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TOPOLOGICAL STRUCTURE OF SOLUTION SET FOR A CLASS OF FRACTIONAL NEUTRAL EVOLUTION EQUATIONS ON THE HALF-LINE

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Abstract. A topological structure of the set of all mild solutions of fractional neutral evolution equations with finite delay on the half-line is investigated. We show that the solution set is an R_δ-set. It is proved on compact intervals by establishing a result on topological structure of fixed point set of Krasnosel'skiĭ type operators. Next, using the inverse limit method, we obtain the same result on the half-line.

1. Introduction

Throughout this paper E denotes a Banach space endowed with the norm $|\cdot|$. Suppose that $A\colon D(A) \subset E \to E$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ of operators on E and $\psi\colon [-r,0] \to E$ is a function belonging to the phase space $\mathcal{C}_0 = C([-r,0];E)$. The main purpose of this paper is to study the topological structure of the set of all mild solutions of fractional neutral evolution equations with finite delay of the form

$$(1.1)\qquad \begin{cases} {}^{\mathrm{C}}D^q[x(t) - h(t,x(t),x_t)] = Ax(t) + f(t,x(t),x_t), & t > 0, \\ x_0(t) = \psi(t), & t \in [-r,0], \end{cases}$$

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where ${}^{\mathrm{C}}D^q$ is the Caputo fractional derivative of order $q \in (0,1)$, the histories $x_t \colon [-r,0] \to E$ are defined by $x_t(s) = x(t+s)$ for all $s \in [-r,0]$, and the functions $h, f \colon \mathbb{R}_+ \times E \times \mathcal{C}_0 \to E$ are given functions satisfying some conditions specified later.

Problem (1.1) on a compact interval has been studied by Zhou and Jiao [28] under a more general framework, namely the condition $x_0 = \psi$ was replaced by the condition

$$x_0(t) + g(x_{t_1}, \ldots, x_{t_n})(t) = \psi(t), \quad t \in [-r,0].$$

They proved an existence result by using the Krasnosel'skiĭ fixed point theorem. In this paper we establish the existence results for problem (1.1) on the half-line and, in particular, we prove that the mild solution set of this problem is an R_δ-set. For the reader's convenience, we recall here some definitions.

Definition 1.1 (see [14]). Let M be a topological space. Then

(a) M is called an *absolute retract* if each continuous map $f \colon B \to M$, where B is a closed subset of some topological space N, possesses a continuous extension over N.
(b) M is called an *R_δ-set* if it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.

Note that any R_δ-set is a nonempty compact connected space. On the other hand, it is acyclic with respect to the Čech homology functor, i.e. it has the same homology as the one point space. It may be not a singleton but, from the point of view of algebraic topology, it is equivalent to a point (see [14]).

In 1890, Peano proved that the Cauchy problem

$$\begin{cases} x'(t) = f(t, x(t)), & 0 < t \le a, \\ x(0) = x_0, \end{cases} \tag{1.2}$$

where $f \colon [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, has local solutions although the uniqueness property does not hold in general. This observation became a motivation for studying the structure of the solution set, Sol, for problem (1.2). Peano also proved that, in the case $n = 1$, the set $\mathrm{Sol}(t) = \{x(t) : x \in \mathrm{Sol}\}$ is nonempty, compact and connected in the standard topology of the real line, for t in some neighbourhood of 0. In 1923, Kneser generalized this result for arbitrary n. Next, in 1928, Hukuhara proved that Sol is a continuum in the Banach space of continuous functions with sup norm. A more precise characterization of Sol has been found in 1942 by Aronszajn [2]. He proved that Sol is an R_δ-set. So Sol is acyclic. The analogous result was obtained for upper-Carathéodory inclusions by De Blasi and Myjak in [8]. For more details, historical remarks and related references, see [1].