

**EXISTENCE OF PERIODIC SOLUTIONS
FOR SOME SINGULAR ELLIPTIC EQUATIONS
WITH STRONG RESONANT DATA**

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ABSTRACT. We prove the existence of at least one T -periodic solution ($T > 0$) for differential equations of the form

$$\left(\frac{u'(t)}{\sqrt{1-u'^2(t)}} \right)' = f(u(t)) + h(t), \quad \text{in } (0, T),$$

where f is a continuous function defined on \mathbb{R} that satisfies a *strong resonance condition*, h is continuous and with zero mean value. Our method uses variational techniques for nonsmooth functionals.

1. Introduction

In this paper we deal with existence of periodic solutions for a class of equations whose model is

$$(1.1) \quad \left(\frac{u'(t)}{\sqrt{1-u'^2(t)}} \right)' = f(u(t)) + h(t), \quad t \in (0, T),$$

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where $T > 0$, f and h are continuous and $\int_0^T h(t)dt = 0$ (i.e. h has zero mean value). This equation is known in literature as *relativistic* forced pendulum.

We recall that the equation of a *classical* forced pendulum provided with periodic boundary conditions can be formulated as

$$(1.2) \quad \begin{cases} u''(t) = A \sin(u(t)) + h(t), & t \in (0, T), \\ u(0) - u(T) = 0 = u'(0) - u'(T), \end{cases}$$

where $A \in \mathbb{R}$ and h is a continuous function with zero mean value.

From the physical point of view, $u(t)$ represents the position of the pendulum and $u''(t)$ its acceleration. The equation in (1.2) is indeed the expression of the classical Newton's law, where the external forces are represented by $h(t)$, the temporal-dependent part, and by $A \sin(u(t))$, which instead depends on the position.

Problem (1.2) has been studied, among others, by Hamel (see the pioneering paper [10]), and later by Willem (see [16]) and Dancer (in [8]). These authors proved the existence of at least a solution by minimizing the energy functional associated to the equation in suitable spaces. Due to the periodicity of this functional, once found the minimum u , we know infinitely many other minima to exist, which have the form $u + j\omega$, for all $j \in \mathbb{Z}$. Later on, Mawhin and Willem (see [12]) exploited this property in order to prove the existence of a second geometrically distinct solution (i.e. a solution v of the problem, such that $v \neq u + j\omega$, $j \in \mathbb{Z}$) which is not a minimum anymore, but has a different nature.

Another contributions were given by Thews in [14], by Ambrosetti and Coti Zelati in [1], by Coti Zelati in [7] and by Arcoya in [3], who studied problem (1.2) replacing the external force $\sin(u)$ with a continuous function $f(u)$, under an assumption on its behavior at infinity, namely:

$$\lim_{|s| \rightarrow +\infty} f(s) = \lim_{|s| \rightarrow +\infty} F(s) = 0,$$

where by F we mean the primitive of f defined by $F(s) = \int_0^s f(\sigma) d\sigma$. In these papers one can find conditions under which the problem is solvable and informations about multiplicity of solutions.

The motivation to generalize the classical pendulum equation comes from the study of a pendulum moved by the laws of relativity. It is well known (see for instance [11]) that the relativistic Newton's law, obtained by use of Lorentz transformation γ , is

$$F = m \frac{d}{dt} \gamma(u'(t)), \quad \text{where } \gamma(s) = \frac{s}{\sqrt{1 - s^2/c^2}},$$

$u'(t)$ is the velocity and c is the speed of light in vacuum. Thus (1.1) represents the equation of a forced pendulum in the relativistic framework (by considering

$m = c = 1$). Moreover, due to the nature of the problem, it seems natural to look for periodic solutions, which leads us to the study of

$$(1.3) \quad \begin{cases} \left(\frac{u'(t)}{\sqrt{1-u'^2(t)}} \right)' = f(u(t)) + h(t), & t \in (0, T), \\ u(0) - u(T) = 0 = u'(0) - u'(T), \end{cases}$$

where f is a continuous function with primitive $F(s) = \int_0^s f(\sigma) d\sigma$.

The main difficulties in dealing with this problem are due both on the presence of a nonlinearity in the right hand side of the equation and on the singularity of the principal part of the operator, which makes the energy functional non-smooth. Let us note that, due to the fact that we handle functions with bounded derivative, the natural framework in which to work is the space of Lipschitz functions, with Lipschitz constant less than 1. The functional is defined as

$$I(u) = \int_0^T \left[1 - \sqrt{1-u'^2} + F(u) + h(t)u \right] dt, \quad \text{if } \|u'\|_\infty \leq 1,$$

and as $+\infty$ otherwise.

The lack of regularity of I makes it necessary to use Szulkin's theory (see [13]) which allows us to define critical points for such a nonsmooth functional (see Definition 2.4) and which makes the study more complicated (notice that, in contrast with the classical case, a critical point does not satisfy a family of identities, but a family of inequalities). Szulkin's theory provides as well an adapted version of the Palais–Smale condition (see Definition 2.6) and of the Mountain Pass Theorem (see Theorem 3.1).

Recently Brezis and Mawhin [6] proved the existence of a solution of (1.3), in the case of 2π -periodic f with zero mean value (or equivalently with a 2π -periodic primitive F) by minimizing the energy functional associated to the equation. They used the direct method of calculus of variations to guarantee that a minimum exists, by proving the boundedness from below and the lower semicontinuity of the functional. They also proved that such a minimum (actually their argument can be extended to any critical point, see Theorem 3.4) is a solution of the problem (in sense of Definition 2.1), so that they obtained the existence of infinitely many solutions (since, as in the classical framework, the energy functional is 2π -periodic).

Bereanu and Torres in [5] extend the existence result of Mawhin and Willem [12] for problem (1.2) to problem (1.3): indeed they prove the existence of a critical point that is not a translation of the minimum found by Brezis and Mawhin. This imply the existence of a second family of critical points that can be either minima or Mountain Pass (in this latter case they use Szulkin's version of Mountain Pass Theorem).

The aim of this paper is to extend problem (1.3) replacing the periodicity of f with a different nonlinearity. We consider an hypothesis that is called in literature as *strong resonance condition*, i.e. $f(s)$ is a real continuous function with primitive $F(s) = \int_0^s f(\sigma)d\sigma$ satisfying

$$(1.4) \quad \lim_{|s| \rightarrow +\infty} f(s) = 0, \quad \lim_{|s| \rightarrow +\infty} F(s) = \alpha,$$

for some $\alpha \in \mathbb{R}$. This assumption allows us to recover some compactness condition for the functional which was guaranteed in case of a periodic f , and which is necessary to prove the existence of a critical point.

Therefore, our main result is the following.

Under hypothesis (1.4), problem (1.3) has at least one solution.

Actually, we prove this existence result for a more general version of problem (1.3), which can be found in Theorem 2.2. In order to prove it, we consider the energy functional associated to the equation of (1.3). We first prove that it satisfies the Palais–Smale condition at every level but one and that it achieves its infimum value on a suitable subspace W of its domain. Two cases may occur: either there exists a function in W in which the functional has a value less or equal than the critical level, or the functional, evaluated on W , always lies above of it. We are able to prove that in both cases a critical point exists, which can be a minimum (if the first case occurs) or a mountain-pass nature critical point (if the second one stands). Applying the regularity result in [6], we prove that such a critical point is a solution of (1.3).

Observe that the above proof provides an alternative: the solution may be a minimizer of the functional, or a mountain pass critical point. In order to give additional informations about this uncertainty, we provide sufficient conditions under which it is ensured the existence of either a minimum (Corollary 2.7) or a mountain pass (Corollary 2.9).

2. Hypotheses and statement of the results

Let us consider the following problem:

$$(2.1) \quad \begin{cases} (\phi(u'(t)))' = f(t, u(t)) + h(t), & \text{in } (0, T), \\ u(0) - u(T) = 0 = u'(0) - u'(T). \end{cases}$$

Let $a > 0$ and let us assume the following hypotheses on the functions involved.

(H $_{\Phi}$) There exists $\Phi: [-a, a] \rightarrow \mathbb{R}$ such that $\Phi \in C[-a, a] \cap C^1(-a, a)$ and $\phi := \Phi': (-a, a) \rightarrow \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$.

(H $_f$) $f: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that

$$\lim_{|s| \rightarrow +\infty} f(t, s) = 0 \quad \text{uniformly with respect to } t.$$

F is the primitive of f defined as

$$F(t, \tau) = \int_0^\tau f(t, s) ds \quad \text{for all } (t, \tau) \in [0, T] \times \mathbb{R},$$

and there exists a constant $\alpha \in \mathbb{R}$ such that:

$$\lim_{|s| \rightarrow +\infty} F(t, s) = \alpha \quad \text{uniformly with respect to } t.$$

(H_h) $h: [0, T] \rightarrow \mathbb{R}$ is a continuous function such that

$$\int_0^T h(t) dt = 0.$$

Observe that, by (H_Φ), without loss of generality we can suppose $\Phi(0) = 0$. Let us notice that (H_f) implies that F is derivable with respect to the second variable and that both f and F are bounded, that is, there exists $C \in \mathbb{R}$ such that:

$$(2.2) \quad |f(t, \tau)| + |F(t, \tau)| \leq C, \quad \text{for all } (t, \tau) \in [0, T] \times \mathbb{R}.$$

We need now to make it clear the sense we give to a solution of problem (2.1).

DEFINITION 2.1. A solution of (2.1) is a function $u \in C^1[0, T]$ with $u(0) = u(T)$, $u'(0) = u'(T)$, such that $\|u'\|_\infty < a$ and $\phi \circ u'$ is absolutely continuous, whose derivative satisfies

$$(\phi(u'))' = f(t, u) + h(t) \quad \text{a.e. } t \in (0, T).$$

Here we state our main results.

THEOREM 2.2. *Under hypotheses (H_Φ), (H_f) and (H_h), there exists at least a solution of problem (2.1).*

Let us stress that we are looking for a function whose derivative is bounded. Therefore, the natural variational framework to work in is the space $W_{\#}^{1,\infty}(0, T)$. This is the Sobolev space of T -periodic functions u such that u and its distributional derivative u' are bounded, equipped with the norm

$$\|u\|_{W_{\#}^{1,\infty}} = \|u\|_\infty + \|u'\|_\infty.$$

Actually we will look for the solution in a convex subset of $W_{\#}^{1,\infty}(0, T)$, namely

$$K = \{v \in W_{\#}^{1,\infty}(0, T) \text{ such that } \|v'\|_\infty \leq a\}.$$

REMARK 2.3. If we find a solution u of (2.1) in K , we can obtain an *a posteriori* information about its regularity. In fact, since $\phi \circ u'$ is absolutely continuous, in particular $\phi(u'(t)) \in C[0, T]$, which, using that ϕ is an homomorphism, clearly implies $u'(t) \in C[0, T]$. In conclusion, $u \in C^1[0, T]$.

Following a standard variational procedure, we will associate a suitable functional to problem (2.1) and prove that it has at least one critical point. Next, we will prove that such a critical point is actually a solution.

Let us introduce the energy functional associated to (2.1):

$$I(v) := \begin{cases} \int_0^T [\Phi(v') + h(t)v + F(t, v)] dt & \text{if } v \in K, \\ +\infty & \text{if } v \in W_{\#}^{1,\infty}(0, T) \setminus K. \end{cases}$$

As we already pointed out, I is a nonsmooth functional and it has the structure required by Szulkin's theory (see [13]), that we briefly recall here.

The functional $I: W_{\#}^{1,\infty}(0, T) \rightarrow \mathbb{R} \cup \{+\infty\}$ is decomposable as

$$I = J + \mathcal{F},$$

where

$$J(v) = \begin{cases} \int_0^T [\Phi(v') + h(t)v] dt, & v \in K, \\ +\infty, & v \in W_{\#}^{1,\infty}(0, T) \setminus K, \end{cases}$$

and

$$\mathcal{F}(v) = \int_0^T F(t, v) dt,$$

for any $v \in W_{\#}^{1,\infty}(0, T)$. Observe that J is convex, proper and lower semicontinuous with respect to the topology of $C[0, T]$ (as it can be seen by using the same argument as in the proof of Lemma 1 of [6]), and \mathcal{F} is C^1 (it is standard to see that it has this required regularity).

According to Szulkin's theory, we have the following definition of critical point of I .

DEFINITION 2.4. A function $u \in W_{\#}^{1,\infty}(0, T)$ is a *critical point* of the functional I if $u \in K$ and it satisfies the inequality

$$J(v) - J(u) + \langle \mathcal{F}'(u), v - u \rangle \geq 0 \quad \text{for all } v \in W_{\#}^{1,\infty}(0, T).$$

We say that $c \in \mathbb{R}$ is a *critical value* of I if there exists a critical point $v \in W_{\#}^{1,\infty}(0, T)$ such that $I(v) = c$.

As we already noticed, dealing with this family of inequalities provides additional difficulties.

The main step in proving Theorem 2.2 is the following result, which ensures the existence of a critical point of the functional I .

THEOREM 2.5. *If assumptions (H_{Φ}) , (H_f) and (H_h) hold, then there exists at least a critical point for I .*

In order to prove it, we need to introduce the notion of Palais–Smale condition at the level $c \in \mathbb{R}$ is in this framework.

DEFINITION 2.6. For every $c \in \mathbb{R}$, a sequence $\{u_n\} \subset W_{\#}^{1,\infty}(0, T)$ is a *Palais–Smale sequence* at level c (in brief $(PS)_c$ -sequence) if

- (a) $I(u_n) = c + \varepsilon_n$,
- (b) $J(v) - J(u_n) + \langle \mathcal{F}'(u_n), v - u_n \rangle \geq -\varepsilon_n \|v - u_n\|_{W_{\#}^{1,\infty}}$, for all v in $W_{\#}^{1,\infty}(0, T)$,

where ε_n tends to 0 as n diverges.

We say that a functional I satisfies the *Palais–Smale condition at the level c* (in brief $(PS)_c$ -condition) if any $(PS)_c$ -sequence has a uniformly convergent subsequence in $[0, T]$.

As we already mentioned in the Introduction, we will give further informations about the nature of the solution we prove to exist. It will be useful to decompose any $u \in W_{\#}^{1,\infty}(0, T)$ as

$$(2.3) \quad u(t) = \bar{u} + \tilde{u}(t), \quad \text{where } \bar{u} = \frac{1}{T} \int_0^T u(t) dt \quad \text{and} \quad \int_0^T \tilde{u}(t) dt = 0.$$

In this way, the entire space can be decomposed as $W_{\#}^{1,\infty}(0, T) = V \oplus W$, where $V = \{v \in W_{\#}^{1,\infty}(0, T) : v \text{ is constant}\}$ and W is its topological and algebraic complement.

We prove (see Remark 3.3), that there exists $\tilde{w} \in W \cap K$ such that

$$m = \min_{\tilde{v} \in W} J(\tilde{v}) = J(\tilde{w}).$$

Notice that, due to the definition of the functional, a minimizer will always belong to K .

We give here some sufficient conditions to have a solution by minimization or of mountain pass nature.

COROLLARY 2.7. *If, in addition to (H_{Φ}) , (H_f) and (H_h) , there exists $\bar{v}_0 \in \mathbb{R}$ such that*

$$(2.4) \quad \int_0^T F(t, \tilde{w} + \bar{v}_0) dt \leq \alpha T,$$

then I has a minimum.

REMARK 2.8. If $sf(t, s) \geq 0$ for all $(t, s) \in [0, T] \times \mathbb{R}$, then $F(t, s) \leq \alpha$ for all $(t, s) \in [0, T] \times \mathbb{R}$, hypothesis (2.4) is satisfied and the functional I attains a minimum.

Let us call $F_0 = \inf_{(t,s) \in [0,T] \times \mathbb{R}} F(t, s)$, and let us note that, by condition (2.2), it is finite.

COROLLARY 2.9. *If, in addition to (H_Φ) , (H_f) and (H_h) , we assume that there exists a positive constant k such that $\Phi(s) \geq ks^2$ for every $s \in [-a, a]$; and it holds*

$$(2.5) \quad F_0T - \frac{T\|h\|_\infty^2}{4k} > m + \alpha T,$$

then I has a critical point of mountain-pass nature.

REMARK 2.10. Observe that in the case of the relativistic model problem (1.1) we have $\Phi(s) = 1 - \sqrt{1 - s^2}$ for $s \in [-1, 1]$, which satisfies (H_Φ) and $\Phi(s) \geq s^2/4$.

3. Proof of the result

Let us first recall an adapted version to Szulkin's theory of the Mountain-Pass theorem, which we will use later.

THEOREM 3.1 (Mountain Pass theorem). *Let X be a Banach space and $I: X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a functional decomposable as $I = J + \mathcal{F}$, with J proper, convex and lower semicontinuous and $\mathcal{F} \in C^1$. Let W be a subspace of X of codimension 1 and suppose that there exist u_1, u_2 belonging to different connected components of $X \setminus W$ (notice that there are only two of them) such that*

$$I(u_i) < \inf_W I, \quad i = 1, 2.$$

If c is given by $c = \inf_{\gamma \in \Gamma} \max_{x \in [0,1]} I(\gamma(x))$, where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_1, \gamma(1) = u_2\}$ and we assume that $(PS)_c$ holds, then c is a critical value of I (greater or equal than $\inf_W I$).

PROOF. This is a slight variant of Szulkin's version of Mountain Pass theorem (see [13, Theorem 3.4]). \square

We now prove that both I and J achieve their infimum value on W .

LEMMA 3.2. *The functional I achieves its infimum value over the space W .*

PROOF. Let us first of all observe that I is bounded from below. Because of the definition of I , we only need to prove that it holds $I(v) \geq C \in \mathbb{R}$ when $v \in K$. In this case it is easy to see that $\|\tilde{v}\|_\infty \leq aT$, using the mean value theorem and the fact that \tilde{v} has zero mean value. Moreover, since Φ achieves its minimum at zero and $\Phi(0) = 0$, F is bounded and h has zero mean value, it stands

$$I(v) = \int_0^T [\Phi(\tilde{v}') + h(t)\tilde{v} + F(v)] dt \geq \left[-a\|h\|_{L^1} + \inf_{[0,T] \times \mathbb{R}} F(t, s) \right] T > -\infty.$$

Our aim is to prove that if we call $\beta = \inf_{\tilde{v} \in W} I(\tilde{v})$, then there exists $\tilde{z} \in W$ such that $I(\tilde{z}) = \beta$.

Let us consider a minimizing sequence $\{\tilde{z}_n\} \subset W \cap K$, i.e. satisfying that $\lim_{n \rightarrow \infty} I(\tilde{z}_n) = \beta$, and let us show that it is bounded. As we already noticed, since any \tilde{z}_n belongs to K , we have that $\|\tilde{z}_n\|_\infty \leq aT$ and that $\|\tilde{z}'_n\|_\infty \leq a$. Thus:

$$\|\tilde{z}_n\|_{W_{\#}^{1,\infty}} = \|\tilde{z}_n\|_\infty + \|\tilde{z}'_n\|_\infty \leq (T + 1)a.$$

Applying now Ascoli–Arzelà theorem, we get the existence of a function $\tilde{z} \in C[0, T]$, such that up to a subsequence $\tilde{z}_n \rightarrow \tilde{z}$ in $C[0, T]$.

Since $\tilde{z}_n \in W \cap K$, the uniform convergence gives that $\tilde{z} \in W \cap K$. By the lower semicontinuity of I , we get $\beta = \liminf_{n \rightarrow \infty} I(\tilde{z}_n) \geq I(\tilde{z})$. It is straightforward $I(\tilde{z}) = \beta$. □

REMARK 3.3. This lemma, with the particular choice $F \equiv 0$, shows that the same result holds for the functional J . In particular,

$$(3.1) \quad \exists \tilde{w} \in W \text{ such that } J(\tilde{w}) = m = \min_W J.$$

PROOF OF THEOREM 2.5.

Step 1. I satisfies the $(PS)_c$ -condition, for every level $c \neq m + \alpha T$.

Let us consider a $(PS)_c$ -sequence in K . According to Definition 2.6, the following two conditions hold true:

$$(PS1) \quad \int_0^T [\Phi(u'_n) + h(t)u_n + F(t, u_n)] dt = c + \varepsilon_n;$$

$$(PS2) \quad \int_0^T [\Phi(v') - \Phi(u'_n)] dt + \int_0^T [f(t, u_n) + h(t)](v - u_n) dt \\ \geq -\varepsilon_n \|v - u_n\|_{W_{\#}^{1,\infty}}, \quad \text{for all } v \in W_{\#}^{1,\infty}(0, T),$$

where ε_n is a sequence converging to 0.

We are interested in proving that $\{u_n\}$ has a uniformly convergent subsequence. Let us note that we only need to show that $\{u_n\}$ is bounded in K . Indeed by Ascoli–Arzelà theorem, we can extract a subsequence (not relabeled) $\{u_n\}$ which converges uniformly in $[0, T]$ to a $u \in C[0, T]$. Moreover, since $\{u_n\} \subset K$ and thanks to the uniform convergence, then $u \in K$.

Let us decompose any $u_n = \tilde{u}_n + \bar{u}_n$ as in (2.3). We already know that $\{\tilde{u}_n\}$ is bounded in K (see proof of Lemma 3.2), so that it only remains to show that $\{\bar{u}_n\}$ is bounded (notice that it is a sequence of real numbers).

Let us suppose, by contradiction, that \bar{u}_n diverges. Let \tilde{w} be the minimizer of J on W given by (3.1) and let us choose $v = \tilde{w} + \bar{u}_n$ in (PS2). So we obtain

$$\int_0^T [\Phi(\tilde{u}'_n) - \Phi(\tilde{w}') + h(t)\tilde{u}_n - h(t)\tilde{w}] dt - \int_0^T f(t, \tilde{u}_n + \bar{u}_n)(\tilde{w} - \tilde{u}_n) dt \\ \leq \varepsilon_n \|\tilde{w} - \tilde{u}_n\|_{W_{\#}^{1,\infty}}.$$

Thus, taking in account the definition of J , recalling that ε_n vanishes as n diverges and being $\|\tilde{w} - \tilde{u}_n\|_{W_{\#}^{1,\infty}}$ bounded, we deduce that (up to subsequences, not relabeled)

$$\limsup_{n \rightarrow \infty} [J(\tilde{u}_n) - J(\tilde{w})] \leq \lim_{n \rightarrow \infty} \int_0^T f(t, \tilde{u}_n + \bar{u}_n)(\tilde{w} - \tilde{u}_n) dt.$$

Thanks to (2.2) we can use Lebesgue Theorem, and by (H_f) the right hand side above tends to zero, so that:

$$\limsup_{n \rightarrow \infty} [J(\tilde{u}_n) - J(\tilde{w})] \leq 0.$$

Consequently, since $J(\tilde{u}_n) \geq J(\tilde{w}) = \min_{\tilde{v} \in W} J(\tilde{v})$ for all n , we get

$$\lim_{n \rightarrow \infty} J(\tilde{u}_n) = J(\tilde{w}) = m.$$

From this equality, thanks again to (2.2), using (H_f) and up to subsequences, we get

$$\lim_{n \rightarrow \infty} I(u_n) = \lim_{n \rightarrow \infty} J(u_n) + \int_0^T F(\tilde{u}_n + \bar{u}_n) = m + \alpha T.$$

On the other hand, by condition (PS1), we know that $\lim_{n \rightarrow \infty} I(u_n) = c$.

Hence, the only constant c which allows the sequence \tilde{u}_n to be unbounded is $c = m + \alpha T$. In conclusion, we have proved that the $(PS)_c$ -condition holds for every $c \neq m + \alpha T$.

Let us now consider the functional I over the subspace W . Thanks to Lemma 3.2, there exists $\tilde{z} \in W$ such that $I(\tilde{z}) = \min_{\tilde{v} \in W} I(\tilde{v})$: there are now two possibilities. Either

$$(3.2) \quad I(\tilde{z}) \leq m + \alpha T,$$

or

$$(3.3) \quad I(\tilde{z}) > m + \alpha T.$$

Step 2. If (3.2) holds, then I has a minimum.

Condition (3.2) implies that $\gamma = \inf_{v \in K} I(v) \leq I(\tilde{z}) \leq m + \alpha T$. We prove that if $\gamma \leq m + \alpha T$ then the infimum of I in K is attained.

Indeed, two cases may occur:

Case 1. If $\gamma = m + \alpha T$, it means that $I(\tilde{z}) = m + \alpha T$. Hence \tilde{z} is a minimizer of I in K and we are done.

Case 2. If $\gamma < m + \alpha T$, we know by Step 1 that I satisfies the $(PS)_{\gamma}$ -condition. Our first aim is to take a minimizing sequence $\{w_n\} \subset K$ and (via the use of Ekeland's variational principle in [9, Theorem 1]) to construct from $\{w_n\}$ a new sequence $\{u_n\}$ which is still minimizing and that is also a $(PS)_{\gamma}$ -sequence.

By definition, it holds $I(w_n) = \gamma + 1/n$. We apply Ekeland's principle (see [13, Proposition 1.6] with the choice $\delta = 1/n$, $\lambda = 1$) and we get the existence of a sequence $\{u_n\}$ satisfying:

- $\gamma \leq I(u_n) \leq I(w_n) = \gamma + 1/n$,
- $I(z) - I(u_n) \geq -(1/n)\|z - u_n\|_{W_{\#}^{1,\infty}}$, for all $z \in W_{\#}^{1,\infty}(0, T)$.

Let us note that the first inequality implies (PS1) and that $\{u_n\}$ is still a minimizing sequence. Let us now work on the second one to show that it leads to (PS2).

For any $v \in W_{\#}^{1,\infty}(0, T)$ and any $n \in \mathbb{N}$, we set $z = (1 - \tau)u_n + \tau v$, $\tau \in (0, 1)$, so that

$$I((1 - \tau)u_n + \tau v) - I(u_n) \geq -\frac{1}{n}\tau\|v - u_n\|_{W_{\#}^{1,\infty}}.$$

On the other hand, using the convexity of Φ , we get

$$\begin{aligned} I((1 - \tau)u_n + \tau v) - I(u_n) &\leq \tau \int_0^T [\Phi(v') - \Phi(u_n') + h(t)(v - u_n)] dt \\ &\quad + \int_0^T [F(t, (u_n + \tau(v - u_n))) - F(t, u_n)] dt, \end{aligned}$$

which yields

$$\begin{aligned} \tau \int_0^T [\Phi(v') - \Phi(u_n') + h(t)(v - u_n)] dt + \int_0^T [F(t, (u_n + \tau(v - u_n))) - F(t, u_n)] dt \\ \geq -\frac{\tau}{n}\|v - u_n\|_{W_{\#}^{1,\infty}}. \end{aligned}$$

Since F is the bounded primitive of f and we can use the mean value theorem, dividing by τ and letting $\tau \rightarrow 0$, we get

$$\begin{aligned} \int_0^T [\Phi(v') - \Phi(u_n') + [f(t, u_n) + h(t)](v - u_n)] dt \geq -\frac{1}{n}\|v - u_n\|_{W_{\#}^{1,\infty}}, \\ \text{for all } v \in W_{\#}^{1,\infty}(0, T), \end{aligned}$$

i.e. condition (PS2). We have now found that $\{u_n\}$ is a minimizing sequence satisfying the $(PS)_{\gamma}$ -condition with $\gamma < m + \alpha T$, so that, by Step 1, it is possible to extract a uniformly convergent subsequence (not relabeled) such that

$$u_n \rightarrow u \quad \text{uniformly on } [0, T], \quad u \in K.$$

Using again the lower semicontinuity of I and $I(u) > -\infty$, we get that

$$\gamma = \liminf_{n \rightarrow \infty} I(u_n) \geq I(u).$$

We have proved eventually that $I(u) = \gamma$ and that u is minimizer of I .

Step 3. If (3.3) holds, then I has a mountain pass.

We want now to apply Theorem 3.1. Let us recall that \tilde{w} is the function such that $J(\tilde{w}) = m$ and, by considering $n \in \mathbb{R}$ and since h has zero mean value, we get

$$I(\tilde{w} \pm n) = \int_0^T [\Phi(\tilde{w}') + h(t)\tilde{w}] dt + \int_0^T F(t, \tilde{w} \pm n) dt = m + \int_0^T F(t, \tilde{w} \pm n) dt.$$

Using now (H_f) , we deduce that both

$$\lim_{n \rightarrow +\infty} I(\tilde{w} + n) = m + \alpha T \quad \text{and} \quad \lim_{n \rightarrow +\infty} I(\tilde{w} - n) = m + \alpha T.$$

That is, for all $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$, such that

$$|I(\tilde{w} + n) - (m + \alpha T)| < \varepsilon \quad \text{and} \quad |I(\tilde{w} - n) - (m + \alpha T)| < \varepsilon, \quad \text{for all } n \geq n_0.$$

Hence, if we fix any $0 < \varepsilon_1 \leq (\beta - (m + \alpha T))/2$, then we find $n_1 \in \mathbb{N}$ such that

$$I(\tilde{w} + n_1) - (m + \alpha T) \leq \varepsilon_1 \quad \text{and} \quad I(\tilde{w} - n_1) - (m + \alpha T) \leq \varepsilon_1.$$

By (3.3), we deduce $I(\tilde{w} \pm n_1) \leq \varepsilon_1 + (m + \alpha T) < \beta$.

Let us call $c = \inf_{\gamma \in \Gamma} \max_{x \in [0,1]} I(\gamma(x))$, where $\Gamma = \{\gamma \in C([0, 1], W_{\#}^{1,\infty}(0, T)) : \gamma(0) = \tilde{w} - n_1, \gamma(1) = \tilde{w} + n_1\}$. Since $c \geq \beta > m + \alpha T$ and by Step 1, condition $(PS)_c$ holds and Theorem 3.1 guarantees that c is a critical value of I . Consequently there exists a function u which is a critical point of I of mountain pass nature. \square

In order to complete the proof of Theorem 2.2, we need the following result about the critical points of the functional I .

THEOREM 3.4. *If we assume hypotheses (H_{Φ}) , (H_h) and f is an essentially bounded function in $(0, T) \times \mathbb{R}$, then every critical point of I is a solution of (2.1).*

PROOF. The proof follows the same strategy which is used in [6]. Let us suppose that u is a critical point of I . Then, according to Szulkin's theory, it satisfies

$$\int_0^T [\Phi(v') - \Phi(u') + (f(t, u) + h(t))(v - u)] dt \geq 0, \quad \text{for all } v \in K.$$

In order to apply Corollary 3 in [4], we rewrite this inequality as

$$(3.4) \quad \int_0^T [\Phi(v') - \Phi(u') + (l_u(t) + u)(v - u)] dt \geq 0, \quad \text{for all } v \in K.$$

where $l_u(t) = f(t, u) + h(t) - u$. According to [6], it is the variational inequality solved by the unique solution of problem

$$(3.5) \quad \begin{cases} (\phi(u'(t)))' = u(t) + l_u(t) & \text{in } (0, T), \\ u(0) - u(T) = 0 = u'(0) - u'(T). \end{cases}$$

We know then, that u is the solution of the above problem and that $u \in C^1(0, T)$, $\phi \circ u'$ is absolutely continuous and $\|u\|_\infty < a$. Moreover, it holds

$$(\phi(u'))' = u + l_u(t) = f(t, u) + h(t)$$

and in conclusion, the critical point u is a solution of (2.1). □

PROOF OF THEOREM 2.2. Because of Theorem 2.5, we know that at least a critical point of I exists. By Theorem 3.4, it is guaranteed that such a point is a solution of (2.1). □

PROOF OF COROLLARY 2.7. Let us note that if (2.4) is satisfied, then

$$I(\tilde{w} + \bar{v}_0) = \int_0^T [\Phi(\tilde{w}') + h(t)\tilde{w}] dt + \int_0^T F(t, \bar{v}_0 + \tilde{w}) dt \leq m + \alpha T,$$

which implies that $\inf_{v \in K} I(v) \leq m + \alpha T$. Consequently, arguing as in Step 2 of the proof of Theorem 2.2, there exists a solution by minimization. □

PROOF OF COROLLARY 2.9. Recall that $F_0 = \inf_{(t,s) \in [0,T] \times \mathbb{R}} F(t, s)$; since $\Phi(s) \geq ks^2$, it holds

$$I(\tilde{v}) = \int_0^T [\Phi(\tilde{v}') + h(t)\tilde{v} + F(t, \tilde{v})] dt \geq k\|\tilde{v}'\|_{L^2}^2 - \|h\|_\infty \int_0^T |\tilde{v}| dt + F_0 T,$$

for any $\tilde{v} \in W$. Notice that in W the Poincaré–Wirtinger inequality holds, i.e.

$$\int_0^T \tilde{v}^2 dt \leq \int_0^T \tilde{v}'^2 dt \quad \text{for all } \tilde{v} \in W.$$

Thus, thanks also to Hölder inequality, we deduce

$$I(\tilde{v}) \geq k\|\tilde{v}'\|_{L^2}^2 - \sqrt{T}\|h\|_\infty\|\tilde{v}'\|_{L^2} + F_0 T = p(\|\tilde{v}'\|_{L^2}),$$

where p is the polynomial defined by $p(s) := ks^2 - \sqrt{T}\|h\|_\infty s + F_0 T$. It is easy to see that p attains its infimum value

$$-\frac{T\|h\|_\infty^2}{4k} + F_0 T s = \frac{\sqrt{T}\|h\|_\infty}{2k},$$

thus we deduce

$$\min_W I \geq -\frac{T\|h\|_\infty^2}{4k} + F_0 T.$$

Hence, if (2.5) holds true, then

$$\min_{\tilde{v} \in W} I(\tilde{v}) > m + \alpha T,$$

i.e. (3.3) is verified. By Step 3 of the proof of Theorem 2.2, the existence of a mountain pass is guaranteed. □

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