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HÖLDER CONTINUOUS RETRACTIONS AND AMENABLE SEMIGROUPS OF UNIFORMLY LIPSCHITZIAN MAPPINGS IN HILBERT SPACES

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Dedicated to Professors Kazimierz Goebel and Lech Górniewicz

Abstract. Suppose that S is a left amenable semitopological semigroup. We prove that if $\mathcal{S} = \{T_t : t \in S\}$ is a uniformly k-Lipschitzian semigroup on a bounded closed and convex subset C of a Hilbert space and $k < \sqrt{2}$, then the set of fixed points of $\mathcal{S}$ is a Hölder continuous retract of C. This gives a qualitative complement to the Ishihara–Takahashi fixed point existence theorem.

1. Introduction

Let C be a nonempty bounded closed and convex subset of a Banach space X. A mapping $T\colon C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for $x, y \in C$. The study of the structure of the fixed-point set $\mathrm{Fix}\, T = \{x \in C : Tx = x\}$ was initiated by R. Bruck (cf. [3], [4]) who proved that if T has a fixed point in every nonempty closed convex subset of C which is invariant under T, and C is convex and weakly compact or separable, then $\mathrm{Fix}\, T$ is a nonexpansive retract of C (i.e. there exists a nonexpansive mapping $R\colon C \to \mathrm{Fix}\, T$ such that

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$R_{| \operatorname{Fix} T} = I$). Notice that the set of fixed points of a k-Lipschitz mapping can be very irregular for any $k > 1$. Indeed, let F be a nonempty closed subset of C, $z \in F$, $\varepsilon > 0$, and define

$$Tx = x + \varepsilon \operatorname{dist}(x, F)(z - x), \quad x \in C.$$

Then the Lipschitz constant of T tends to 1 if $\varepsilon \to 0$, but $\operatorname{Fix} T = F$ (see, e.g. [17]).

In 1973, Goebel and Kirk [7] introduced the class of uniformly Lipschitzian mappings and proved a fixed point theorem which was later studied by several authors. A mapping $T\colon C \to C$ is said to be k-uniformly Lipschitzian if

$$\|T^n x - T^n y\| \leq k\|x - y\|$$

for every $x, y \in C$ and $n \in \mathbb{N}$. Recall that a Banach space X is uniformly convex if $\delta_X(\varepsilon) > 0$ for $\varepsilon > 0$, where $\delta_X\colon [0,2] \to [0,1]$ is the modulus of convexity of X defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x - y\| \geq \varepsilon\right\}.$$

Theorem 1.1 (cf. [7]). *Let X be a uniformly convex Banach space with the modulus of convexity δ_X and let C be a bounded, closed and convex subset of X. Suppose $T\colon C \to C$ is uniformly k-Lipschitzian and*

$$k\left(1 - \delta_X\left(\frac{1}{k}\right)\right) < 1. \tag{1.1}$$

Then T has a fixed point in C.

It was shown in [17] that under the assumptions of Theorem 1.1, the fixed-point set $\operatorname{Fix} T$ is a (continuous) retract of C. Recently, Górnicki (cf. [8], [9]) proved several structural results concerning uniformly Lipschitzian mappings but many questions remain open. In [16], Pérez García and Fetter Nathansky gave conditions under which $\operatorname{Fix} T$ is a Hölder continuous retract and applied them to the study of n-periodic mappings in Hilbert spaces.

Notice that in a Hilbert space a solution to (1.1) gives $k < \sqrt{5}/2$. Lifschitz [13] improved this estimation and showed that in a Hilbert space a uniformly k-Lipschitzian mapping with $k < \sqrt{2}$ has a fixed point. The Lifschitz theorem was generalized to uniformly k-Lipschitzian semigroups in [6], [10], [15]. The aim of this note is to give a qualitative complement to the above results in the case of left amenable (semitopological) semigroups which partially extends a result of Górnicki (cf. [9, Corollary 14]). We show that if $\mathcal{S} = \{T_t : t \in S\}$ is a uniformly k-Lipschitzian semigroup on C and $k < \sqrt{2}$, then $\operatorname{Fix} \mathcal{S} = \bigcap_{t \in S} \{x \in C : T_t x = x\}$, the set of (common) fixed points of $\mathcal{S}$, is a Hölder continuous retract of C.

2. Fixed point theorem

We start with the following variant of a well known result (see, e.g.[2, Proposition 1.10], [16, Lemma 2.1]).

Lemma 2.1. *Let (X,d) be a complete bounded metric space and let $L\colon X\to X$ be a k-Lipschitz mapping. Suppose there exist $0<\gamma<1$ and $c>0$ such that $d(L^{n+1}x, L^n x)\le c\gamma^n$ for every $x\in X$ and $n\in\mathbb{N}$. Then $Rx=\lim_{n\to\infty} L^n x$ is a Hölder continuous mapping on X.*

Proof. We may assume that $\operatorname{diam} X=1$. Notice that for every $x\in X$ and $n,m\in\mathbb{N}$,

$$d(L^{n+m}x, L^n x)\le c\frac{\gamma^n}{1-\gamma}.$$

Hence, for every $x,y\in X$,

$$d(Rx,Ry)\le d(Rx,L^n x)+d(L^n x,L^n y)+d(L^n y,Ry)\le 2c\,\frac{\gamma^n}{1-\gamma}+k^n d(x,y).$$

Take $\alpha<1$ such that $k\le\gamma^{1-\alpha^{-1}}$ and fix $x,y\in X$, $x\neq y$. Then, there exist $n\in\mathbb{N}$ and $0<r\le 1$ such that $\gamma^{n-r}=d(x,y)^\alpha$. Furthermore, $k^{n-1}\le(\gamma^{1-\alpha^{-1}})^{n-r}$. It follows that

$$d(Rx,Ry)\le 2c\,\frac{\gamma^{n-r}}{1-\gamma}+k(\gamma^{n-r})^{1-\alpha^{-1}}d(x,y)=\left(\frac{2c}{1-\gamma}+k\right)d(x,y)^\alpha. \qquad \square$$

Let S be a semigroup and $\ell^\infty(S)$ the Banach space of bounded real valued functions on S with the supremum norm. For $s\in S$ and $f\in\ell^\infty(S)$, we define elements $l_s f$ and $r_s f$ in $\ell^\infty(S)$ by

$$l_s f(t)=f(st),\qquad r_s f(t)=f(ts)$$

for every $t\in S$. An element μ of $(\ell^\infty(S))^*$ is said to be a mean on X if $\|\mu\|=\mu(I_S)=1$, where $I_S(t)=1$ for all $t\in S$. It is well known that μ is a mean if and only if

$$\inf_{t\in S} f(t)\le\mu(f)\le\sup_{t\in S} f(t)$$

for each $f\in\ell^\infty(S)$. A mean μ on $\ell^\infty(S)$ is said to be left (resp. right) invariant if $\mu(l_s f)=\mu(f)$ (resp. $\mu(r_s f)=\mu(f)$) for each $s\in S$ and $f\in\ell^\infty(S)$. A Banach limit is a special case of an invariant mean, where S equals the semigroup of natural numbers. A semigroup S is called left (resp. right) amenable if there exists a left (resp. right) invariant mean on $\ell^\infty(S)$. The term "amenable semigroup" was coined by M.M. Day in his celebrated paper [5].

Let C be a bounded closed convex subset of a Hilbert space H. For simplicity, we consider only a real Hilbert space. A family $\mathcal{S}=\{T_t : t\in S\}$ of mappings from

C into C is said to be a uniformly k-Lipschitzian semigroup on C if $T_{ts}x = T_t T_s x$ and

$$\|T_t x - T_t y\| \leq k\|x - y\| \quad \text{for all } t, s \in S \text{ and } x \in C.$$

The following construction is well known (see, e.g. [1], [18]). Let μ be a mean on $\ell^\infty(S), x \in C$, and consider a functional $F(y) = \int \langle T_t x, y\rangle \, d\mu(t)$, $y \in H$, where $\int \langle T_t x, y\rangle \, d\mu(t)$ denotes the value of μ at the function $t \to \langle T_t x, y\rangle$. It is not difficult to see that F is linear and continuous since

$$\left\| \int \langle T_t x, y\rangle \, d\mu(t) \right\| \leq \sup_{t\in S} \|\langle T_t x, y\rangle\| \leq \sup_{t\in S} \|T_t x\|\|y\|.$$

By the Riesz theorem, there exists a unique element $\overline{x}$ such that

$$\int \langle T_t x, y\rangle \, d\mu(t) = \langle \overline{x}, y\rangle \tag{2.1}$$

for every $y \in H$. Furthermore, by the separation theorem, $\overline{x} \in C$. Thus we obtain a mapping $\overline{T}_\mu : C \to C$ by putting $\overline{T}_\mu x = \overline{x}$. Notice that if a semigroup $\mathcal{S} = \{T_t : t \in S\}$ is uniformly k-Lipschitzian, then

$$\langle \overline{T}_\mu x - \overline{T}_\mu y, v\rangle = \int \langle T_t x - T_t y, v\rangle \, d\mu(t) \leq \sup_{t\in S} \|T_t x - T_t y\|\|v\| \leq k\|x - y\|\|v\|$$

for every $x, y \in C$, $v \in H$, and hence

$$\|\overline{T}_\mu x - \overline{T}_\mu y\| \leq k\|x - y\|$$

for every $x, y \in C$, i.e. $\overline{T}_\mu$ is k-Lipschitz.

The above notions may be generalized to the case of semitopological semigroups. Recall that S is a topological semigroup if there exists a Hausdorff topology on S such that the mapping $S \times S \ni (s, t) \to st$ is (jointly) continuous. A semigroup S is semitopological if for each $t \in S$ the mappings $S \ni s \to ts$ and $S \ni s \to st$ are continuous. Notice that every semigroup can be equipped with the discrete topology and then it is called a discrete semigroup. Let X be a linear closed subspace of $\ell^\infty(S)$ containing I_S such that $l_s(X) \subset X$ and $r_s(X) \subset X$ for each $s \in S$. Let X^* be its topological dual. Then X is said to be left (resp. right) amenable if there exists a left (resp. right) invariant mean on X.

The following theorem is a qualitative version of (a slight generalization of) Theorem 1 in [10].

Theorem 2.2. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and $\mathcal{S} = \{T_t : t \in S\}$ a uniformly k-Lipschitzian semigroup on C. Suppose that X is a left amenable subspace of $\ell^\infty(S)$ such that the functions $g(t) = \langle T_t x, y\rangle$ and $h(t) = \|T_t x - y\|^2$ belong to X for each $x \in C$ and $y \in H$.*

If $k < \sqrt{2}$, then $\mathcal{S}$ has a fixed point in C and $\operatorname{Fix} \mathcal{S} = \bigcap_{t \in S} \{x \in C : T_t x = x\}$ *is a Hölder continuous retract of C.*

PROOF. Let μ be a left invariant mean μ on X. Then there exists a mapping $\overline{T}_\mu : C \to C$ such that

$$\int \langle T_t x, y\rangle \, d\mu(t) = \langle \overline{T}_\mu x, y\rangle$$

for every $y \in H$ (see (2.1)). Following [10, Theorem 1], fix $x_0 \in C$ and put $x_{n+1} = \overline{T}_\mu x_n$, $n = 0, 1, \ldots$ Then

$$\|T_t x_n - y\|^2 = \|T_t x_n - x_{n+1}\|^2 + \|x_{n+1} - y\|^2 + 2\langle T_t x_n - x_{n+1}, x_{n+1} - y\rangle,$$
$$\int \langle T_t x_n - x_{n+1}, x_{n+1} - y\rangle \, d\mu(t) = \langle \overline{T}_\mu x_n - x_{n+1}, x_{n+1} - y\rangle = 0,$$

and hence

$$\int \|T_t x_n - y\|^2 \, d\mu(t) = \int \|T_t x_n - x_{n+1}\|^2 \, d\mu(t) + \|x_{n+1} - y\|^2$$

for every $y \in H$ and $n = 0, 1, \ldots$ It follows (writing $y = x_n$) that

$$\int \|T_t x_n - x_{n+1}\|^2 \, d\mu(t) \le \int \|T_t x_n - x_n\|^2 \, d\mu(t). \tag{2.2}$$

Furthermore, taking $y = T_s x_{n+1}$, we have

$$\begin{aligned} \|T_s x_{n+1} - x_{n+1}\|^2 &= \int \|T_t x_n - T_s x_{n+1}\|^2 \, d\mu(t) - \int \|T_t x_n - x_{n+1}\|^2 \, d\mu(t) \\ &= \int \|T_{st} x_n - T_s x_{n+1}\|^2 \, d\mu(t) - \int \|T_t x_n - x_{n+1}\|^2 \, d\mu(t) \\ &\le (k^2 - 1) \int \|T_t x_n - x_{n+1}\|^2 d\mu(t) \end{aligned}$$

for any $s \in S$, since μ is left invariant and $\|T_{st} x_n - T_s x_{n+1}\| \le k\|T_t x_n - x_{n+1}\|$. Thus

$$\int \|T_s x_{n+1} - x_{n+1}\|^2 \, d\mu(s) \le (k^2 - 1) \int \|T_t x_n - x_{n+1}\|^2 \, d\mu(t). \tag{2.3}$$

Combining (2.2) with (2.3) yields

$$\int \|T_t x_{n+1} - x_{n+1}\|^2 \, d\mu(t) \le (k^2 - 1) \int \|T_t x_n - x_n\|^2 \, d\mu(t).$$

for $n = 0, 1, \ldots$ Hence

$$\begin{aligned} \|\overline{T}_\mu^{n+1} x_0 - \overline{T}_\mu^n x_0\|^2 &= \|x_{n+1} - x_n\|^2 \\ &\le 2 \int \|x_{n+1} - T_t x_n\|^2 \, d\mu(t) + 2 \int \|T_t x_n - x_n\|^2 \, d\mu(t) \\ &\le 4 \int \|T_t x_n - x_n\|^2 d\mu(t) \le 4(k^2 - 1)^n \operatorname{diam} C \end{aligned}$$

for every $x_0 \in C$ and $n = 0, 1, \dots$ Since $\overline{T}_\mu$ is k-Lipschitz and $k < \sqrt{2}$, it follows from Lemma 2.1 that $Rx = \lim_{n\to\infty} \overline{T}_\mu^n x$ is Hölder continuous on C. We show that R is a retraction onto $\mathrm{Fix}\, \mathcal{S}$. It is clear that if $x \in \mathrm{Fix}\, \mathcal{S}$, then $Rx = \overline{T}_\mu x = x$. Furthermore, it follows from the generalized parallelogram law that for every $x, y, z \in H$,

$$\begin{aligned}\|x+y+z\|^2 &= \|x+y\|^2 + \|y+z\|^2 + \|z+x\|^2 - \|x\|^2 - \|y\|^2 - \|z\|^2 \\ &\le 2(\|x\|^2 + \|y\|^2) + 2(\|y\|^2 + \|z\|^2) \\ &\quad + 2(\|z\|^2 + \|x\|^2) - \|x\|^2 - \|y\|^2 - \|z\|^2 \\ &= 3(\|x\|^2 + \|y\|^2 + \|z\|^2).\end{aligned}$$

Therefore, for every $x \in C$ and $n = 0, 1, \dots,$

$$\begin{aligned}&\int \|T_t Rx - Rx\|^2 \, d\mu(t) \\ &\quad \le 3 \int (\|T_t Rx - T_t \overline{T}_\mu^n x\|^2 + \|T_t \overline{T}_\mu^n x - \overline{T}_\mu^n x\|^2 + \|\overline{T}_\mu^n x - Rx\|^2) \, d\mu(t) \\ &\quad \le 3(k^2+1)\|Rx - \overline{T}_\mu^n x\|^2 + 3\mu_t \|T_t \overline{T}_\mu^n x - \overline{T}_\mu^n x\|^2.\end{aligned}$$

Letting n go to infinity, $\int \|T_t Rx - Rx\|^2 \, d\mu(t) = 0$. Therefore, for each $x \in C$ and $s \in S$,

$$\begin{aligned}\|T_s Rx - Rx\|^2 &\le 2 \int \|T_s Rx - T_t Rx\|^2 \, d\mu(t) + 2 \int \|T_t Rx - Rx\|^2 \, d\mu(t) \\ &= 2 \int \|T_s Rx - T_{st} Rx\|^2 \, d\mu(t) \le 2k^2 \int \|Rx - T_t Rx\|^2 \, d\mu(t) = 0.\end{aligned}$$

Thus $Rx \in \mathrm{Fix}\, \mathcal{S}$ for every $x \in C$ and the proof is complete. □

Remark 2.3. Notice that a detailed analysis of the proof of Lemma 2.1 gives the estimation $\alpha = 1/(1 - \log_{k^2-1} k)$ for the exponent and $c = k + 8 \operatorname{diam} C/(2 - k^2)$ for the constant of the Hölder continuous retraction R, $1 < k < \sqrt{2}$.

In particular, the above theorem is applicable to commutative semigroups since every commutative semigroup is amenable.

Example 2.4. Let G be an unbounded subset of $[0, \infty)$ such that $t+s, t-s \in G$ for all $t, s \in G$ with $t > s$ (e.g. $G = [0, \infty)$ or $G = \mathbb{N}$). Suppose that $\mathcal{T} = \{T_t : t \in G\}$ is a family of uniformly k-Lipschitzian mappings from C into C such that $T_{t+s}x = T_t T_s x$ for all $t, s \in G$ and $x \in C$. If $\sup_{t\in G} |T_t| < \sqrt{2}$, where $|T_t| = \inf\{c > 0 : \|T_t x - T_t y\| \le c\|x - y\| \text{ for all } x, y \in C\}$ denotes the Lipschitz constant of a mapping T_t, then $\mathrm{Fix}\, \mathcal{T} = \bigcap_{t\in G} \{x \in C : T_t x = x\}$ is a Hölder continuous retract of C. Notice that we do not assume that the semigroup $\mathcal{T}$ is continuous in t (i.e. strongly continuous in the terminology of C_0-semigroups).

Now consider a more general case.

Example 2.5. Let S be a semitopological semigroup and let $\mathrm{CB}(S)$ denote the closed subalgebra of $\ell^\infty(S)$ consisting of continuous functions. Let $\mathrm{LUC}(S)$ (resp. $\mathrm{RUC}(S)$) be the space of left (resp. right) uniformly continuous functions on S, i.e. all $f \in \mathrm{CB}(S)$ such that the mapping $S \ni s \to l_s f$ (resp. $s \to r_s f$) from S to $\mathrm{CB}(S)$ is continuous when $\mathrm{CB}(S)$ has the sup norm topology. It is known (see [14]) that $\mathrm{LUC}(S)$ and $\mathrm{RUC}(S)$ are left and right translation invariant closed subalgebras of $\mathrm{CB}(S)$ containing constants. Notice that if a uniformly k-Lipschitzian semigroup $\mathcal{S} = \{T_t : t \in S\}$ is continuous in t on C (i.e. the mapping $S \ni t \to T_t x$ is continuous for each $x \in C$), then the functions $g(t) = \langle T_t x, y\rangle$ and $h(t) = \|T_t x - y\|^2$ belong to $\mathrm{RUC}(S)$ for every $x \in C$ and $y \in H$. Indeed (see [11]),

$$\begin{aligned}\|r_s g - r_u g\| &= \sup_{t\in S} \|(r_s g)(t) - (r_u g)(t)\| = \sup_{t\in S} \|g(ts) - g(tu)\| \\ &= \sup_{t\in S} \|\langle T_t T_s x - T_t T_u x, y\rangle\| \le k\|T_s x - T_u x\|\|y\|\end{aligned}$$

and

$$\begin{aligned}\|r_s h - r_u h\| &= \sup_{t\in S} \|h(ts) - h(tu)\| = \sup_{t\in S} \|\|T_{ts}x - y\|^2 - \|T_{tu}x - y\|^2\| \\ &= \sup_{t\in S} \|(\|T_{ts}x - y\| + \|T_{tu}x - y\|)(\|T_{ts}x - y\| - \|T_{tu}x - y\|)\| \\ &\le 2\sup_{t\in S} \|T_t x - y\| \sup_{t\in S} \|T_t T_s x - T_t T_u x\| \\ &\le 2k \sup_{t\in S} \|T_t x - y\|\|T_s x - T_u x\|\end{aligned}$$

for any $s, u \in S$, $x \in C$ and $y \in H$. Thus Theorem 2.2 is applicable with $X = \mathrm{RUC}(S)$ if we assume that $\mathcal{S} = \{T_t : t \in S\}$ is (separately) continuous.

Recall that a semitopological semigroup S is said to be left reversible if any two closed right ideals of S have a non-void intersection. In this case $(S, \le)$ is a directed set with the relation $a \le b$ if and only if $\{a\} \cup \overline{aS} \supset \{b\} \cup \overline{bS}$. In general, the conditions "$\mathrm{RUC}(S)$ has a left invariant mean" and "S is left reversible" are independent of each other. However, if $\mathrm{RUC}(S)$ has sufficiently many functions to separate closed sets, then the former condition implies the latter (see, e.g. [12, Lemma 3.1]).

Recently, Górnicki (cf. [9, Corollary 14]) proved that if S is left reversible and $\mathcal{S} = \{T_t : t \in S\}$ is a uniformly k-Lipschitzian semigroup on C, then the set of fixed points of $\mathcal{S}$ is a retract of C. It is not clear how to extend Theorem 2.2 to left reversible semigroups. There are many other open problems in this area. Perhaps the most natural is the following.

Problem. Under the assumptions of Theorem 1.1, is the fixed-point set $\operatorname{Fix} \mathcal{S}$ a Lipschitz or Hölder continuous retract of C?

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