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BIFURCATION AND MULTIPLICITY RESULTS FOR CLASSES OF p,q-LAPLACIAN SYSTEMS

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Abstract. We study positive solutions to boundary value problems of the form

$$\begin{cases} -\Delta_p u = \lambda\{u^{p-1-\alpha} + f(v)\} & \text{in } \Omega, \\ -\Delta_q v = \lambda\{v^{q-1-\beta} + g(u)\} & \text{in } \Omega, \\ u = 0 = v & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_m u := \operatorname{div}(|\nabla u|^{m-2}\nabla u)$, $m > 1$, is the m-Laplacian operator of u, $\lambda > 0$, $p,q > 1$, $\alpha \in (0,p-1)$, $\beta \in (0,q-1)$ and Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$. Here $f,g\colon [0,\infty) \to \mathbb{R}$ are nondecreasing continuous functions with $f(0) = 0 = g(0)$. We first establish that for $\lambda \approx 0$ there exist positive solutions bifurcating from the trivial branch $(\lambda, u \equiv 0, v \equiv 0)$ at $(0,0,0)$. We further discuss an existence result for all $\lambda > 0$ and a multiplicity result for a certain range of λ under additional assumptions on f and g. We employ the method of sub-super solutions to establish our results.

1. Introduction

Consider boundary value problems of the form

$$\begin{cases} -\Delta_p u = \lambda \widetilde{f}(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, is the p-Laplacian operator of u, $\lambda > 0$ and Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$. Here $\widetilde{f}\colon [0,\infty) \to \mathbb{R}$ is a nondecreasing continuous function. When $\widetilde{f}(0) > 0$, there is a rich history on the study of positive solutions. The authors in [5] have considered such problems in the Laplacian case ($p = 2$) and established an existence result for all $\lambda > 0$ and a multiplicity result for a certain range of λ under additional assumptions on $\widetilde{f}$. Later in [7], these results were extended to the p-Laplacian case ($p > 1$). In particular, the authors in [7] proved the existence of a positive solution for all $\lambda > 0$ when $\widetilde{f}$ is p sublinear at ∞, and multiplicity results for a certain range of λ when there exist a and b such that $0 < a < b$ and $(a^{p-1}/\widetilde{f}(a))/(b^{p-1}/\widetilde{f}(b))$ is sufficiently large. See also [1], [2] and [6] for related results in the case $\widetilde{f}(0) > 0$. Here, we focus on the case $\widetilde{f}(0) = 0$. If $\widetilde{f}(0) > 0$, then $u \equiv 0$ is a very useful nonnegative strict subsolution to help with the study of establishing positive solutions. In this paper, $u \equiv 0$ is a solution for each $\lambda > 0$ and hence we lack the presence of this trivial nonnegative strict subsolution. However, we use the presence of the term $u^{p-1-\alpha}$ as our advantage to overcome this difficulty and show that positive solutions bifurcate at $(0,0)$ from the trivial branch $(\lambda, u \equiv 0)$. Under additional properties on $\widetilde{f}$, we establish further existence and multiplicity results. We also extend these results to classes of p,q-Laplacian systems. In particular, we consider boundary value problems of the form

$$
\begin{cases}
-\Delta_p u = \lambda\{u^{p-1-\alpha} + f(v)\} & \text{in } \Omega,\\
-\Delta_q v = \lambda\{v^{q-1-\beta} + g(u)\} & \text{in } \Omega,\\
u = 0 = v & \text{on } \partial\Omega,
\end{cases}
\tag{1.1}
$$

where $\Delta_m u := \operatorname{div}(|\nabla u|^{m-2}\nabla u)$, $m > 1$, is the m-Laplacian operator of u, $\lambda > 0$, $p, q > 1$, $\alpha \in (0, p-1)$, $\beta \in (0, q-1)$ are parameters and Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$. Here $f, g\colon [0,\infty) \to \mathbb{R}$ are nondecreasing continuous functions with $f(0) = 0 = g(0)$. Clearly for all λ, $(u \equiv 0,\ v \equiv 0)$ is a solution of (1.1). In this paper, we are interested in the study of solution $(u,v) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ with $u, v > 0$ in Ω. We first establish:

Theorem 1.1. *There exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, (1.1) has a positive solution (u,v) such that $\|u\|_\infty \to 0$, $\|v\|_\infty \to 0$ as $\lambda \to 0$ (see Figure 1).*

Next we consider the case when f, g satisfy the following combined p,q sublinear condition at ∞:

(H$_1$) $\displaystyle\lim_{s\to\infty} \frac{f(Mg(s)^{1/(q-1)})}{s^{p-1}} = 0$, for all $M > 0$,

and establish: