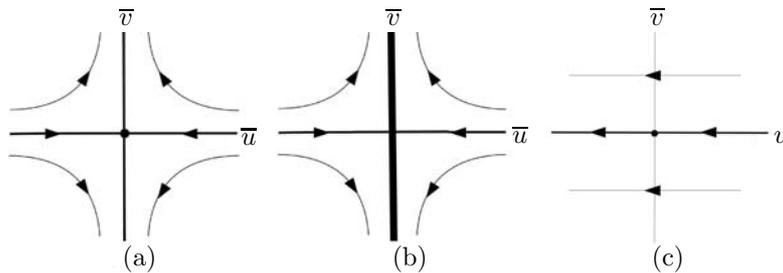


FIGURE 16. Case $a > 0$ and $c = b = 0$. (a) Blow-up in the positive u -direction, system (6.25), (b) system (6.24), (c) sectorial decomposition of system (6.2) at the origin.

PROOF. Suppose $a \neq 0$, $b = c = 0$. After the blow-up in the positive u -direction defined by $u = \bar{u}^2$, $v = \bar{u}^3 \bar{v}$ system (6.2) becomes

$$(6.24) \quad \begin{aligned} \dot{\bar{u}} &= -\frac{1}{2}\bar{u}^{11}[(1 + \bar{u}^4)\bar{v}^4 + a], \\ \dot{\bar{v}} &= -\bar{u}^{10}\bar{v}(\bar{u}^4\bar{v}^4 + a) + \frac{3}{2}\bar{u}^{10}\bar{v}[(1 + \bar{u}^4)\bar{v}^4 + a]. \end{aligned}$$

Rescaling the independent variable by the rule $dt = \bar{u}^{10} ds$, system (6.24) becomes

$$(6.25) \quad \bar{u}' = -\frac{1}{2}\bar{u}[(1 + \bar{u}^4)\bar{v}^4 + a], \quad \bar{v}' = \frac{1}{2}\bar{v}(\bar{u}^4\bar{v}^4 + 3\bar{v}^4 + a).$$

The singular points of system (6.25) with $\bar{u} = 0$ are isolated and given by $(0, \bar{v}_0)$, where $\bar{v}_0 = 0$ or \bar{v}_0 is a solution of the equation $3\bar{v}_0^4 = -a$. The Jacobian matrix of the vector-field defined by (6.25) evaluated at $(0, \bar{v}_0)$ is

$$\begin{pmatrix} -(\bar{v}_0^4 + a)/2 & 0 \\ 0 & (15\bar{v}_0^4 + a)/2 \end{pmatrix}.$$

If $a < 0$ then there are three singular points of system (6.25) with $\bar{u} = 0$, moreover $(0, 0)$ is a hyperbolic saddle point, and $(0, \pm \sqrt[4]{-a/3})$ are hyperbolic unstable nodes. The blow-up in the positive u -direction is given by Figures 15 (a)–(b). From Remark 6.6 it follows that the sectorial decomposition of system (6.2) at the origin is given by Figure 15 (c).

If $a > 0$ then the origin is the unique singular point of system (6.25) with $\bar{u} = 0$, and it is a hyperbolic saddle point. We present the blow-up in the positive u -direction in Figures 16 (a)–(b). From Remark 6.6 it follows that the sectorial decomposition of system (6.2) at the origin is given by Figure 16 (c). \square

7. Proof of Theorem 1.1

The proof of Theorem 1.1 is a combination of Lemmas 7.1, 7.2 and Remark 7.3. In Lemma 7.1 we use a linear change of coordinates to find normal forms of system (1.2). Finally, in Lemma 7.2 we determine the possibilities for the global phase portrait in the Poincaré disk of each normal form.

LEMMA 7.1. *All systems of the form (1.2) are topologically equivalent to one of the normal forms (i)–(viii) for an appropriate choice of $A \in \{-1, 1\}$, $B, C \in \mathbb{R} \setminus \{0\}$. The correspondence between parameters of system (1.2) and parameters of its respective normal form is given in Table 2.*

PROOF. Fixed $\delta \neq 0$, after the linear change of coordinates $(x, y) \mapsto (\delta x, \delta y)$ system (1.2) becomes

$$\dot{x} = -y, \quad \dot{y} = x + \frac{a}{\delta^4}x^5 + \frac{b}{\delta^4}x^3y^2 + \frac{c}{\delta^4}xy^4.$$

Considering the appropriate choice of $\delta = \sqrt[4]{\pm a}$, $\delta = \sqrt[4]{\pm b}$ or $\delta = \sqrt[4]{\pm c}$, we obtain Table 2. \square

System (1.2)	Normal form (i)–(viii)
$a = b = c = 0$	(i)
$a = b = 0, c \neq 0$	(ii), $A = c/ c $
$a = 0, b \neq 0, c = 0$	(iii), $A = b/ b $
$a = 0, b \neq 0, c \neq 0$	(iv), $A = b/ b , C = c/ b $
$a \neq 0, b = c = 0$	(v), $A = a/ a $
$a \neq 0, b = 0, c \neq 0$	(vi), $A = a/ a , C = c/ a $
$a \neq 0, b \neq 0, c = 0$	(vii), $A = a/ a , B = b/ a $
$a \neq 0, b \neq 0, c \neq 0$	(viii), $A = a/ a , B = b/ a , C = c/ a $

TABLE 2. Correspondence between the parameters of system (1.2) and the parameters of normal forms.

LEMMA 7.2. *The phase portrait in the Poincaré disk of each normal form (i)–(viii) is topologically equivalent to one of the phase portraits of Figure 1. The possibilities for the global phase portrait in the Poincaré disk for each normal form (i)–(viii) are presented in the first and third columns of Table 1.*

PROOF. From [5] it follows that the origin is a center of all normal forms (i)–(viii). Combining Remarks 5.1, 6.1, Lemmas 6.2–6.9 and the symmetries of system (1.2), we obtain the local phase portraits in the Poincaré disk presented in Figure 17. The correspondence between normal forms and local phase portraits of Figure 17 is given in Table 3. Since system (1.2) is linear when $a = b = c = 0$, the global phase portrait in the Poincaré disk of the vector field corresponding to Figure 17(a) is given by Figure 1(a).

Now we will determine the global phase portrait in the Poincaré disk of the normal forms corresponding to Figure 17(b). In this case, the finite singularities of X are the origin, which is a center, and two hyperbolic saddle points, see Remark 5.1. Notice that from the definition of system (1.2) it follows that $\dot{x} < 0$ and $\dot{y} = 0$ if $x = 0$ and $y > 0$; $\dot{x} > 0$ and $\dot{y} = 0$ if $x = 0$ and $y < 0$. Moreover, $\dot{x} = 0$ and $\dot{y} > 0$ when $x \in (-\infty, -1) \cup (0, 1)$ and $y = 0$; $\dot{x} = 0$ and $\dot{y} < 0$ when $x \in (-1, 0) \cup (1, +\infty)$ and $y = 0$. Combining this with the symmetries of system (1.2), we obtain that the global phase portrait in the Poincaré disk of the normal forms corresponding Figure 17(b) is given by Figure 1(b).

In the same way we can prove that the global phase portrait in the Poincaré disk of normal forms corresponding to Figures 17(a)–(i) is given by Figures 1(a)–(i), respectively. Also, the global phase portrait in the Poincaré disk of normal forms corresponding to Figures 17(k), (l), (n) is given by Figures 1(m), (n), and (r), respectively.

Local Phase Portrait of Figure 17	Normal Form
(a)	(i)
(b)	(v), $A = -1$ (viii), $A = -1, B > 0, C < 0, B^2 + 4C < 0$ (viii), $A = -1, B < 0, C < 0$ (vii), $A = -1, B < 0$ (vi), $A = -1, C < 0$
(c)	(viii), $A = 1, B < 0, C > 0, B^2 - 4C < 0$ (viii), $A = 1, B > 0, C > 0$ (vi), $A = 1, C > 0$
(d)	(v), $A = 1$ (vii), $A = 1, B > 0$
(e)	(ii), $A = 1$ (iv), $A = 1, C > 0$
(f)	(iii), $A = 1$
(g)	(ii), $A = -1$ (iii), $A = -1$ (iv), $A = -1, C < 0$
(h)	(viii), $A = 1, B < 0, C > 0, B^2 - 4C = 0$
(i)	(viii), $A = -1, B > 0, C < 0, B^2 + 4C = 0$
(j)	(viii), $A = -1, B > 0, C > 0$ (viii), $A = -1, B < 0, C > 0$ (vi), $A = -1, C > 0$
(k)	(viii), $A = 1, B \neq 0, C < 0$ (vii), $A = 1, B < 0$ (vi), $A = 1, C < 0$
(l)	(vii), $A = -1, B > 0$
(m)	(iv), $A = -1, C > 0$
(n)	(iv), $A = 1, C < 0$
(o)	(viii), $A = 1, B < 0, C > 0, B^2 - 4C > 0$
(p)	(viii), $A = -1, B > 0, C < 0, B^2 + 4C > 0$

TABLE 3. Classification of local phase portraits in the Poincaré disk of normal forms (i)–(viii).

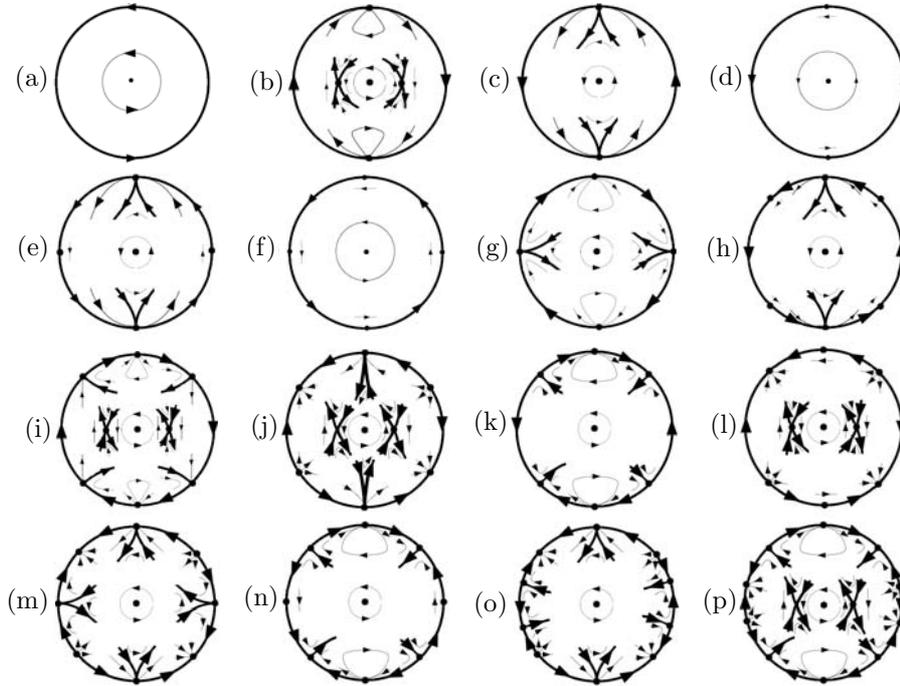


FIGURE 17. Local phase portraits in the Poincaré disk of normal forms (i)–(viii).

With the same arguments we can prove that for each normal form corresponding to Figures 17 (j), (m), (o) there exist three possibilities for the global phase portrait in the Poincaré disk, which are presented in Figures 1 (j), (k), (l), Figures 1 (o), (p), (q) and Figures 1 (s), (t), (u), respectively.

Finally, we will prove that the global phase portrait in the Poincaré disk of normal form corresponding to Figure 17 (p) is given by Figure 1 (v). In fact, first observe that in this case for the corresponding system (6.1) in \mathbb{R}^2 the union of points (u, v) such that $\dot{u} = 0$ is the union Λ of the graphics of the functions $v = \pm \sqrt[4]{-p(u)/(1+u^2)}$, the points (u, v) for which $\dot{u} > 0$ are given by the region R^+ in Figure 18. The region R^- corresponds to the points (u, v) such that $\dot{u} < 0$.

To complete the proof, it is sufficient to show that $\alpha(\Gamma) = \{F\}$ in Figure 19 (a), where $\alpha(\Gamma)$ denotes the α -limit of the orbit Γ . Suppose that this assertion is false, then $\alpha(\Gamma) = \{D\}$ or $\alpha(\Gamma) = \{E\}$. The phase portraits of system (6.1) in \mathbb{R}^2 , for cases $\alpha(\Gamma) = \{D\}$ and $\alpha(\Gamma) = \{E\}$, are presented in Figures 19 (b)–(c), respectively. If $\alpha(\Gamma) = \{D\}$ then an integral curve which connects points E and H provides a contradiction with the sign of \dot{u} in the region R^+ presented in Figure 18. In the same way, if $\alpha(\Gamma) = \{E\}$ then Γ contradicts Figure 18. □

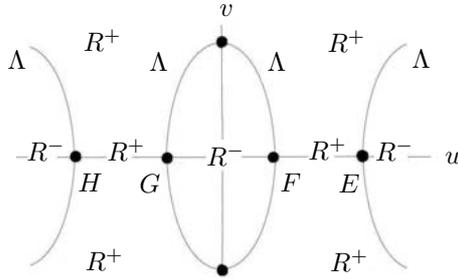


FIGURE 18. Signs of \dot{u} for system (6.1) corresponding to case of Figure 17 (p).

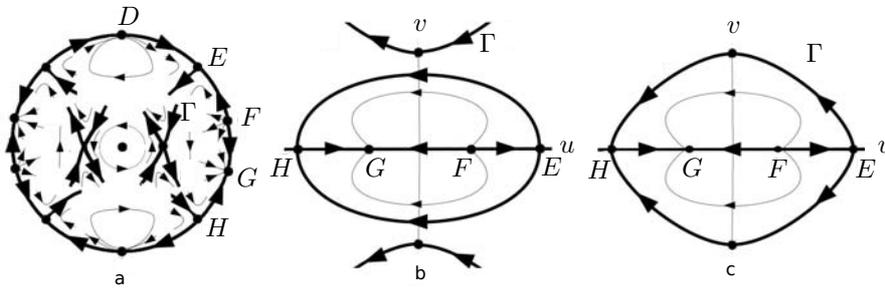


FIGURE 19. (a) Local phase portrait in the Poincaré disk (b) global phase portrait in \mathbb{R}^2 of system (6.1), if $\alpha(\Gamma) = \{D\}$, (c) global phase portrait in \mathbb{R}^2 of system (6.1), if $\alpha(\Gamma) = \{E\}$.

REMARK 7.3. Using Program P4, we can check that the global phase portraits in the Poincaré disk of Figures 1 (j), (l) are realized by choosing $a = -1$, $b \in \{-1, 0, 1\}$, $c = 20$ and $a = -1$, $b \in \{-1, 0, 1\}$, $c = 1$ in system (1.2), respectively. Then, by continuity, choosing $a = -1$ and $b \in \{-1, 0, 1\}$, the global phase portrait in the Poincaré disk of Figure 1 (k) is realized for the appropriate choice of $c \in (1, 20)$. In the same way, the global phase portraits in the Poincaré disk of Figures 1 (o), (q) are realized by the choice $a = 0$, $b = -1$, $c = 10$ and $a = 0$, $b = -1$, $c = 0.1$, respectively. By continuity, the global phase portrait in the Poincaré disk of Figure 1 (p) is realized for $a = 0$, $b = -1$ and the appropriate choice of $c \in (0.1, 10)$. Finally, the global phase portraits in the Poincaré disk of Figures 1 (s), (u) are realized choosing $a = 1$, $b = -2.1$, $c = 1$ and $a = 1$, $b = -10$, $c = 1$, respectively. By continuity, the global phase portrait in the Poincaré disk of Figure 1 (t) is realized for $a = -1$, $c = 1$ and the appropriate choice of $c \in (-10, -2.1)$.

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Manuscript received March 5, 2015

accepted April 30, 2015

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 TMNA : VOLUME 48 – 2016 – N° 1