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ON THE TAIL PRESSURE

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Abstract. In this paper, we give two equivalent definitions of tail pressure involving open covers and establish a variational principle which exhibits the relationship between tail pressure and measure-theoretic tail entropy.

1. Introduction

Topological tail entropy quantifies the complexity of a dynamical system at arbitrarily small scales. It captures the entropy near any single orbit. This quantity was first introduced by Misiurewicz in [9] and was thoroughly studied by many others (e.g. see [1], [2], [5], [8]). (Historically, Misiurewicz and Buzzi called it the topological conditional entropy and local entropy respectively.) It is well known that the variational principle plays a fundamental role in ergodic theory and dynamical systems. In [6], Ledrappier obtained a variational principle of topological tail entropy, and Downarowicz ([5]) established a variational principle between the topological tail entropy and the entropy structure. Later, Burguet ([1]) presented a direct proof of Downarowicz's results and extended them to a noninvertible case. Recently, there appeared some works which study the tail entropy of dynamics of group actions (e.g. see [4], [15], [16]).

As a natural generalization of topological entropy, topological pressure is a quantity which belongs to one of the concepts in thermodynamic formalism. This generalization was first done by Rulle in [11] and next by many others (e.g.

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see [3], [10], [12], [14]). In [8], Li–Chen–Cheng extended the tail entropy to tail pressure for continuous transformations. In fact, they proved a tail variational principle which exhibits the relationship between the tail pressure and the tail entropy function and gave some applications of tail pressure.

In this paper, we give two equivalent definitions of tail pressure involving open covers and establish a variational principle which exhibits the relationship between the tail pressure and the measure-theoretic tail entropy. Let (X, d) be a compact metric space and $T\colon X \to X$ be a homeomorphism. For $\varepsilon > 0, n \in \mathbb{N}$ and $x \in X$, the Bowen's ball of order n, radius ε and center x is defined by

$$B(x, n, \varepsilon) = \{y \in X : d(T^k(x), T^k(y)) < \varepsilon, \text{ for all } k = 0, \dots, n-1\}.$$

Given $K \subset X$, a set $E \subset X$ is said to be an (n, ε)-spanning subset for K if

$$K \subset \bigcup_{x \in E} B(x, n, \varepsilon),$$

and an (n, ε)-separated subset of K if for all $x \neq y \in E$ there is $0 \leq k \leq n-1$ such that $d(T^k(x), T^k(y)) \geq \varepsilon$.

Now we recall the concept of tail pressure which was defined by Li–Chen–Cheng in [8]. Let $C(X, \mathbb{R})$ be the space of real-valued continuous functions of X. For $f \in C(X, \mathbb{R})$, denote by

$$(S_n f)(x) = \sum_{i=0}^{n-1} f(T^i(x)), \quad \text{for all } x \in X.$$

Let $f \in C(X, \mathbb{R})$, $n \in \mathbb{N}$, $\varepsilon > 0, \delta > 0$ and $x \in X$. Write

$$Q_n(T, f, x, \delta, \varepsilon) = \inf\left\{ \sum_{y \in F} e^{(S_n f)(y)} : F \text{ is an } (n, \delta)\text{-spanning set for } B(x, n, \varepsilon)\right\}$$

and

$$P_n(T, f, x, \delta, \varepsilon) = \sup\left\{ \sum_{y \in E} e^{(S_n f)(y)} : E \text{ is an } (n, \delta)\text{-separated set of } B(x, n, \varepsilon)\right\}.$$

The *tail pressure* $P^*(T, f)$ is defined by

$$P^*(T, f) = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \sup_{x \in X} \ln P_n(T, f, x, \delta, \varepsilon).$$

Write

$$Q^*(T, f) = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \sup_{x \in X} \ln Q_n(T, f, x, \delta, \varepsilon).$$

By Lemma 3.3 of [8],

$$P^*(T, f) = Q^*(T, f). \tag{1.1}$$