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GENERALIZED TOPOLOGICAL TRANSITION MATRIX

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Abstract. This article represents a major step in the unification of the theory of algebraic, topological and singular transition matrices by introducing a definition which is a generalization that encompasses all of the previous three. When this more general transition matrix satisfies the additional requirement that it covers flow-defined Conley-index isomorphisms, one proves algebraic and connection-existence properties. These general transition matrices with this covering property are referred to as generalized topological transition matrices and are used to consider connecting orbits of Morse–Smale flows without periodic orbits, as well as those in a continuation associated to a dynamical spectral sequence.

1. Introduction

A challenging question in the study of dynamical systems is that of the existence of global bifurcations. The difficulty in detecting such bifurcation orbits is the fact that one must analyze the dynamical system globally. Topological techniques for global analysis are, therefore, a perfect fit for such an investigation. In particular, Conley index theory has proven to be quite useful in this role, as

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can be seen by the ample use of connection and transition matrices in bifurcation-related results. See [3], [4], [7]–[10] and [17].

Connection matrices have been extensively studied and can be computed by numerical techniques, see [1], [2] and [6]. Their continuation properties have proven to be useful in detecting global bifurcations. In particular, the continuation theorem [10] states that the connection matrices of an admissible ordering are invariant under local continuation. Yet, under global continuation, sets of connection matrices can undergo change. For instance, if there is a continuation between parameters with unique but different connection matrices, then within the continuation there must be a parameter value with nonunique connection matrices. At such a parameter value the system typically has a global bifurcation.

In other words, Morse decompositions and connection matrices provide a supporting structure within which global bifurcations can be detected, particularly via changes in the associated algebraic structures. These differences that occur in connection matrices under continuation, which can naturally be identified algebraically, were the main motivation for the introduction of transition matrices as a combinatorial mechanism to keep track of these changes. These transition matrices have since appeared in the literature under several guises: singular [22], topological [17], algebraic [11], and directional [15]. These four types of matrices are defined differently (particularly under contrasting conditions) and have distinct properties. On the other hand, due to underlying similarities in the definitions and their corresponding properties, a unified theory for transition matrices has long been called for.

In this paper we briefly introduce the generalization which unifies the theory. We focus on an initial and important step toward understanding the properties of this newly defined and more general transition matrix, which has the additional property that it covers flow-defined Conley-index isomorphisms. We refer to these matrices as generalized topological transition matrices and prove several properties they possess.

In contrast to the classical case, in our definition of a (generalized) topological transition matrix in Section 2, we do not require that there are no connections at the initial and final parameters of a continuation. As a consequence, this theory can be applied to a much broader class of dynamical systems than the classical topological transition matrix. We also establish properties of the generalized topological transition matrices – including connecting orbit existence results – corresponding to those of the classical topological transition matrix. In Section 3, we apply this new theory to Morse–Smale flows without periodic orbits. In this setting one demonstrates uniqueness and provides a simple way to compute the generalized topological transition matrix. In the last section, we see how the