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ON PROPERTIES OF SOLUTIONS FOR A FUNCTIONAL EQUATION

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Abstract. This paper studies properties of solutions for a functional equation arising in dynamic programming of multistage decision processes. Using the Banach fixed point theorem and the Mann iterative methods, we prove the existence and uniqueness of solutions and convergence of sequences generated by the Mann iterative methods for the functional equation in the Banach spaces $BC(S)$ and $B(S)$ and the complete metric space $BB(S)$, and discuss behaviors of solutions for the functional equation in the complete metric space $BB(S)$. Four examples illustrating the results presented in this paper are also provided.

1. Introduction

In the past decades, the existence of solutions for some classes of functional equations arising in dynamic programming in the complete metric space $BB(S)$ has been discussed by several authors, see, for instance, [1]–[15] and the references quoted therein. Bellman [1], Bhakta and Choudhury [4], Liu and Kang [8], [9], Liu et al. [12] and Liu et al. [15] studied the existence of solutions for the functional equations

$$f(x) = \inf_{y\in D} \max\{u(x,y), f(a(x,y))\}, \quad \text{for all } x \in S, \tag{1.1}$$

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and

$$f(x) = \inf_{y\in D} \max\{u(x,y), v(x,u)f(a(x,y))\}, \quad \text{for all } x \in S \tag{1.2}$$

in the complete metric space $BB(S)$. Liu et al. [12] and Liu and Ume [11] obtained the existence, uniqueness and iterative approximations of solutions for the functional equations

$$f(x) = \sup_{y\in D} \max\{u(x,y), f(a(x,y))\}, \quad \text{for all } x \in S, \tag{1.3}$$

$$f(x) = \operatorname*{opt}_{y\in D} \operatorname{opt}\{u(x,y), f(a(x,y))\}, \quad \text{for all } x \in S, \tag{1.4}$$

in the complete metric space $BB(S)$, where opt stands for the sup or inf. However, relatively little is known about the existence of continuous bounded solutions and bounded solutions for the functional equations (1.1)–(1.4).

The aim of this paper is to introduce and study a more general functional equation arising in dynamic programming of multistage decision processes

$$\begin{aligned} f(x) = {}& \lambda \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)f(a(x,y))\} \\ & + (1-\lambda) \inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)f(b(x,y))\}, \quad \text{for all } x \in S, \end{aligned} \tag{1.5}$$

where $\lambda \in [0,1]$ is a constant, x and y stand for the state and decision vectors, respectively, a and b denote the transformations of the processes, and $f(x)$ is the optimal return function with initial state x. It is clear that the functional equation (1.5) includes the functional equations (1.1)–(1.4) as special cases. Based on the Banach fixed point theorem, the Mann iterative methods and some techniques in nonlinear analysis, we provide sufficient conditions which guarantee the existence, uniqueness and iterative approximations of continuous bounded solutions, bounded solutions and solutions for the functional equation (1.5) in the Banach spaces $BC(S)$ and $B(S)$ and the complete metric space $BB(S)$, respectively, and prove some error estimates between the sequences generated by the Mann iterative methods and these solutions. Furthermore, we deduce properties of solutions for the functional equation (1.5) in the complete metric space $BB(S)$. Four example illustrating the results presented in this paper are included. Our results extend, improve and unify the corresponding results in [4], [6], [8], [9], [11], [12] and [15].

2. Preliminaries

In this section we introduce some notation and preliminary lemmas.

Throughout this paper, $(X, \|\cdot\|)$ and $(Y, \|\cdot\|')$ denote real Banach spaces, $S \subset X$ and $D \subset Y$ stand for the state space and decision space, respectively,

$\mathbb{N}$ is the set of all positive integers, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{R}^- = (-\infty, 0]$. Define

$$\begin{aligned}
\Phi_1 &= \{\varphi : \varphi\colon \mathbb{R}^+ \to \mathbb{R}^+ \text{ is nondecreasing}\},\\
\Phi_2 &= \left\{(\varphi, \psi) : \varphi, \psi \in \Phi_1,\ \sum_{n=0}^{\infty} \psi(\varphi^n(t)) < +\infty \text{ and } \psi(t) > 0 \text{ for all } t > 0\right\},\\
B(S) &= \{h : h : S \to \mathbb{R} \text{ is bounded}\},\\
BC(S) &= \{h : h \in B(S) \text{ is continuous}\},\\
BB(S) &= \{h : h\colon S \to \mathbb{R} \text{ is bounded on each bounded subsets of } S\}.
\end{aligned}$$

Clearly, $(B(S), \|\cdot\|_1)$ and $(BC(S), \|\cdot\|_1)$ are Banach spaces with the norm $\|h\|_1 = \sup_{x \in S} |h(x)|$. For each $k \in \mathbb{N}$ and $f, g \in BB(S)$, put

$$d_k(h, g) = \sup\{|h(x) - g(x)| : x \in \overline{B}(0, k)\}, \qquad d(h, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(h, g)}{1 + d_k(h, g)},$$

where $\overline{B}(0, k) = \{x : x \in S \text{ and } \|x\| \le k\}$. Obviously, $\{d_k\}_{k \in \mathbb{N}}$ is a countable family of pseudometrics in $BB(S)$. A sequence $\{x_k\}_{k \in \mathbb{N}}$ in $BB(S)$ is said to *converge* to a point $x \in BB(S)$ if $d_k(x_n, x) \to 0$ as $n \to \infty$ and $\{x_n\}_{n \in \mathbb{N}}$ is a *Cauchy sequence* if $d_k(x_n, x_m) \to 0$ as $n, m \to \infty$ for each $k \in \mathbb{N}$. It is clear that $(BB(S), d)$ is a complete metric space.

Lemma 2.1 ([9]). *Let E be a set, p and $q\colon E \to \mathbb{R}$ be mappings. If $\operatorname{opt}_{y \in E} p(y)$ and $\operatorname{opt}_{y \in E} q(y)$ are bounded, then*

$$\left| \operatorname*{opt}_{y \in E} p(y) - \operatorname*{opt}_{y \in E} q(y) \right| \le \sup_{y \in E} |p(y) - q(y)|.$$

Lemma 2.2 ([11]). *Let α, β, γ and δ be in $\mathbb{R}$. Then*

$$|\operatorname{opt}\{\alpha, \beta\} - \operatorname{opt}\{\gamma, \delta\}| \le \max\{|\alpha - \gamma|, |\beta - \delta|\}.$$

3. Properties of solutions for the functional equation (1.5)

Now we derive the existence, uniqueness and iterative approximations of continuous bounded solutions and bounded solutions for the functional equation (1.5) in the Banach spaces $BC(S)$ and $B(S)$, respectively.

Theorem 3.1. *Let $\lambda \in [0, 1]$ and $\alpha \in (0, 1)$ be constants. Assume that S is compact, $p, q, u, v\colon S \times D \to \mathbb{R}$ and $a, b\colon S \times D \to S$ satisfy that*

(C1) *p and q are bounded in $S \times D$;*

(C2) $\sup_{(x,y) \in S \times D} \max\{|u(x, y)|, |v(x, y)|\} \le \alpha$;

(C3) *for each* $(x_0, g) \in S \times \{u, v, p, q, a, b\}$,

$$\lim_{x \to x_0} g(x, y) = g(x_0, y) \quad \textit{uniformly for } y \in D.$$

Then the functional equation (1.5) *possesses a unique continuous bounded solution* $w \in BC(S)$ *such that*

(C4) *for each* $w_0 \in BC(S)$, *the Mann iterative sequence* $\{w_n\}_{n \in \mathbb{N}_0}$ *defined by*

$$\begin{aligned}(3.1)\quad w_{n+1}(x) = (1-\alpha_n)w_n(x) + \alpha_n \Big(\lambda \sup_{y \in D} \operatorname{opt}\{p(x,y), u(x,y)w_n(a(x,y))\} \\ + (1-\lambda)\inf_{y \in D} \operatorname{opt}\{q(x,y), v(x,y)w_n(b(x,y))\}\Big),\end{aligned}$$

for all $(x, n) \in S \times \mathbb{N}_0$ *and*

$$(3.2)\qquad \{\alpha_n\}_{n \in \mathbb{N}_0} \subset [0,1] \quad \textit{and} \quad \sum_{n=0}^{\infty} \alpha_n = +\infty$$

converges to w *and has the error estimate:*

$$(3.3)\qquad \|w_{n+1} - w\|_1 \le e^{-(1-\alpha)\sum_{i=0}^{n}\alpha_i}\|w_0 - w\|_1, \quad \textit{for all } n \in \mathbb{N}_0.$$

Proof. Define a mapping $H: BC(S) \to BC(S)$ by

$$\begin{aligned}(3.4)\quad Hh(x) = \lambda \sup_{y \in D} \operatorname{opt}\{p(x,y), u(x,y)h(a(x,y))\} \\ + (1-\lambda)\inf_{y \in D} \operatorname{opt}\{q(x,y), v(x,y)h(b(x,y))\},\end{aligned}$$

for all $(x, h) \in S \times BC(S)$.

Firstly, we show that H is a self mapping in $BC(S)$. Let $(x_0, h) \in S \times BC(S)$ and $\varepsilon > 0$. In light of (C1), (C3) and the compactness of S, we infer that there exist constants $M \ge 1$, $\delta > 0$ and $\delta_1 > 0$ such that

$$(3.5)\qquad \sup_{(x,y) \in S \times D} \max\{|h(x)|, |p(x,y)|, |q(x,y)|\} \le M;$$

$$\begin{aligned}(3.6)\quad \max\{|r(x,y) - r(x_0,y)| : r \in \{u, v, p, q\}\} < \frac{\varepsilon}{2M}, \\ \text{for all } (x,y) \in S \times D \text{ with } \|x - x_0\| < \delta;\end{aligned}$$

$$(3.7)\qquad |h(x_1) - h(x_2)| < \frac{\varepsilon}{2M}, \quad \text{for all } x_1, x_2 \in S \text{ with } \|x_1 - x_2\| < \delta_1;$$

$$\begin{aligned}(3.8)\quad \max\{\|t(x,y) - t(x_0,y)\| : t \in \{a, b\}\} < \delta_1, \\ \text{for all } (x,y) \in S \times D \text{ with } \|x - x_0\| < \delta.\end{aligned}$$

In terms of (C2), (3.4)–(3.8), Lemmas 2.1 and 2.2, we know that

$$
\begin{aligned}
|Hh(x) - Hh(x_0)| = \Big| & \lambda \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)h(a(x,y))\} \\
& + (1-\lambda) \inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)h(b(x,y))\} \\
& - \lambda \sup_{y\in D} \operatorname{opt}\{p(x_0,y), u(x_0,y)h(a(x_0,y))\} \\
& - (1-\lambda) \inf_{y\in D} \operatorname{opt}\{q(x_0,y), v(x_0,y)h(b(x_0,y))\}\Big| \\
\le & \lambda \sup_{y\in D} \big| \operatorname{opt}\{p(x,y), u(x,y)h(a(x,y))\} - \operatorname{opt}\{p(x_0,y), u(x_0,y)h(a(x_0,y))\}\big| \\
& + (1-\lambda) \sup_{y\in D} \big| \operatorname{opt}\{q(x,y), v(x,y)h(b(x,y))\} \\
& - \operatorname{opt}\{q(x_0,y), v(x_0,y)h(b(x_0,y))\}\big| \\
& \le \lambda \sup_{y\in D} \max\{|p(x,y) - p(x_0,y)|, |u(x,y)h(a(x,y)) - u(x_0,y)h(a(x_0,y))|\} \\
& + (1-\lambda) \sup_{y\in D} \max\{|q(x,y) - q(x_0,y)|,
\end{aligned}
$$

$$
\begin{aligned}
|v(x,y)h(b(x,y)) & - v(x_0,y)h(b(x_0,y))|\} \\
& \le \lambda \sup_{y\in D} \max\left\{\frac{\varepsilon}{2M}, |h(a(x,y))||u(x,y) - u(x_0,y)|\right. \\
& \quad \left. + |u(x_0,y)||h(a(x,y)) - h(a(x_0,y))|\right\} \\
& \quad + (1-\lambda) \sup_{y\in D} \max\left\{\frac{\varepsilon}{2M}, |h(b(x,y))||v(x,y) - v(x_0,y)|\right. \\
& \quad \left. + |v(x_0,y)||h(b(x,y)) - h(b(x_0,y))|\right\} \\
& < \lambda \max\left\{\frac{\varepsilon}{2M}, M\cdot\frac{\varepsilon}{2M} + M\cdot\frac{\varepsilon}{2M}\right\} \\
& \quad + (1-\lambda) \max\left\{\frac{\varepsilon}{2M}, M\cdot\frac{\varepsilon}{2M} + M\cdot\frac{\varepsilon}{2M}\right\} = \varepsilon,
\end{aligned}
$$

for all $x \in S$ with $\|x - x_0\| < \delta$ and

$$
\begin{aligned}
|Hh(x)| = \Big| & \lambda \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)h(a(x,y))\} \\
& + (1-\lambda) \inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)h(b(x,y))\}\Big| \\
\le & \lambda \sup_{y\in D} |\operatorname{opt}\{p(x,y), u(x,y)h(a(x,y))\}| \\
& + (1-\lambda) \sup_{y\in D} |\operatorname{opt}\{q(x,y), v(x,y)h(b(x,y))\}|
\end{aligned}
$$

$$
\begin{aligned}
&\le \lambda \sup_{y\in D} \max\{|p(x,y)|, |u(x,y)||h(a(x,y))|\} \\
&\qquad + (1-\lambda) \sup_{y\in D} \max\{|q(x,y)|, |v(x,y)||h(b(x,y))|\} \\
&\le \lambda \max\{M, \alpha M\} + (1-\lambda)\max\{M, \alpha M\} = M,
\end{aligned}
$$

for all $x \in S$, which yield that Hh is continuous and bounded in S. That is, H maps $BC(S)$ into $BC(S)$.

Secondly, we show that H is a contraction mapping in $BC(S)$. Notice that (3.4), Lemmas 2.1 and 2.2 ensure that

$$
\begin{aligned}
|Hh(x) - Hg(x)| = \Big| &\lambda \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)h(a(x,y))\} \\
&+ (1-\lambda) \inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)h(b(x,y))\} \\
&- \lambda \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)g(a(x,y))\} \\
&- (1-\lambda) \inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)g(b(x,y))\}\Big| \\
\le\, &\lambda \sup_{y\in D} |\operatorname{opt}\{p(x,y), u(x,y)h(a(x,y))\} \\
&- \operatorname{opt}\{p(x,y), u(x,y)g(a(x,y))\}| \\
&+ (1-\lambda) \sup_{y\in D} |\operatorname{opt}\{q(x,y), v(x,y)h(b(x,y))\} \\
&- \operatorname{opt}\{q(x,y), v(x,y)g(b(x,y))\}| \\
\le\, &\lambda \sup_{y\in D}\{|u(x,y)||h(a(x,y)) - g(a(x,y))|\} \\
&+ (1-\lambda) \sup_{y\in D}\{|v(x,y)||h(b(x,y)) - g(b(x,y))|\} \\
\le\, &\lambda\alpha\|h-g\|_1 + (1-\lambda)\alpha\|h-g\|_1 = \alpha\|h-g\|_1,
\end{aligned}
$$

for all $x \in S$, $h, g \in BC(S)$, which means that

$$
\|Hh - Hg\|_1 \le \alpha\|h-g\|_1, \quad \text{for all } h, g \in BC(S), \tag{3.9}
$$

that is, H is a contraction mapping in $BC(S)$. Thus the Banach fixed point theorem guarantees that H has a unique fixed point $w \in BC(S)$, which is a unique continuous bounded solution of the functional equation (1.5) in $BC(S)$.

Thirdly, we show (C4). Note that, for all $x \in S$,

$$
\begin{aligned}
w(x) = \lambda &\sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)w(a(x,y))\} \\
&+ (1-\lambda) \inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)w(b(x,y))\}.
\end{aligned} \tag{3.10}
$$

By virtue of (3.1), (3.4), (3,9) and (3.10), we derive that

$$
|w_{n+1}(x) - w(x)| = \Big|(1-\alpha_n)w_n(x) + \alpha_n\Big(\lambda \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)w_n(a(x,y))\}
$$

$$
\begin{aligned}
&+ (1-\lambda) \inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)w_n(b(x,y))\}\Big) - w(x)\Big| \\
&\le (1-\alpha_n)|w_n(x) - w(x)| + \alpha_n \Big| \lambda \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)w_n(a(x,y))\} \\
&+ (1-\lambda) \inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)w_n(b(x,y))\} \\
&- \lambda \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)w(a(x,y))\} \\
&- (1-\lambda) \inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)w(b(x,y))\}\Big| \\
&\le (1-\alpha_n)|w_n(x) - w(x)| + \alpha_n |Hw_n(x) - Hw(x)| \\
&\le (1-\alpha_n)\|w_n - w\|_1 + \alpha_n\alpha|w_n(x) - w(x)| \le (1-(1-\alpha)\alpha_n)\|w_n - w\|_1,
\end{aligned}
$$

for all $(x,n) \in S \times \mathbb{N}_0$, which implies that

$$
\begin{aligned}
\|w_{n+1} - w\|_1 &\le (1-(1-\alpha)\alpha_n)\|w_n - w\|_1 \\
&\le e^{-(1-\alpha)\alpha_n}\|w_n - w\|_1 \le e^{-(1-\alpha)\sum_{i=0}^{n}\alpha_i}\|w_0 - w\|_1,
\end{aligned}
$$

for all $n \in \mathbb{N}_0$, which together with (3.3) gives that the Mann iterative sequence $\{w_n\}_{n\in\mathbb{N}_0}$ converges to w. This completes the proof. □

As in the proof of Theorem 3.1, we infer immediately the following result.

Theorem 3.2. *Let $\lambda \in [0,1]$ and $\alpha \in (0,1)$ be constants. Assume that $p,q,u,v\colon S\times D \to \mathbb{R}$ and $a,b\colon S\times D \to S$ satisfy* (C1) *and* (C2). *Then the functional equation* (1.5) *possesses a unique bounded solution $w \in B(S)$ and for each $w_0 \in B(S)$, the Mann iterative sequence $\{w_n\}_{n\in\mathbb{N}_0}$ defined by* (3.1) *and* (3.2) *converges to w and satisfies* (C4).

Now we give two examples as applications of Theorems 3.1 and 3.2.

Example 3.3. Consider the functional equation

$$
\begin{aligned}
f(x) = \lambda \sup_{y\in\mathbb{R}^+} \operatorname{opt}\left\{\frac{x^4}{x+y^3}, \frac{\cos(x^2y^3)}{y^4+3} f\left(\frac{x^4+y^2}{x^3+y^2}\right)\right\} \\
+ (1-\lambda) \inf_{y\in\mathbb{R}^+} \operatorname{opt}\left\{\frac{x^6}{x^2+y^2}, \frac{x^6\sin(x^5y^2)}{2x^6+y^3} f\left(\frac{x^2+15x+y^2}{x^2+y^2}\right)\right\},
\end{aligned}
\tag{3.11}
$$

for all $x \in [1,20]$.

Let $X = Y = \mathbb{R}$, $S = [1,20]$, $D = \mathbb{R}^+$, $\alpha = 5/6$, $\lambda \in [0,1]$, $p,q,u,v\colon S\times D \to \mathbb{R}$ and $a,b\colon S\times D \to S$ be defined by

$$
\begin{aligned}
p(x,y) &= \frac{x^4}{x+y^3}, & q(x,y) &= \frac{x^6}{x^2+y^2}, \\
u(x,y) &= \frac{\cos(x^2y^3)}{y^4+3}, & v(x,y) &= \frac{x^6\sin(x^5y^2)}{2x^6+y^3}, \\
a(x,y) &= \frac{x^4+y^2}{x^3+y^2}, & b(x,y) &= \frac{x^2+15x+y^2}{x^2+y^2},
\end{aligned}
$$

for all $(x,y) \in S \times D$. Clearly, the conditions of Theorem 3.1 hold. It follows from Theorem 3.1 that the functional equation (3.11) has a unique continuous bounded solution $w \in BC(S)$, and for each $w_0 \in BC(S)$, the Mann iterative sequence $\{w_n\}_{n\in\mathbb{N}_0}$ defined by

$$\begin{aligned} w_{n+1}(x) = &\bigg(1 - \frac{1}{\sqrt{5n+6+(-1)^n \sin n}}\bigg) w_n(x) + \frac{1}{\sqrt{5n+6+(-1)^n \sin n}} \\ &\times \bigg(\lambda \sup_{y\in\mathbb{R}^+} \operatorname{opt}\bigg\{\frac{x^4}{x+y^3}, \frac{\cos(x^2y^3)}{y^4+3} w_n\bigg(\frac{x^4+y^2}{x^3+y^2}\bigg)\bigg\} \\ &+ (1-\lambda) \inf_{y\in\mathbb{R}^+} \operatorname{opt}\bigg\{\frac{x^6}{x^2+y^2}, \frac{x^6 \sin(x^5y^2)}{2x^6+y^3} w_n\bigg(\frac{x^2+15x+y^2}{x^2+y^2}\bigg)\bigg\}\bigg), \end{aligned}$$

for all $(x,n) \in [1,20] \times \mathbb{N}_0$ and $\alpha_n = 1/\sqrt{5n+6+(-1)^n \sin n}$, for all $n \in \mathbb{N}_0$ converges to w and has the error estimate:

$$\|w_{n+1} - w\|_1 \le e^{-\frac{1}{6}\sum_{i=0}^{n} \frac{1}{\sqrt{5i+6+(-1)^i \sin i}}} \|w_0 - w\|_1, \quad \text{for all } n \in \mathbb{N}_0.$$

Example 3.4. Consider the functional equation

$$\begin{aligned} \text{(3.12)}\quad f(x) = &\lambda \sup_{y\in\mathbb{R}^-} \operatorname{opt}\bigg\{\frac{x^7y^4 + \ln\big(1+\frac{1}{1+x^3y^8}\big)}{x^7y^4+1}, \frac{xy^2+1}{x^2+y^4+2} f\bigg(\frac{x^5+x^3y^8+y^2}{x^2y^2+1}\bigg)\bigg\} \\ &+ (1-\lambda) \inf_{y\in\mathbb{R}^-} \operatorname{opt}\bigg\{\frac{x^3+y}{(x+1)^3+y^2}, \frac{x^3y^2\cos^5(x^2y^9)}{4x^3y^2+1} f\bigg(\frac{x^3y^4}{x^4|y|+1}\bigg)\bigg\}, \end{aligned}$$

for all $x \in \mathbb{R}^+$. Let $X = Y = \mathbb{R}$, $S = \mathbb{R}^+$, $D = \mathbb{R}^-$, $\alpha = 3/4$, $\lambda \in [0,1]$, $p,q,u,v\colon S \times D \to \mathbb{R}$ and $a,b\colon S \times D \to S$ be defined by

$$\begin{aligned} p(x,y) &= \frac{x^7y^4 + \ln\big(1+\frac{1}{1+x^3y^8}\big)}{x^7y^4+1}, & q(x,y) &= \frac{x^3+y}{(x+1)^3+y^2}, \\ u(x,y) &= \frac{xy^2+1}{x^2+y^4+2}, & v(x,y) &= \frac{x^3y^2\cos^5(x^2y^9)}{4x^3y^2+1}, \\ a(x,y) &= \frac{x^5+x^3y^8+y^2}{x^2y^2+1}, & b(x,y) &= \frac{x^3y^4}{x^4|y|+1}, \end{aligned}$$

for all $(x,y) \in S \times D$. Obviously, the conditions of Theorem 3.2 hold. Thus Theorem 3.2 means that the functional equation (3.12) has a unique bounded solution $w \in B(S)$, and for any $w_0 \in BC(S)$, the Mann iterative sequence $\{w_n\}_{n\in\mathbb{N}_0}$ defined by

$$\begin{aligned} w_{n+1}(x) = &\bigg(1 - \frac{n^2+3n+1}{n^3+4n^2-3n+4+\ln(n+3)}\bigg) w_n(x) \\ &+ \frac{n^2+3n+1}{n^3+4n^2-3n+4+\ln(n+3)} \end{aligned}$$

$$\times\Bigg(\lambda \sup_{y\in\mathbb{R}^-} \operatorname{opt}\left\{\frac{x^7y^4+\ln\left(1+\frac{1}{1+x^3y^8}\right)}{x^7y^4+1}, \frac{xy^2+1}{x^2+y^4+2}w_n\left(\frac{x^5+x^3y^8+y^2}{x^2y^2+1}\right)\right\}$$
$$+(1-\lambda)\inf_{y\in\mathbb{R}^-} \operatorname{opt}\left\{\frac{x^3+y}{(x+1)^3+y^2}, \frac{x^3y^2\cos^5(x^2y^9)}{4x^3y^2+1}w_n\left(\frac{x^3y^4}{x^4|y|+1}\right)\right\}\Bigg),$$

for all $(x,n)\in\mathbb{R}^+\times\mathbb{N}_0$ and

$$\alpha_n = \frac{n^2+3n+1}{n^3+4n^2-3n+4+\ln(n+3)}, \quad \text{for all } n\in\mathbb{N}_0$$

converges to w and has the error estimate:

$$\|w_{n+1}-w\|_1 \le e^{-\frac{1}{4}\sum\limits_{i=0}^{n}\frac{i^2+3i+1}{i^3+4i^2-3i+4+\ln(i+3)}}\|w_0-w\|_1, \quad \text{for all } n\in\mathbb{N}_0.$$

Next we study the behaviors of solutions and iterative approximations for the functional equation (1.5) in the complete metric space $BB(S)$.

Theorem 3.5. *Let $\lambda\in[0,1]$ and $\alpha\in(0,1)$ be constants and* (3.2) *hold. Assume that $p,q,u,v\colon S\times D\to\mathbb{R}$ and $a,b\colon S\times D\to S$ satisfy that*

(C5) *p and q are bounded on $\overline{B}(0,k)\times D$, for all $k\in\mathbb{N}$;*
(C6) $\sup\limits_{(x,y)\in\overline{B}(0,k)\times D}\max\{|u(x,y)|,|v(x,y)|\}\le\alpha$, *for all $k\in\mathbb{N}$;*
(C7) $\sup\limits_{(x,y)\in\overline{B}(0,k)\times D}\max\{\|a(x,y)\|,\|b(x,y)\|\}\le k$, *for all $k\in\mathbb{N}$.*

Then the functional equation (1.5) *possesses a unique solution $w\in BB(S)$ such that*

(C8) *for each $w_0\in BB(S)$, the Mann iterative sequence $\{w_n\}_{n\in\mathbb{N}_0}$ defined by*

$$w_{n+1}(x) = (1-\alpha_n)w_n(x)+\alpha_n\Big(\lambda\sup_{y\in D}\operatorname{opt}\{p(x,y),u(x,y)w_n(a(x,y))\} + (1-\lambda)\inf_{y\in D}\operatorname{opt}\{q(x,y),v(x,y)w_n(b(x,y))\}\Big), \tag{3.13}$$

for all $(x,k,n)\in\overline{B}(0,k)\times\mathbb{N}\times\mathbb{N}_0$, converges to w and has the error estimate:

$$d_k(w_{n+1},w)\le e^{-(1-\alpha)\sum\limits_{i=0}^{n}\alpha_i}d_k(w_0,w), \quad \textit{for all } (k,n)\in\mathbb{N}\times\mathbb{N}_0. \tag{3.14}$$

Proof. Define a mapping $H\colon BB(S)\to BB(S)$ by

$$Hh(x)=\lambda\sup_{y\in D}\operatorname{opt}\{p(x,y),u(x,y)h(a(x,y))\} + (1-\lambda)\inf_{y\in D}\operatorname{opt}\{q(x,y),v(x,y)h(b(x,y))\}, \tag{3.15}$$

for all $(x,k,h) \in \overline{B}(0,k) \times \mathbb{N} \times BB(S)$. Put $(k,h) \in \mathbb{N} \times BB(S)$. It follows from (C5) and (C7) that there exists $\gamma > 0$ such that

$$\sup_{(x,y)\in\overline{B}(0,k)\times D} \max\{|p(x,y)|, |q(x,y)|, |h(a(x,y)|, |h(b(x,y))|\} \le \gamma,$$

which together with (C6), (3.15) and Lemma 2.1 yields that

$$\begin{aligned} |Hh(x)| &= \Big|\lambda \operatorname*{opt}_{y\in D} \sup\{p(x,y), u(x,y)h(a(x,y))\} \\ &\qquad + (1-\lambda)\inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)h(b(x,y))\}\Big| \\ &\le \lambda \sup_{y\in D} \max\{|p(x,y)|, |u(x,y)||h(a(x,y))|\} \\ &\qquad + (1-\lambda)\sup_{y\in D} \max\{|q(x,y)|, |v(x,y)||h(b(x,y))|\} \\ &\le \lambda \max\{\gamma, \alpha\gamma\} + (1-\lambda)\max\{\gamma,\alpha\gamma\} = \gamma, \end{aligned}$$

for all $(x,k,h) \in \overline{B}(0,k) \times \mathbb{N} \times BB(S)$, which means that H is a self mapping in $BB(S)$.

On account of (3.15), (C6), (C7), Lemmas 2.1 and 2.2, we derive that

$$\begin{aligned} |Hh(x) - Hg(x)| &= \Big|\lambda \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)h(a(x,y))\} \\ &\quad + (1-\lambda)\inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)h(b(x,y))\} \\ &\quad - \lambda \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)g(a(x,y))\} \\ &\quad - (1-\lambda)\inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)g(b(x,y))\}\Big| \\ &\le \lambda \sup_{y\in D} |\operatorname{opt}\{p(x,y), u(x,y)h(a(x,y))\} - \operatorname{opt}\{p(x,y), u(x,y)g(a(x,y))\}| \\ &\quad + (1-\lambda)\sup_{y\in D} |\operatorname{opt}\{q(x,y), v(x,y)h(b(x,y))\} \\ &\quad - \operatorname{opt}\{q(x,y), v(x,y)g(b(x,y))\}| \\ &\le \lambda \sup_{y\in D}\{|u(x,y)||h(a(x,y)) - g(a(x,y))|\} \\ &\quad + (1-\lambda)\sup_{y\in D}\{|v(x,y)||h(b(x,y)) - g(b(x,y))|\} \\ &\le \lambda\alpha d_k(h,g) + (1-\lambda)\alpha d_k(h,g) = \alpha d_k(h,g), \end{aligned}$$

for all $(x,k,h,g) \in \overline{B}(0,k) \times \mathbb{N} \times BB(S) \times BB(S)$, which yields that

$$d_k(Hh, Hg) \le \alpha d_k(h,g), \quad \text{for all } (k,h,g) \in \mathbb{N} \times BB(S) \times BB(S). \tag{3.16}$$

For $z_0 = 0 \in BB(S)$, put

$$z_n(x) = \lambda \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)z_{n-1}(a(x,y))\} \\ + (1-\lambda) \inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)z_{n-1}(b(x,y))\}, \tag{3.17}$$

for all $(x,n) \in S \times \mathbb{N}$. In view of (3.15), (3.16) and (3.17), we obtain that

$$\begin{aligned} d_k(z_n, z_{n+m}) &\le \sum_{i=n}^{n+m-1} d_k(z_i, z_{i+1}) = \sum_{i=n}^{n+m-1} d_k(Hz_{i-1}, Hz_i) \\ &\le \sum_{i=n}^{n+m-1} \alpha d_k(z_{i-1}, z_i) \le \sum_{i=n}^{n+m-1} \alpha^i d_k(z_0, z_1) \le \frac{\alpha^n}{1-\alpha} d_k(z_0, z_1), \end{aligned}$$

for all $(k,n,m) \in \mathbb{N} \times \mathbb{N}_0 \times \mathbb{N}$, which implies that $\{z_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in $BB(S)$. Consequently, there exists some $w \in BB(S)$ such that $\{z_n\}_{n\in\mathbb{N}_0}$ converges to w. Notice that (3.16) ensures that

$$d_k(w, Hw) \le d_k(w, z_{n+1}) + d_k(Hz_n, Hw) \le d_k(w, z_{n+1}) + \alpha d_k(z_n, w) \to 0,$$

for all $k \in \mathbb{N}$ as $n \to \infty$, that is, $d_k(w, Hw) = 0$, for all $k \in \mathbb{N}$, which yields that

$$d(w, Hw) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(w, Hw)}{1 + d_k(w, Hw)} = 0,$$

that is, $w = Hw$.

Suppose that there exists $t \in BB(S) \setminus \{w\}$ with $t = Ht$. It follows that $d_{k_0}(w,t) > 0$ for some $k_0 \in \mathbb{N}$. In terms of (3.16), we see that

$$0 < d_{k_0}(w,t) = d_{k_0}(Hw, Ht) \le \alpha d_{k_0}(w,t) < d_{k_0}(w,t),$$

which is impossible. Hence the mapping H has a unique fixed point $w \in BB(S)$, which is a unique solution of the functional equation (1.5) in $BB(S)$.

Finally, we show that (C8) holds. Using (3.13), (3.15), (3.16) and Lemma 2.1, we infer that

$$\begin{aligned} |w_{n+1}(x) - w(x)| &= \Big|(1-\alpha_n)w_n(x) + \alpha_n\Big(\lambda \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)w_n(a(x,y))\} \\ &\qquad + (1-\lambda) \inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)w_n(b(x,y))\}\Big) - w(x)\Big| \\ &\le (1-\alpha_n)|w_n(x) - w(x)| \\ &\qquad + \alpha_n\Big|\lambda \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)w_n(a(x,y))\} \\ &\qquad + (1-\lambda) \inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)w_n(b(x,y))\} \\ &\qquad - \lambda \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)w(a(x,y))\} \end{aligned}$$

$$
\begin{aligned}
&\quad - (1-\lambda)\inf_{y\in D}\operatorname{opt}\{q(x,y), v(x,y)w(b(x,y))\}\Big| \\
&\le (1-\alpha_n)d_k(w_n,w) + \alpha_n d_k(Hw_n,Hw) \\
&\le (1-(1-\alpha)\alpha_n)d_k(w_n,w),
\end{aligned}
$$

for all $(x,k,n)\in \overline{B}(0,k)\times\mathbb{N}\times\mathbb{N}_0$, which yields that

$$
d_k(w_{n+1},w) \le e^{-(1-\alpha)\alpha_n} d_k(w_n,w) \le e^{-(1-\alpha)\sum\limits_{i=0}^{n}\alpha_i} d_k(w_0,w),
$$

for all $(k,n)\in\mathbb{N}\times\mathbb{N}_0$, which together with (3.2) yields that $\{w_n\}_{n\in\mathbb{N}_0}$ converges to w. This completes the proof. $\square$

Remark 3.6. Theorem 3.5 extends and improves Theorems 3.4 and 3.5 in [5]. The example below shows that Theorem 3.5 extends substantially Theorems 3.4 and 3.5 in [5].

Example 3.7. Consider the functional equation

$$
\begin{aligned}
(3.18)\quad f(x) = {} & \lambda\sup_{y\in\mathbb{R}}\operatorname{opt}\left\{\frac{x^{17}y^6\sin^9(x^5y^7)}{x^4y^6+2}, \frac{\cos^5(x^4y^3)}{2+\ln(x^2y^2+2)} f\left(\frac{x^9y^2\sin(x^5y^4)}{x^8y^2+1}\right)\right\} \\
& + (1-\lambda)\inf_{y\in\mathbb{R}}\operatorname{opt}\left\{\frac{x^{51}-\sqrt{x^{42}+1}}{(x^2-y^3)^2+1}, \frac{\sin(x^2y)}{3+\cos^7(x^4y^9)} f\left(\frac{x^2y^3\cos(x^3y^6)}{|xy^3|+1}\right)\right\},
\end{aligned}
$$

for all $x\in\mathbb{R}$. Put $X=Y=S=D=\mathbb{R}$, $\lambda\in[0,1]$ and $\alpha = 2/(2+\ln 2)$. Let $p,q,u,v\colon S\times D\to\mathbb{R}$ and $a,b\colon S\times D\to S$ be defined by

$$
\begin{aligned}
p(x,y) &= \frac{x^{17}y^6\sin^9(x^5y^7)}{x^4y^6+2}, & q(x,y) &= \frac{x^{51}-\sqrt{x^{42}+1}}{(x^2-y^3)^2+1}, \\
u(x,y) &= \frac{\cos^5(x^4y^3)}{2+\ln(x^2y^2+2)}, & v(x,y) &= \frac{\sin(x^2y)}{3+\cos^7(x^4y^9)}, \\
a(x,y) &= \frac{x^9y^2\sin(x^5y^4)}{x^8y^2+1}, & b(x,y) &= \frac{x^2y^3\cos(x^3y^6)}{|xy^3|+1},
\end{aligned}
$$

for all $(x,y)\in S\times D$. It is clear that the conditions of Theorem 3.5 hold. It follows from Theorem 3.5 that the functional equation (3.18) possesses a unique solution $w\in BB(S)$, and for each $w_0\in BB(S)$, the Mann iterative sequence $\{w_n\}_{n\in\mathbb{N}_0}$ defined by

$$
\begin{aligned}
w_{n+1}(x) = {} & \frac{\sqrt{4n+2}+3+(-1)^n}{\sqrt{n^2+1}+\sqrt{4n+2}+5} w_n(x) + \frac{\sqrt{n^2+1}+2-(-1)^n}{\sqrt{n^2+1}+\sqrt{4n+2}+5} \\
& \times \left(\lambda\sup_{y\in D}\operatorname{opt}\left\{\frac{x^{17}y^6\sin^9(x^5y^7)}{x^4y^6+2}, \frac{\cos^5(x^4y^3)}{2+\ln(x^2y^2+2)} w_n\left(\frac{x^9y^2\sin(x^5y^4)}{x^8y^2+1}\right)\right\}\right. \\
& \left. + (1-\lambda)\inf_{y\in D}\operatorname{opt}\left\{\frac{x^{51}-\sqrt{x^{42}+1}}{(x^2-y^3)^2+1}, \frac{\sin(x^2y)}{3+\cos^7(x^4y^9)} w_n\left(\frac{x^2y^3\cos(x^3y^6)}{|xy^3|+1}\right)\right\}\right),
\end{aligned}
$$

for all $(x,k,n) \in \overline{B}(0,k) \times \mathbb{N} \times \mathbb{N}_0$ and

$$\alpha_n = \frac{\sqrt{n^2+1}+2-(-1)^n}{\sqrt{n^2+1}+\sqrt{4n+2}+5}, \quad \text{for all } n \in \mathbb{N}_0$$

converges to w and has the error estimate:

$$d_k(w_{n+1},w) \le e^{-\frac{\ln 2}{2+\ln 2}\sum_{i=0}^{n}\frac{\sqrt{i^2+1}+2-(-1)^i}{\sqrt{i^2+1}+\sqrt{4i+2}+5}} d_k(w_0,w), \quad \text{for all } (k,n) \in \mathbb{N} \times \mathbb{N}_0.$$

But Theorems 3.4, 3.5 in [5] are unapplicable for the functional equation (3.18).

THEOREM 3.8. *Let $\lambda \in [0,1]$ and $\beta \in (0,1]$ be constants and $(\varphi,\psi) \in \Phi_2$. Assume that $p,q,u,v\colon S \times D \to \mathbb{R}$, $a,b\colon S \times D \to S$ and $c\colon S \to S$ satisfy that:*

(C9) $\sup\limits_{(x,y)\in\overline{B}(0,k)\times D} \max\{|p(x,y)|,|q(x,y)|\} \le \psi(\|x\|)$, *for all $k \in \mathbb{N}$;*

(C10) $\sup\limits_{(x,y)\in\overline{B}(0,k)\times D} \max\{|u(x,y)|,|v(x,y)|\} \le 1$, *for all $k \in \mathbb{N}$;*

(C11) $\max\Big\{\sup\limits_{x\in\overline{B}(0,k)} \|c(x)\|, \sup\limits_{(x,y)\in\overline{B}(0,k)\times D} \max\{\|a(x,y)\|,\|b(x,y)\|\}\Big\} \le \varphi(\|x\|)$, *for all $k \in \mathbb{N}$.*

Then the functional equation (1.5) *possesses a solution $w \in BB(S)$ such that*

(C12) *for each $w_0 \in BB(S)$ with $|w_0(x)| \le \psi(\|x\|)$, for all $(x,k) \in \overline{B}(0,k)\times\mathbb{N}$, the iterative sequence $\{w_n\}_{n\in\mathbb{N}_0}$ defined by*

$$\begin{aligned} w_{n+1}(x) &= (1-\beta)w_n(c(x)) \\ &\quad + \beta\Big(\lambda \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)w_n(a(x,y))\} \\ &\qquad + (1-\lambda)\inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)w_n(b(x,y))\}\Big), \end{aligned} \tag{3.19}$$

for all $(x,k,n) \in \overline{B}(0,k) \times \mathbb{N} \times \mathbb{N}_0$, converges to w and

$$d_k(w_n,w) \le \sum_{j=n-1}^{\infty} \psi(\varphi^j(k)), \quad \textit{for all } (k,n) \in \mathbb{N}\times\mathbb{N};$$

(C13) $|w(x)| \le \sum\limits_{n=0}^{\infty} \psi(\varphi^n(\|x\|))$, *for all $(x,k) \in \overline{B}(0,k) \times \mathbb{N}$;*

(C14) $\lim\limits_{n\to\infty} w(x_n) = 0$, *for all $(x_0,k) \in \overline{B}(0,k) \times \mathbb{N}$, $\{y_n\}_{n\in\mathbb{N}} \subset D$ with $x_n \in \{c(x_{n-1}), a(x_{n-1},y_n), b(x_{n-1},y_n)\}$, for all $n \in \mathbb{N}$;*

(C15) *w is a unique solution of the functional equation* (1.5) *relative to* (C14).

PROOF. Now we show that

$$\varphi(t) < t, \quad \text{for all } t > 0. \tag{3.20}$$

Suppose that there exists some $t_0 > 0$ such that $\varphi(t_0) \geq t_0 > 0$. Observe that $\sum_{n=0}^{\infty} \psi(\varphi^n(t_0)) < +\infty$ and $(\varphi, \psi) \in \Phi_2$. It follows that

$$0 < \psi(t_0) \leq \psi(\varphi(t_0)) \leq \ldots \leq \psi(\varphi^{n-1}(t_0)) \leq \psi(\varphi^n(t_0)), \quad \text{for all } n \in \mathbb{N},$$

which implies that $\lim_{n\to\infty} \psi(\varphi^n(t_0)) \geq \psi(t_0) > 0$, which yields that $\sum_{n=0}^{\infty} \psi(\varphi^n(t_0)) = +\infty$, which is absurd. Hence (3.20) holds.

Define a mapping $H\colon BB(S) \to BB(S)$ by

$$Hh(x) = \lambda Ah(x) + (1-\lambda)Bh(x), \quad \text{for all } (x,k,h) \in \overline{B}(0,k)\times\mathbb{N}\times BB(S), \tag{3.21}$$

where, for all $(x,k,h) \in \overline{B}(0,k) \times \mathbb{N} \times BB(S)$,

$$\begin{aligned} Ah(x) &= \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)h(a(x,y))\}, \\ Bh(x) &= \inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)h(b(x,y))\}. \end{aligned} \tag{3.22}$$

It follows from $(\varphi, \psi) \in \Phi_2$ and (3.20) that (C9) and (C11) imply (C5) and (C7), respectively. Similar to the proof of Theorem 3.5, by (C10) we deduce that the mapping H maps $BB(S)$ into $BB(S)$ and satisfies that

$$d_k(Hh, Hg) \leq d_k(h,g), \quad \text{for all } (k,h,g) \in \mathbb{N} \times BB(S) \times BB(S),$$

which means that, for all $(h,g) \in BB(S) \times BB(S)$,

$$d(Hh,Hg) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(Hh,Hg)}{1+d_k(Hh,Hg)} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(h,g)}{1+d_k(h,g)} = d(h,g), \tag{3.23}$$

that is, the mapping H is nonexpansive in $BB(S)$. Now we show that for each $n \in \mathbb{N}_0$

$$|w_n(x)| \leq \sum_{j=0}^{n} \psi(\varphi^j(\|x\|)), \quad \text{for all } (x,k) \in \overline{B}(0,k) \times \mathbb{N}. \tag{3.24}$$

It is clear that (3.24) holds for $n = 0$. Assume that (3.24) holds for some $n \in \mathbb{N}_0$. In view of (C9), (C11), (C12), (3.19), $(\varphi, \psi) \in \Phi_2$ and Lemma 2.1, we derive that

$$\begin{aligned} |w_{n+1}(x)| = \bigg|&(1-\beta)w_n(c(x)) + \beta\bigg(\lambda \sup_{y\in D} \operatorname{opt}\{p(x,y), u(x,y)w_n(a(x,y))\} \\ &+ (1-\lambda) \inf_{y\in D} \operatorname{opt}\{q(x,y), v(x,y)w_n(b(x,y))\}\bigg)\bigg| \\ \leq{}& (1-\beta)|w_n(c(x))| + \beta\bigg(\lambda \sup_{y\in D} |\operatorname{opt}\{p(x,y), u(x,y)w_n(a(x,y))\}| \\ &+ (1-\lambda) \sup_{y\in D} |\operatorname{opt}\{q(x,y), v(x,y)w_n(b(x,y))\}|\bigg) \end{aligned}$$

$$
\begin{aligned}
&\le (1-\beta)\sum_{j=0}^{n}\psi(\varphi^j(\|c(x)\|))\\
&\quad+\beta\Big(\lambda\sup_{y\in D}\max\{|p(x,y)|,|u(x,y)||w_n(a(x,y))|\}\\
&\quad+(1-\lambda)\sup_{y\in D}\max\{|q(x,y)|,|v(x,y)||w_n(b(x,y))|\}\Big)\\
&\le (1-\beta)\sum_{j=0}^{n}\psi(\varphi^{j+1}(\|x\|))\\
&\quad+\beta\Big(\lambda\sup_{y\in D}\max\Big\{\psi(\|x\|),\sum_{j=0}^{n}\psi(\varphi^j(\|a(x,y)\|))\Big\}\\
&\quad+(1-\lambda)\sup_{y\in D}\max\Big\{\psi(\|x\|),\sum_{j=0}^{n}\psi(\varphi^j(\|b(x,y)\|))\Big\}\Big)\\
&\le (1-\beta)\sum_{j=0}^{n}\psi(\varphi^{j+1}(\|x\|))+\beta\Big(\lambda\max\Big\{\psi(\|x\|),\sum_{j=0}^{n}\psi(\varphi^{j+1}(\|x\|))\Big\}\\
&\quad+(1-\lambda)\max\Big\{\psi(\|x\|),\sum_{j=0}^{n}\psi(\varphi^{j+1}(\|x\|))\Big\}\Big)\\
&\le (1-\beta)\sum_{j=0}^{n}\psi(\varphi^{j+1}(\|x\|))\\
&\quad+\beta\Big(\lambda\sum_{j=0}^{n+1}\psi(\varphi^j(\|x\|))+(1-\lambda)\sum_{j=0}^{n+1}\psi(\varphi^j(\|x\|))\Big)\le\sum_{j=0}^{n+1}\psi(\varphi^j(\|x\|)),
\end{aligned}
$$

for all $(x,k,n)\in\overline{B}(0,k)\times\mathbb{N}\times\mathbb{N}_0$, that is, (3.24) holds for $n+1$. Consequently (3.24) holds for each $n\in\mathbb{N}_0$.

Let $\varepsilon>0$, $k,n,p\in\mathbb{N}$ and $x_0\in\overline{B}(0,k)$. It follows from (3.22) that there exist $y,y_0,z,z_0\in D$ with

$$
\begin{aligned}
Aw_{n+p-1}(x_0)-2^{-1}\varepsilon &< \mathrm{opt}\{p(x_0,y),u(x_0,y)w_{n+p-1}(a(x_0,y))\},\\
Aw_{n-1}(x_0)-2^{-1}\varepsilon &< \mathrm{opt}\{p(x_0,y_0),u(x_0,y_0)w_{n-1}(a(x_0,y_0))\},\\
Aw_{n+p-1}(x_0) &\ge \mathrm{opt}\{p(x_0,y_0),u(x_0,y_0)w_{n+p-1}(a(x_0,y_0))\},\\
Aw_{n-1}(x_0) &\ge \mathrm{opt}\{p(x_0,y),u(x_0,y)w_{n-1}(a(x_0,y))\}
\end{aligned}
\tag{3.25}
$$

and

$$
\begin{aligned}
Bw_{n+p-1}(x_0)+2^{-1}\varepsilon &> \mathrm{opt}\{q(x_0,z),v(x_0,z)w_{n+p-1}(b(x_0,z))\},\\
Bw_{n-1}(x_0)+2^{-1}\varepsilon &> \mathrm{opt}\{q(x_0,z_0),v(x_0,z_0)w_{n-1}(b(x_0,z_0))\},\\
Bw_{n+p-1}(x_0) &\le \mathrm{opt}\{q(x_0,z_0),v(x_0,z_0)w_{n+p-1}(b(x_0,z_0))\},\\
Bw_{n-1}(x_0) &\le \mathrm{opt}\{q(x_0,z),v(x_0,z)w_{n-1}(b(x_0,z))\}.
\end{aligned}
\tag{3.26}
$$

Combining (C10), (3.25) and (3.26) and using Lemma 2.2, we gain that

$$\begin{aligned}Aw_{n+p-1}(x_0) &- Aw_{n-1}(x_0)\\ &< \operatorname{opt}\{p(x_0,y),u(x_0,y)w_{n+p-1}(a(x_0,y))\}\\ &\quad - \operatorname{opt}\{p(x_0,y),u(x_0,y)w_{n-1}(a(x_0,y))\} + 2^{-1}\varepsilon\\ &\le |u(x_0,y)||w_{n+p-1}(a(x_0,y)) - w_{n-1}(a(x_0,y))| + 2^{-1}\varepsilon\\ &\le |w_{n+p-1}(a(x_0,y)) - w_{n-1}(a(x_0,y))| + 2^{-1}\varepsilon;\end{aligned}$$

$$\begin{aligned}Aw_{n+p-1}(x_0) &- Aw_{n-1}(x_0)\\ &> \operatorname{opt}\{p(x_0,y_0),u(x_0,y_0)w_{n+p-1}(a(x_0,y_0))\}\\ &\quad - \operatorname{opt}\{p(x_0,y_0),u(x_0,y_0)w_{n-1}(a(x_0,y_0))\} - 2^{-1}\varepsilon\\ &\ge -|u(x_0,y_0)||w_{n+p-1}(a(x_0,y_0)) - w_{n-1}(a(x_0,y_0))| - 2^{-1}\varepsilon\\ &\ge -|w_{n+p-1}(a(x_0,y_0)) - w_{n-1}(a(x_0,y_0))| - 2^{-1}\varepsilon;\end{aligned}$$

$$\begin{aligned}Bw_{n+p-1}(x_0) &- Bw_{n-1}(x_0)\\ &> \operatorname{opt}\{q(x_0,z),v(x_0,z)w_{n+p-1}(b(x_0,z))\}\\ &\quad - \operatorname{opt}\{q(x_0,z),v(x_0,z)w_{n-1}(b(x_0,z))\} - 2^{-1}\varepsilon\\ &\ge -|v(x_0,z)||w_{n+p-1}(b(x_0,z)) - w_{n-1}(b(x_0,z))| - 2^{-1}\varepsilon\\ &\ge -|w_{n+p-1}(b(x_0,z)) - w_{n-1}(b(x_0,z))| - 2^{-1}\varepsilon\end{aligned}$$

and

$$\begin{aligned}Bw_{n+p-1}(x_0) &- Bw_{n-1}(x_0)\\ &< \operatorname{opt}\{q(x_0,z_0),v(x_0,z_0)w_{n+p-1}(b(x_0,z_0))\}\\ &\quad - \operatorname{opt}\{q(x_0,z_0),v(x_0,z_0)w_{n-1}(b(x_0,z_0))\} + 2^{-1}\varepsilon\\ &\le |v(x_0,z_0)||w_{n+p-1}(b(x_0,z_0)) - w_{n-1}(b(x_0,z_0))| + 2^{-1}\varepsilon\\ &\le |w_{n+p-1}(b(x_0,z_0)) - w_{n-1}(b(x_0,z_0))| + 2^{-1}\varepsilon,\end{aligned}$$

which imply that

$$\begin{aligned}|Aw_{n+p-1}(x_0) &- Aw_{n-1}(x_0)|\\ &< \max\{|w_{n+p-1}(a(x_0,y)) - w_{n-1}(a(x_0,y))|,\\ &\qquad |w_{n+p-1}(a(x_0,y_0)) - w_{n-1}(a(x_0,y_0))|\} + 2^{-1}\varepsilon,\end{aligned}\tag{3.27}$$

$$\begin{aligned}|Bw_{n+p-1}(x_0) &- Bw_{n-1}(x_0)|\\ &< \max\{|w_{n+p-1}(b(x_0,z)) - w_{n-1}(b(x_0,z))|,\\ &\qquad |w_{n+p-1}(b(x_0,z_0)) - w_{n-1}(b(x_0,z_0))|\} + 2^{-1}\varepsilon.\end{aligned}\tag{3.28}$$

In light of (3.19), (3.21), (3.22), (3.27) and (3.28), we infer that

$$
\begin{aligned}
(3.29)\quad |w_{n+p}(x_0) - w_n(x_0)| &= |(1-\beta)w_{n+p-1}(c(x_0)) + \beta(\lambda A w_{n+p-1}(x_0) \\
&\quad + (1-\lambda)Bw_{n+p-1}(x_0)) - (1-\beta)w_{n-1}(c(x_0)) \\
&\quad - \beta\big(\lambda A w_{n-1}(x_0) + (1-\lambda)Bw_{n-1}(x_0)\big)| \\
&\le (1-\beta)|w_{n+p-1}(c(x_0)) - w_{n-1}(c(x_0))| \\
&\quad + \beta\big(\lambda|Aw_{n+p-1}(x_0) - Aw_{n-1}(x_0)| \\
&\quad + (1-\lambda)|Bw_{n+p-1}(x_0) - Bw_{n-1}(x_0)|\big) \\
&< (1-\beta)|w_{n+p-1}(c(x_0)) - w_{n-1}(c(x_0))| \\
&\quad + \beta(\lambda \max\{|w_{n+p-1}(a(x_0,y)) - w_{n-1}(a(x_0,y))|, \\
&\qquad |w_{n+p-1}(a(x_0,y_0)) - w_{n-1}(a(x_0,y_0))|\} + 2^{-1}\lambda\varepsilon \\
&\quad + (1-\lambda)\max\{|w_{n+p-1}(b(x_0,z)) - w_{n-1}(b(x_0,z))|, \\
&\qquad |w_{n+p-1}(b(x_0,z_0)) - w_{n-1}(b(x_0,z_0))|\} + 2^{-1}(1-\lambda)\varepsilon) \\
&\le \max\{|w_{n+p-1}(c(x_0)) - w_{n-1}(c(x_0))|, \\
&\qquad |w_{n+p-1}(a(x_0,y)) - w_{n-1}(a(x_0,y))|, \\
&\qquad |w_{n+p-1}(a(x_0,y_0)) - w_{n-1}(a(x_0,y_0))|, \\
&\qquad |w_{n+p-1}(b(x_0,z)) - w_{n-1}(b(x_0,z))|, \\
&\qquad |w_{n+p-1}(b(x_0,z_0)) - w_{n-1}(b(x_0,z_0))|\} + 2^{-1}\varepsilon \\
&= |w_{n+p-1}(x_1) - w_{n-1}(x_1)| + 2^{-1}\varepsilon
\end{aligned}
$$

for some $y_1 \in \{y, y_0, z, z_0\}$ and $x_1 \in \{c(x_0), a(x_0,y_1), b(x_0,y_1)\}$. Similarly, we get that for each $i \in \{1, \dots, n\}$ there exist $y_i \in D$ and $x_i \in \{c(x_{i-1}), a(x_{i-1},y_i), b(x_{i-1},y_i)\}$ satisfying

$$
(3.30)\quad
\begin{aligned}
&|w_{n+p-1}(x_1) - w_{n-1}(x_1)| < |w_{n+p-2}(x_2) - w_{n-2}(x_2)| + 2^{-2}\varepsilon, \\
&|w_{n+p-2}(x_2) - w_{n-2}(x_2)| < |w_{n+p-3}(x_3) - w_{n-3}(x_3)| + 2^{-3}\varepsilon, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&|w_{p+1}(x_{n-1}) - w_1(x_{n-1})| < |w_p(x_n) - w_0(x_n)| + 2^{-n}\varepsilon.
\end{aligned}
$$

Next we show that $\{w_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in $(BB(S), d)$. According to $(\varphi, \psi) \in \Phi_2$, (C10), (3.24), (3.29) and (3.30), we gain that

$$
\begin{aligned}
|w_{n+p}(x_0) - w_n(x_0)| &< |w_p(x_n) - w_0(x_n)| + \varepsilon \le \sum_{j=0}^{p} \psi(\varphi^j(\|x_n\|)) + \psi(\|x_n\|) + \varepsilon \\
&\le \sum_{j=0}^{p} \psi(\varphi^{j+n}(k)) + \psi(\varphi^n(k)) + \varepsilon \le \sum_{j=n-1}^{p+n} \psi(\varphi^j(k)) + \varepsilon,
\end{aligned}
$$

which means that $d_k(w_{n+p}, w_n) \le \sum_{j=n-1}^{n+p} \psi(\varphi^j(k)) + \varepsilon$, taking $\varepsilon \to 0^+$ in the above inequality, we infer that

$$d_k(w_{n+p}, w_n) \le \sum_{j=n-1}^{n+p} \psi(\varphi^j(k)). \tag{3.31}$$

Observe that $\sum_{n=0}^{\infty} \psi(\varphi^n(t)) < +\infty$ for each $t > 0$. Thus (3.31) guarantees that $\{w_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in $(BB(S), d)$ and it converges to some $w \in BB(S)$. Letting $p \to \infty$ in (3.31), we know immediately that

$$d_k(w_n, w) \le \sum_{j=n-1}^{\infty} \psi(\varphi^j(k)), \quad \text{for all } (k, n) \in \mathbb{N} \times \mathbb{N}.$$

In terms of (3.23), we see that

$$d(Hw, w) \le d(Hw, Hw_n) + d(w_{n+1}, w) \le d(w, w_n) + d(w_{n+1}, w) \to 0 \quad \text{as } n \to \infty,$$

which yields that $Hw = w$, that is, the functional equation (1.5) possesses a solution $w \in BB(S)$.

Next we show (C13). Let $(x, k) \in \overline{B}(0, k) \times \mathbb{N}$. Using (C10), (3.24) and $(\varphi, \psi) \in \Phi_2$, we conclude that

$$|w(x)| \le |w(x) - w_n(x)| + |w_n(x)| \le d_k(w, w_n) + \sum_{j=0}^{n} \psi(\varphi^j(\|x\|)) \to \sum_{j=0}^{\infty} \psi(\varphi^j(\|x\|))$$

as $n \to \infty$, that is, (C13) holds.

Next we show (C14). Given $(x_0, k) \in \overline{B}(0, k) \times \mathbb{N}$ and $\{y_n\}_{n\in\mathbb{N}} \subset D$ with $x_n \in \{c(x_{n-1}), a(x_{n-1}, y_n), b(x_{n-1}, y_n)\}$ for each $n \in \mathbb{N}$. It follows from (C11), (3.20) and $(\varphi, \psi) \in \Phi_2$ that

$$\begin{aligned} \|x_n\| &\le \max\{\|a(x_{n-1}, y_n)\|, \|b(x_{n-1}, y_n)\|, \|c(x_{n-1})\|\} \\ &\le \varphi(\|x_{n-1}\|) \le \ldots \le \varphi^n(\|x_0\|) \le \varphi^n(k) < k, \end{aligned}$$

for all $n \in \mathbb{N}$, which together with (3.24) implies that

$$\begin{aligned} |w(x_n)| &\le |w(x_n) - w_n(x_n)| + |w_n(x_n)| \\ &\le d_k(w, w_n) + \sum_{j=0}^{n} \psi(\varphi^j(\|x_n\|)) \le d_k(w, w_n) + \sum_{j=n}^{2n} \psi(\varphi^j(k)) \to 0 \end{aligned}$$

as $n \to \infty$, which yields that $\lim_{n\to\infty} w(x_n) = 0$.

Finally, we show (C15). Suppose that the functional equation (1.5) has another solution $h \in BB(S)$ that satisfies (C14).

Given $(x_0, k) \in \overline{B}(0, k) \times \mathbb{N}$ and $\varepsilon > 0$. It follows from (3.22) that there exist $y, y_0, z, z_0 \in D$ with

$$\begin{aligned}
Aw(x_0) - 2^{-1}\varepsilon &< \operatorname{opt}\{p(x_0, y), u(x_0, y)w(a(x_0, y))\},\\
Ah(x_0) - 2^{-1}\varepsilon &< \operatorname{opt}\{p(x_0, y_0), u(x_0, y_0)h(a(x_0, y_0))\},\\
Aw(x_0) &\geq \operatorname{opt}\{p(x_0, y_0), u(x_0, y_0)w(a(x_0, y_0))\},\\
Ah(x_0) &\geq \operatorname{opt}\{p(x_0, y), u(x_0, y)h(a(x_0, y))\},\\
Bw(x_0) + 2^{-1}\varepsilon &> \operatorname{opt}\{q(x_0, z), v(x_0, z)w_{n+p-1}(b(x_0, z))\},\\
Bh(x_0) + 2^{-1}\varepsilon &> \operatorname{opt}\{q(x_0, z_0), v(x_0, z_0)h(b(x_0, z_0))\},\\
Bw(x_0) &\leq \operatorname{opt}\{q(x_0, z_0), v(x_0, z_0)w_{n+p-1}(b(x_0, z_0))\},\\
Bh(x_0) &\leq \operatorname{opt}\{q(x_0, z), v(x_0, z)h(b(x_0, z))\},
\end{aligned}$$

which together with (C10), (3.21) and (3.22) yield that

$$\begin{aligned}
|w(x_0) - h(x_0)| &= |\lambda Aw(x_0) + (1-\lambda)Bw(x_0) - \lambda Ah(x_0) - (1-\lambda)Bh(x_0)|\\
&\leq \lambda|Aw(x_0) - Ah(x_0)| + (1-\lambda)|Bw(x_0) - Bh(x_0)|\\
&< \lambda \max\{|u(x_0, y)||w(a(x_0, y)) - h(a(x_0, y))|,\\
&\qquad |u(x_0, y_0)||w(a(x_0, y_0)) - h(a(x_0, y_0))|\} + 2^{-1}\lambda\varepsilon\\
&\quad + (1-\lambda)\max\{|v(x_0, z)||w(b(x_0, z)) - h(b(x_0, z))|,\\
&\qquad |v(x_0, z_0)||w(b(x_0, z_0)) - h(b(x_0, z_0))|\} + 2^{-1}(1-\lambda)\varepsilon\\
&\leq \max\{|w(a(x_0, y)) - h(a(x_0, y))|, |w(a(x_0, y_0)) - h(a(x_0, y_0))|\\
&\qquad |w(b(x_0, z)) - h(b(x_0, z))|, |w(b(x_0, z_0)) - h(b(x_0, z_0))|\} + 2^{-1}\varepsilon\\
&= |w(x_1) - h(x_1)| + 2^{-1}\varepsilon
\end{aligned}$$

for some $x_1 \in \{a(x_0, y_1), b(x_0, y_1)\}$ and $y_1 \in \{y, y_0, z, z_0\}$, that is,

$$|w(x_0) - h(x_0)| \leq |w(x_1) - h(x_1)| + 2^{-1}\varepsilon. \tag{3.32}$$

Proceeding in this way, we get that for each $n \in \mathbb{N} \setminus \{1\}$, there exist $x_i \in \{a(x_{i-1}, y_i), b(x_{i-1}, y_i)\}$ and $y_i \in D$, $i \in \{2, \dots, n\}$, satisfying

$$\begin{aligned}
|w(x_1) - h(x_1)| &\leq |w(x_2) - h(x_2)| + 2^{-2}\varepsilon,\\
|w(x_2) - h(x_2)| &\leq |w(x_3) - h(x_3)| + 2^{-3}\varepsilon,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
|w(x_{n-1}) - h(x_{n-1})| &\leq |w(x_n) - h(x_n)| + 2^{-n}\varepsilon.
\end{aligned} \tag{3.33}$$

Using (3.32) and (3.33), we infer that $|w(x_0) - h(x_0)| \leq |w(x_n) - h(x_n)| + \varepsilon \to \varepsilon$ as $n \to \infty$, letting $\varepsilon \to 0^+$ in the above inequality, we derive that $w(x_0) = h(x_0)$.□

Remark 3.9. Theorem 3.8 generalizes and unifies Theorems 3.4 and 3.5 in [4], Theorem 3.5 in [6], Corollaries 2.2 and 2.3 in [8], Theorem 3.5 in [9], Corollaries 3.1, 3.3 and 3.4 in [11], Theorems 2.3 and 2.4 in [12], Theorem 3.4 in [15]. The example below shows that Theorem 3.8 extends properly the corresponding results in [4], [6], [8], [9], [11], [12] and [15].

Example 3.10. Consider the functional equation

$$\begin{aligned} f(x) = {} & \lambda \sup_{y\in\mathbb{R}} \operatorname{opt}\left\{\frac{x^9y^2\sin(x^8-y^6)}{x^2y^2+1}, \sin^3(\sqrt{x}y^2) f\left(\frac{x\ln\left(1+\frac{1}{1+x^2y^6}\right)}{4+\cos^9(x^5y^7)}\right)\right\} \\ & + (1-\lambda)\inf_{y\in\mathbb{R}} \operatorname{opt}\left\{\frac{x^8\arctan^7(x^5y^3)}{x+y^2+1}, \cos^5(x^9y^4) f\left(\frac{x^6\sin^2(x^2y^4)}{3x^5+y^8+1}\right)\right\}, \end{aligned} \tag{3.34}$$

for all $x \in \mathbb{R}^+$. Let $\lambda \in [0,1]$, $\beta \in (0,1], X = Y = \mathbb{R}$, $S = \mathbb{R}^+$, $D = \mathbb{R}$. Let $p, q, u, v\colon S\times D \to \mathbb{R}$, $a, b\colon S\times D \to S$, $c\colon S \to S$ and $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$\begin{aligned} p(x,y) &= \frac{x^9y^2\sin(x^8-y^6)}{x^2y^2+1}, & q(x,y) &= \frac{x^8\arctan^7(x^5y^3)}{x+y^2+1}, \\ u(x,y) &= \sin^3(\sqrt{x}y^2), & v(x,y) &= \cos^5(x^9y^4), \\ a(x,y) &= \frac{x\ln\left(1+\frac{1}{1+x^2y^6}\right)}{4+\cos^9(x^5y^7)}, & b(x,y) &= \frac{x^6\sin^2(x^2y^4)}{3x^5+y^8+1}, \\ c(x) &= \frac{\sin^2\sqrt{x}}{5+2\cos(x^9-8x^7+3x^6-x-1)}, & & \text{for all } (x,y)\in S\times D, \\ \varphi(t) &= \frac{t}{3}, \quad \psi(t) = 36t^7, & & \text{for all } t \in \mathbb{R}^+. \end{aligned}$$

For any $w_0 \in BB(S)$ with $|w_0(x)| \le \psi(\|x\|)$, for all $(x,k) \in \overline{B}(0,k)\times\mathbb{N}$, define the iterative sequence $\{w_n\}_{n\in\mathbb{N}_0}$ by

$$\begin{aligned} w_{n+1}(x) = {} & (1-\beta) w_n\left(\frac{\sin^2\sqrt{x}}{5+2\cos(x^9-8x^7+3x^6-x-1)}\right) \\ & + \beta\left[\lambda \sup_{y\in D} \operatorname{opt}\left\{\frac{x^9y^2\sin(x^8-y^6)}{x^2y^2+1}, \sin^3(\sqrt{x}y^2) w_n\left(\frac{x\ln\left(1+\frac{1}{1+x^2y^6}\right)}{4+\cos^9(x^5y^7)}\right)\right\}\right. \\ & \left. + (1-\lambda)\inf_{y\in D} \operatorname{opt}\left\{\frac{x^8\arctan^7(x^5y^3)}{x+y^2+1}, \cos^5(x^9y^4) w_n\left(\frac{x^6\sin^2(x^2y^4)}{3x^5+y^8+1}\right)\right\}\right], \end{aligned}$$

for all $(x,k,n) \in \overline{B}(0,k)\times\mathbb{N}\times\mathbb{N}_0$. Obviously, the conditions of Theorem 3.4 hold. It follows from Theorem 3.4 that the functional equation (3.34) has a unique solution $w \in BB(S)$ satisfying (C12)–(C15). In particular, the iterative sequence $\{w_n\}_{n\in\mathbb{N}_0}$ converges to w. However, Theorems 3.4 and 3.5 in [4], Theorem 3.5 in [6], Corollaries 2.2 and 2.3 in [8], Theorem 3.5 in [9], Corollaries 3.1, 3.3 and 3.4 in [11], Theorems 2.3 and 2.4 in [12], Theorem 3.4 in [15] are not applicable for the functional equation (3.34).

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