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ON STABILITY AND CONTROLLABILITY FOR SEMIGROUP ACTIONS

Josiney A. Souza — Hélio V.M. Tozatti — Victor H.L. Rocha

Abstract. This paper deals with stability and controllability for semigroup actions by using the topological method of admissible family of open coverings. The main results state a relationship of stable sets and control sets. The classical notion of controllability relates to the Poisson stability. The concept of prolongational control set relates to the Lyapunov stability.

1. Introduction

The present paper studies stability and controllability for semigroup actions on topological spaces. We introduce the notions of Poisson stable set, nonwandering set, and Lagrange stable set, which extend respectively the concepts of Poisson stable point, nonwandering point, and Lagrange stable motion. These concepts of stability theory can be connected by means of limit sets. We also present an aspect of controllability by prolongations that is related to the nonwandering points. The so-called prolongational control set is the link between Lyapunov stability and controllability.

The concept of stable set for dynamical systems on metric spaces was extensively studied by Bhatia and Szegő [2] and [3]. Bhatia and Hajek [1] developed a theory of stability for local semidynamical systems on topological spaces. Recently, Braga Barros, Souza and Rocha [6] introduced a theory of Lyapunov

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stability of sets for semigroup actions on topological spaces, extending several concepts and results of Lyapunov stable sets from [1]–[3]. On the other hand, the notion of controllability was extended from the setting of control systems to the setting of semigroup actions by San Martin [10]–[12]. Thus stability and controllability are both currently concepts of semigroup actions on topological spaces.

A questioning on the relationship of stability and controllability was incited in the paper [17], where a connection between the notions of Poisson stability and control set were stated, although the Poisson stability was named as the Poincaré recurrence. This connection was established by means of control sets depending on a family of subsets of the semigroup, which has stated an asymptotic aspect of controllability related to the well-known concept of topological transitivity. However, there is no study relating Lyapunov stability to controllability in the literature. This is due to the fact that the aspects of Lyapunov stability are totally different from the aspects of controllability. A Lyapunov stable set need not be controllable, while a control set need not be Lyapunov stable. The basic difference between these concepts is that the Lyapunov stability concerns the dynamics in each neighbourhood of a set, while the controllability considers an equivalence class by almost transitivity among the points inside the set. Our intention in the present paper is to explain a situation in what these concepts are connected.

Let (S, M) be a semigroup action on the topological space M. A set $X \subset M$ is controllable if $X \subset \mathrm{cls}(Sx)$ for every $x \in X$. On the other hand, if $X \subset M$ is equistable then $\mathrm{D}(x, S) \subset X$ for all $x \in X$, where $\mathrm{D}(x, S)$ is the forward prolongation of x, which implies $\mathrm{cls}(Sx) \subset X$ for every $x \in X$. Thus controllable and equistable sets are technically distinct, although both concepts concern the orbits through the points of the set. Then we introduce two objects which approximate controllability and stability: the weak prolongational control set and the minimal equistable set. A set $X \subset M$ is minimal equistable if it is nonempty, closed, equistable, and has no proper subset satisfying these properties. A subset $E \subset M$ is weak controllable by prolongations if each point y in E can be reachable by prolongations from another point x in E. In other words, E is weak controllable by prolongations if $E \subset \mathrm{D}(x, S)$ for every $x \in E$. The set E is called a weak prolongational control set if it is weak controllable by prolongations and is maximal satisfying this property. In general, if $\mathcal{F}$ is a family of subsets of the semigroup S, then E is a weak prolongational $\mathcal{F}$-control set if $E \subset \mathrm{J}(x, \mathcal{F})$ for every $x \in E$, where $\mathrm{J}(x, \mathcal{F})$ is the forward $\mathcal{F}$-prolongational limit set of x, and E is maximal with this property. The notions of nonwandering point and dispersiveness are involved in this study. A point $x \in M$ is $\mathcal{F}$-nonwandering if $x \in \mathrm{J}(x, \mathcal{F})$, while the action is dispersive if $\mathrm{J}(x, \mathcal{F}) = \emptyset$ for every $x \in M$ (see [19]). Thus