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A HARTMAN–NAGUMO TYPE CONDITION FOR A CLASS OF CONTRACTIBLE DOMAINS

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Abstract. We generalize an existence result on second order systems with a nonlinear term satisfying the so-called Hartman–Nagumo condition. The generalization is based on the use of Gauss second fundamental form and continuation techniques.

1. Introduction

In 1960, Hartman [5] showed that the second order system in $\mathbb{R}^n$ for a vector function $x: I = [0,1] \to \mathbb{R}^n$ satisfying

$$\begin{cases} x'' = f(t,x,x'), \\ x(0) = x_0, \\ x(1) = x_1, \end{cases} \tag{1.1}$$

with $f: I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ continuous, has at least one solution when f satisfies the following conditions:

$$\langle f(t,x,y), x\rangle + |y|^2 > 0 \quad \text{for all } (t,x,y) \in I \times \mathbb{R}^n \times \mathbb{R}^n, \tag{1.2}$$

with $|x| = R$, $\langle x, y\rangle = 0$ for some $R \geq |x_0|, |x_1|$.

$$|f(t,x,y)| \leq \phi(|y|) \quad \text{where } \phi: [0,\infty) \to \mathbb{R}^+ \quad \text{and} \quad \int_0^\infty \frac{x}{\phi(x)}\, dx = \infty, \tag{1.3}$$

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$$(1.4)\qquad |f(t,x,y)| \le \alpha(\langle f(t,x,y),x\rangle + |y|^2) + C, \quad \text{where } \alpha, C > 0.$$

A stronger version of (1.2) is easier to understand:

$$(1.5)\qquad \langle f(t,x,y),x\rangle > 0 \quad \text{for all } (t,x,y) \in I\times\mathbb{R}^n\times\mathbb{R}^n,$$

with $|x| = R$, $\langle x,y\rangle = 0$.

Indeed, this condition means that whenever $x \in \partial B(0,R)$, the vector field f points outwards the ball $B(0,R)$. Condition (1.2) allows f to point inwards, but not too much if the velocity is small.

The proof basically uses the Schauder fixed point theorem. It can also be proved using Leray–Schauder continuation theorem [9] in the open set of curves lying inside $B(0,R)$.

The key argument is that a solution u cannot be tangent to the ball of radius R from inside because (from (1.2)) the second derivative of $|u|^2$ is positive when $|u|$ is close to R. Conditions (1.3) and (1.4) guarantee that the C^1 norm of the solutions remains bounded during the continuation.

Hartman's result has been extended in several ways, for different boundary conditions (see [7] for a first result of this type under periodic conditions) and for more general second order operators (see e.g. [10], [13] and the references therein). However, less generalizations are known if one replaces the ball $B(0,R)$ by an arbitrary domain D.

In view of the geometrical interpretation of (1.5), it is not difficult to prove existence of solutions using (1.3)–(1.5) when D is convex. Condition (1.5) takes, in consequence, the following form:

$$\langle f(t,x,y),n_x\rangle > 0 \quad \text{for all } (t,x,y)\in I\times\mathbb{R}^n\times\mathbb{R}^n, \text{ with } x\in\partial D,\ \langle n_x,y\rangle = 0$$

where n_x is an outer normal of ∂D at the point x. For periodic conditions, this result has been obtained by Bebernes and Schmitt in [1] assuming, instead of (1.4) and (1.5), that f has some specific subquadratic growth on y. In this work, we extend the result for a more general (not necessarily convex) domain $D \subset \mathbb{R}^n$.

Some results in this direction have been given in [12] and [3] (see also [4] and the survey [11]), where the concept of *curvature bound set* is introduced in order to ensure that solutions starting inside an appropriate domain remain there all the time, thus allowing the use of the continuation method. Roughly speaking, at any point of the boundary of such a set D there exists a smooth surface that is tangent from outside and measures the curvature of the solutions touching ∂D from inside.

In this work we shall show that, in some sense, if D has C^2 boundary and the role of the surfaces in the previous definition is assumed by ∂D itself, then a precise geometric condition involving its second fundamental form yields. In this context, our version of (1.2) reads as follows:

$$\langle f(t,x,y), n_x\rangle > \mathbb{I}_x(y) \quad \text{for all } (t,x,y) \in I \times T\partial D \tag{1.6}$$

where $T\partial D$ is the tangent vector bundle identified, as usual, with a subset of $\mathbb{R}^n \times \mathbb{R}^n$, $\mathbb{I}_x(y)$ is the second fundamental form of the hypersurface and n_x is the outer-pointing normal unit vector field. This condition requires f to point outside D as much as ∂D is "bended outside" in the direction of the velocity. In particular, when $D = B(0,R)$ its curvature is constantly $1/R^{n-1}$; moreover, $\mathbb{I}_x(y) = -|y|^2/R$ and $n_x = x/R$, so our new Hartman condition takes the form of the original one.

The paper is organized as follows. In the next section, we recall the basic facts about Gaussian curvature and state some preliminary results concerning the generalized Hartman condition (1.6). In Section 3, we introduce some growth conditions that extend (1.3) and (1.4) on the one hand, and the growth condition in [1] (used also in [4]), on the other hand. In Section 4 we establish and prove our main results on existence of solutions under Dirichlet and periodic conditions using the Leray–Schauder continuation method. Finally, in Section 5 we prove that the growth conditions force the domain D to be contractible, thus restricting the class of examples to which the main theorems are applicable.

2. Curvature

Let D be an open subset of $\mathbb{R}^n$ such that $M := \partial D$ is a C^2 oriented manifold and let n_x be the outer unit normal vector at $x \in M$. The application $x \mapsto n_x$ defines a smooth function $n: M \to S^{n-1}$ and its differential defines a linear map $T_xM \to T_{n_x}S^{n-1}$. Since both linear spaces are orthogonal to n_x, they may be identified and we obtain a linear endomorphism known as the Gauss map $g_x: T_xM \to T_xM$. This map is easily seen to be self-adjoint with respect to the inner product inherited from $\mathbb{R}^n$. The associated quadratic form $\mathbb{I}_x(v) = -\langle g_x(v), v\rangle$ is called the second fundamental form of the hypersurface. It is important to remark that $\mathbb{I}_x$ is independent -up to a sign- of the orientation given by n.

The next lemma is essentially proved in do Carmo's book [2]:

Lemma 2.1. *Let $\alpha: \mathbb{R} \to \overline{D}$ be a C^2 curve such that $\alpha(0) = p \in M$. Let n_p be the outer unit normal vector at p. Then $\langle \alpha''(0), n_p\rangle \le \mathbb{I}_p(\alpha'(0))$.*

Proof. As a direct application of the inverse function theorem, we obtain near p a coordinate system of the form $(m,\lambda) \in M \times \mathbb{R}$ given by $x = m(x) + \lambda(x) n_{m(x)}$. The curve α may be written as $\alpha(t) = \gamma(t) + \lambda(t) n_{\gamma(t)}$ for some C^1 functions γ, λ. Let $n(t) = n_{\gamma(t)}$ so $n'(t) = g_{\gamma(t)}(\gamma'(t))$ and compute

$$\langle n(t), \alpha'(t)\rangle = \langle n(t), \gamma'(t)\rangle + \lambda'(t) + \lambda(t)\langle n(t), n'(t)\rangle = \lambda'(t),$$

since $\gamma'(t)$ and $n'(t)$ are orthogonal to $n(t)$. In particular, λ is a C^2 function.

Moreover, as the image of α is contained in D, its λ-coordinate is always non-positive. But $\lambda(0) = 0$, and hence $\lambda''(0) \le 0$. We deduce that

$$\frac{d}{dt}\langle n(t), \alpha'(t)\rangle\bigg|_{t=0} \le 0.$$

Now $\langle n(t), \alpha''(t)\rangle = \langle n(t), \alpha'(t)\rangle' - \langle n'(t), \alpha'(t)\rangle$; thus,

$$\langle n(0), \alpha''(0)\rangle \le -\langle g_{\alpha(0)}(\alpha'(0)), \alpha'(0)\rangle = \mathbb{I}_p(\alpha'(0)). \qquad \square$$

Corollary 2.2. *Let $f: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be such that* (1.6) *holds. Then there are no solutions of the differential equation $x'' = f(t, x, x')$ inside $\overline{D}$ touching the boundary.*

The following definition will be useful.

Definition 2.3. We define $\operatorname{curv}(D, x) = \lambda_1$, where $\lambda_1 \le \ldots \le \lambda_{n-1}$ are the eigenvalues of the self-adjoint operator g_x.

Remark 2.4. (a) It is clear from the definition that $-|y|^2\operatorname{curv}(D, x) \ge \mathbb{I}_x(y)$, and equality holds when y is an eigenvector of g_x associated with the eigenvalue λ_1.

(b) If x is a point of convexity of the surface, then $curv(D, x) \ge 0$.

(c) It may be deduced, as in the proof of Lemma 2.1, that:

- If $\operatorname{curv}(D, x) < 0$ then $-\operatorname{curv}(D, x)^{-1}$ is the radius of the largest ball which is tangent from outside to $\overline{D}$ at the point x.
- If $\operatorname{curv}(D, x) > 0$ then $\operatorname{curv}(D, x)^{-1}$ is the radius of the smallest ball B such that D is tangent from inside to ∂B at x.

3. Growth conditions

In order to apply the Leray–Schauder method [9], it is necessary to find *a priori* bounds of the solutions during the continuation. As the nonlinear term depends on u and u', we will need estimates for the C^1 norm.

To this end, following [5], we shall impose some growth conditions on f. These conditions must be compatible with the deformations used in the main theorems.

We shall make use of the next two lemmas, proved in [5] (Lemmas 2 and 3, respectively), conveniently adapted to our situation. Without loss of generality, we may assume that $D = \eta^{-1}(-\infty, 0)$ where $\eta: \mathbb{R}^n \to \mathbb{R}$ is a C^2 function and 0 is a regular value of η, then $M = \partial D = \eta^{-1}(0)$. We can use the function η itself to replace r in Lemma 3 of [5].

Lemma 3.1. *Let R, C be non-negative constants and $\phi\colon [0,\infty) \to (0,\infty)$ a continuous function such that*

$$\int_0^\infty \frac{s}{\phi(s)}\, ds = \infty. \tag{3.1}$$

Then there exists a constant $N = N(R, C, \|\eta\|_\infty, \phi)$ such that if $x \in C^2(I, \mathbb{R}^n)$ satisfies

$$|x| \le R, \quad |x''| \le \eta(x(t))'' + C \quad \text{and} \quad |x''| \le \phi(|x'|)$$

then $|x'| \le N$.

Now $\eta(x(t))''$ may be easily calculated as

$$d^2_{x(t)}\eta(x'(t), x'(t)) + d_{x(t)}\eta(x''(t)),$$

where $d^2_x\eta$ stands for the quadratic form induced by the Hessian. The condition thus obtained is

$$\begin{aligned} |f(t,x,y)| &\le d^2_x\eta(y,y) + d_x\eta(f(t,x,y)) + C, \\ |f(t,x,y)| &\le \phi(|y|). \end{aligned} \tag{3.2}$$

This condition obviously generalizes the original assumptions (1.3) and (1.4) given in [5], setting $\eta(x) = |x|^2 - R^2$ and $D = B(0,R)$.

Remark 3.2. (a) The function η might also depend on the time t, although the expression for $\eta(x(t))''$ in this case would become more complicated.

(b) As $d^2_x\eta(y,y) + d_x\eta(f(t,x,y)) = (-\mathbb{I}_x(y) + \langle f(t,x,y), n_x\rangle).|\nabla_x\eta|$ for $x \in \partial D$, the fact that the expression $\langle f(t,x,y), x\rangle + |y|^2$ appears both in (1.2) and (1.4) is not a coincidence.

From the discussion in [5, Corollary 1], we get a simpler (but somewhat more restrictive) growth condition on f. Let us firstly recall the following

Lemma 3.3. *Let R, γ, C be non-negative constants where $R\gamma < 1$. Then there exist $N = N(R, \gamma, C)$ such that if $x \in C^2(I, \mathbb{R}^n)$ satisfies*

$$|x| \le R \quad \text{and} \quad |x''| \le \gamma |x'|^2 + C$$

then $|x'| \le N$.

With this last result in mind, we may impose, instead of (3.2), the condition:

$$|f(t,x,y)| \le \gamma |y|^2 + C \tag{3.3}$$

for every $x \in \overline{D}$, where C and γ are constants with $\gamma R_D < 1$. Here, R_D is the radius of D, defined as the radius of the smallest ball containing D (notice that $\operatorname{diam}(D) \le 2R_D$). In particular, (3.3) generalizes the growth condition imposed in [1].

Unfortunately our version of the Hartman condition combined with (3.3) requires that $R_D^{-1}|y|^2 + C > \mathbb{I}_x(y)$ for all y, so the theorem is not applicable for arbitrary domains. For instance, taking y as an eigenvector of g_x such that $|y| \gg 0$, it is easy to see that (3.3) together with (1.6) implies that $R_D curv(D,x) > -1$. Thus, our results cannot be applied if for example $D = B(0,R) \setminus B(0,r)$ for some $r < R$.

In Section 5 we shall prove that, furthermore, the domain D must be contractible. The same happens with condition (3.2), independently of (1.6).

Remark 3.4. It is worth observing that as far as an *a priori* bound N is obtained for the derivative of the solutions, condition (1.6) can be relaxed to consider only those points $(x,y) \in T\partial D$ such that $|y| \le N$. This shows that, in fact, condition (3.3) is not necessarily incompatible with (1.6) when the domain is non-contractible. We shall not pursue this direction here; some interesting consequences of this fact will take part in a forthcoming paper.

The following result is a refinement of Lemma 3.3 that shall be needed for the proof of Theorem 4.2.

Lemma 3.5. *Let R, γ, C be non-negative constants where $R\gamma < 1$, and $0 < T \le 1$. Then there exist $N = N(R,\gamma,C)$ (independent of T) such that if $x \in C^2([0,T],\mathbb{R}^n)$ satisfies:*

$$|x| \le R, \quad |x''| \le \gamma |x'|^2 + C \quad \textit{and} \quad x(0) = x(T) = x_0$$

then $|x'| \le N$.

Proof. Following the remarks of [5, Lemma 3] and the proof of [5, Lemma 2], let $\rho\colon [0,T] \to \mathbb{R}$ be defined by

$$\rho(t) = \alpha |x|^2 + \frac{K}{2} t^2 \quad \text{where } \alpha = \frac{\gamma}{2(1-\gamma R)}, \quad K = \frac{C}{1-\gamma R}.$$

Then $\|\rho\|_\infty \le M_1(C,\gamma,R)$, and $|x''| \le \rho$, since

$$\begin{aligned} \rho''(t) &= 2\alpha(\langle x'', x\rangle + |x'|^2) + K \\ &\ge 2\alpha(|x'|^2 - R(\gamma|x'|^2 + C)) + K = |x'|^2 2\alpha(1-\gamma R) - 2\alpha RC + K \\ &\ge \frac{|x''| - C}{\gamma} 2\alpha(1-\gamma R) - 2\alpha RC + K = |x''| - C - 2\alpha RC + K = |x''|. \end{aligned}$$

From the discussion in [5, sections 3–5], we obtain the formula:

$$|\Phi(|x'(t)|) - \Phi(|x'(T/2)|)| \le \int_t^{T/2} M_2(C,\gamma,R)/T \pm \rho'(s)\, ds$$

for all $t \in [0,T]$ (the $\pm$ sign depends on whether $t < T/2$ or $t > T/2$). Hence,

$$|\Phi(|x'(t)|) - \Phi(|x'(T/2)|)| \le M_2(C,\gamma,R) + 2\|\rho\|_\infty. \tag{3.4}$$

Unless x is constant, as $x(0) = x(T)$ there must exist a tangent ball $B \supseteq \operatorname{Im}(x)$ of radius R to x, at a point $x(t_0) \neq x_0$. Now, let n be the outer unit normal vector of B at $x(t_0)$. Then

$$\gamma|x'(t_0)|^2 + C \geq |x''(t_0)| \geq -\langle x''(t_0), n\rangle \geq \frac{1}{R}\,|x'(t_0)|^2,$$
$$|x'(t_0)| \leq \sqrt{\frac{C}{1/R - \gamma}}.$$

Using inequality (3.4) twice, for $t = t_0$ and for arbitrary t, we get

$$|\Phi(|x'(t_0)|) - \Phi(|x'(t)|)| \leq 2M_2(C, \gamma, R) + 4\|\rho\|_\infty$$

and hence

$$|\Phi(|x'(t)|)| \leq |\Phi(|x'(t_0)|)| + 2M_2(C, \gamma, R) + 4\|\rho\|_\infty \leq M_3(C, \gamma, R). \qquad \square$$

Next we show an example where the main theorems are applicable for a function on a non-convex set in the plane $D \subset \mathbb{R}^2$. In this situation the boundary may be described locally by C^2 curves $b\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^2$. The second fundamental form is simply $\mathbb{I}_x(y) = -k(x)|y|^2$ where k is the curvature of b. When b is parametrized by arc-length, $|k|$ is the norm of the vector b''.

Example 3.6. Let

$$\eta(x) = (|x|^2 - 1)\left(|x + (\delta, 0)|^2 - \frac{1}{\gamma^2}\right),$$

where $\gamma \in (0, 1), \delta \in (1/\gamma - 1, 1/\gamma + 1)$ are fixed constants. Take D_ε to be the connected component of $\eta^{-1}(-\infty, -\varepsilon)$ on the right side, where $\varepsilon > 0$ is small enough.

Observe that $D_0 = B(0,1)\setminus\overline{B((-\delta, 0), 1/\gamma)}$. The singular points of η are the intersections of $\partial B(0, 1)$ with $\partial B((-\delta, 0), 1/\gamma))$ and the centers of these circles.

Define

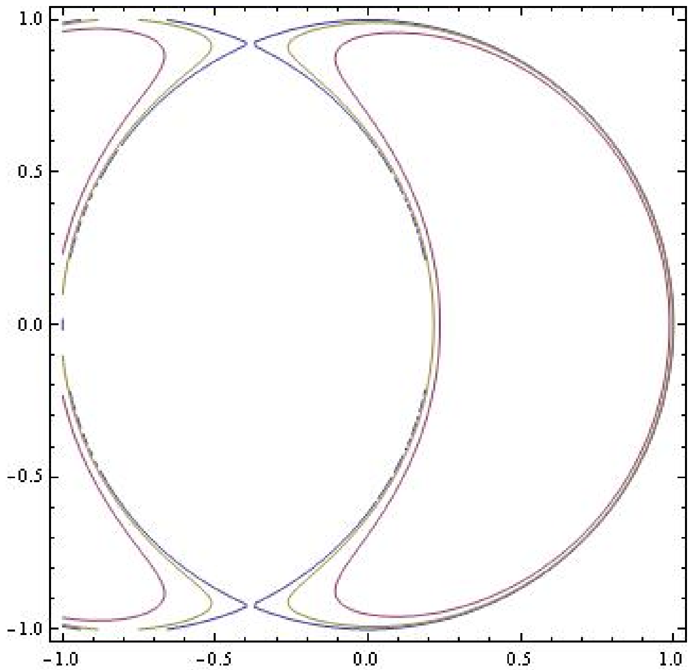
$$f(t, x, y) = |y|^2 \frac{\nabla\eta}{|\nabla\eta|} + p(t, x, y)$$

where p is bounded.

We have always $\overline{D_\varepsilon} \subseteq D_0 \subseteq B(0, 1)$, so $R_{D_\varepsilon} < 1$ and (3.3) is satisfied since $\gamma < 1$. Next, we need to check condition (1.6).

Let $k(x)$ be the curvature of the curve $\partial D_{\eta(x)}$ in the point x. It is obvious that k is a continuous function of x for the regular points of η.

From the choice of f, we only need to show that $k(x) > -1$. This is true for all regular points of ∂D_0 because of Remark 2.4(c). The differential $d^2\eta$ can be explicitely calculated to show that $k(x) > 0$ for points of ∂D_ε near the singular points of D_0.



Condition (1.6) takes the form

$$\langle n_x, p(t,x,y)\rangle > -|y|^2(k(x)+1)$$

Moreover, $k+1$ is strictly positive, so if for example $\langle n_x, p(t,x,y)\rangle \geq 0$, then the periodic and Dirichlet problems admit at least one solution.

Now we give a counter-example showing that the growth conditions cannot be easily dropped.

Let $x_0, x_1 \in \mathbb{R}^n$ and $\varepsilon, r > 0$ such that $x_1 \notin B(x_0, r+2\varepsilon)$. Let $\eta\colon \mathbb{R}^n \to \mathbb{R}$ be a C^∞, bounded function such that

$$\eta(x) = \begin{cases} 1 & \text{if } ||x|-r| \leq \varepsilon, \\ 0 & \text{if } ||x|-r| \geq 2\varepsilon. \end{cases}$$

Let $f_0(t,x,y) = -Ky(|y|^2+1)\eta(x)$ where $K > 2\pi/\varepsilon$. Finally, let $g(t,x,y)$ be any function equal to 0 for $x \in B(x_0, r+2\varepsilon)$.

Claim 3.7. *There is no solution of the problem*

$$\begin{cases} x'' = (f_0+g)(t,x,x'), \\ x(0) = x_0, \\ x(1) = x_1. \end{cases}$$

Proof. Let x be a solution and $t_0 \in I$ such that $|x(t_0) - x_0| = r$. For all t such that $|x(t) - x(t_0)| \le \varepsilon$ we have

$$x'' = -Kx'(|x'|^2 + 1).$$

Let $w(t) = |x'|$. We compute:

$$\begin{aligned} 2ww' &= (w^2)' = 2\langle x', x''\rangle = -2K|x'|^2(|x'|^2+1) = -2Kw^2(w^2+1), \\ w' &= -w(w^2+1). \end{aligned}$$

The unique solution of this differential equation is

$$w(t)^2 = ((1 + w^{-2}(t_0))e^{2Kt} - 1)^{-1}$$

which satisfies

$$\int_{t_0}^{t} w(s)\,ds = \frac{1}{K}\mathrm{ArcTan}\,\left(\sqrt{(1 + w^{-2}(t_0))e^{2Kt} - 1}\right)\Big|_{t_0}^{t} \le \frac{\pi}{K}.$$

Since $K > 2\pi/\varepsilon$ we have

$$|x(t) - x(t_0)| = \left|\int_{t_0}^{t} w(s)\,ds\right| \le \frac{\varepsilon}{2}.$$

Then we know that x lies in $B(x(t_0), \varepsilon/2)$ whenever x lies in $B(x(t_0), \varepsilon)$. This means that x lies in $B(x(t_0), \varepsilon/2)$ for all $t \ge t_0$, a contradiction. □

For any domain D and points $x_0, x_1 \in D$ we may choose ε, r, g for which the counter-example applies and $f_0 + g$ satisfies condition (1.6). For example, for r, ε small and $g = 0$ we have a counter-example to the original Hartman condition when $D = B(0, R)$.

Other counter-examples for scalar equations (also with cubic growth in $|x'|$) can be found in [8].

A more delicate question is whether if condition $\gamma R_D < 1$ in (3.3) may be relaxed or dropped.

Let us use complex notation for $\mathbb{R}^2$. Define

$$f(t, x, y) = -\frac{x}{|x|^2}|y|^2 + 2ix.$$

Claim 3.8. *The system $x'' = f(t, x, x')$ has no classical periodic solutions.*

Proof. Let x be a periodic solution. Notice that

$$\frac{d^2}{dt^2}|x|^2 = 2(\langle f(t, x, x'), x\rangle + |x'|^2) = 0$$

so by periodicity, $|x| = r$ is constant. Writing $x(t) = re^{i\theta(t)}$ it follows that $\theta \equiv 2$, and hence $x'(t)$ cannot be equal to $x'(0)$ for any $t > 0$, a contradiction. □

It is seen that

$$\left\langle f(t,x,y), \frac{x}{|x|} \right\rangle = -\frac{|y|^2}{|x|},$$

so if

$$D = \{x \in \mathbb{R}^2 : r_1 < |x| < r_2\} \quad \text{for } r_2 > r_1 > 0,$$

then (1.6) is satisfied for suitable perturbations of f. Condition (3.3) is not satisfied because of the requirement $\gamma R_D < 1$.

The general theory of compact perturbations of the identity in Banach spaces implies that there are no periodic solutions in $\overline{D}$ of the equations $x'' = g(t,x,x')$ for g in a neighbourhood of f.

For example, one may take $g\colon \mathbb{R}^n \to \mathbb{R}^n$ continuous such that

$$\begin{aligned} \langle g(u), u\rangle &> 0 \quad \text{if } |u| = r_2, \\ \langle g(u), u\rangle &< 0 \quad \text{if } |u| = r_1, \end{aligned}$$

then (1.6) holds for $f_n := f + g/n$. If x_n is a periodic solution of the problem $x'' = f_n(t,x,x')$ such that $x_n(t) \in D$ for all t, then $x_n = \overline{x}_n + K(f_n(x_n))$ where $\overline{x}$ denotes the average of the function x and K is the right inverse of the operator $Lx := x''$ satisfying $\overline{K\varphi} = 0$ for $\varphi \in C([0,1])$ with $\overline{\varphi} = 0$. As $\{x_n\}$ is bounded, passing to a subsequence we may assume that $\overline{x}_n + K(f_n(x_n))$ converges to some function x, so $x_n \to x$, and x is a solution of the problem, a contradiction.

We conclude that condition (3.3) is sharp.

4. Main theorems

4.1. Dirichlet conditions. Throughout this section, we shall use the following notations:

$T = C(I \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, equipped with the usual compact open topology.

$C^i = C^i(I, \mathbb{R}^n)$ as Banach spaces with the standard norms.

$X_0 = \{x \in X : x(0) = x(1) = 0\}$ for $X = C, C^1, \ldots$

It is well-known that the map $L\colon C_0^2 \to C, Lu = u''$ is a Banach space isomorphism; let $K\colon C \to C_0^2$ be its inverse and $\iota\colon C^2 \to C^1$ the compact inclusion.

Moreover, let $N : T \times C^1 \to C$ be the nonlinear operator

$$\mathcal{N}(f,x)(t) = f(t, x(t), x'(t)),$$

which is clearly continuous. Finally, let $B\colon \mathbb{R}^n \times \mathbb{R}^n \to C^1$ be the segment $B(x,y)(t) = ty + (1-t)x$ and $F\colon T \times C^1 \times \mathbb{R}^n \times \mathbb{R}^n \to C^1$ the operator defined by $F(f,u,x,y) = u - \iota K \mathcal{N}(f,u) - B(x,y)$.

LEMMA 4.1. *$F(f,u,x,y) = 0$ if and only if u is a solution of the nonlinear problem:*

$$\begin{cases} u'' = f(t,u,u'), \\ u(0) = x, \\ u(1) = y. \end{cases}$$

Thus, our main theorem for Dirichlet conditions reads as follows:

THEOREM 4.2. *Let D be a bounded domain in $\mathbb{R}^n$ with C^2 boundary, $x_0, x_1 \in D$ and $f: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that* (1.6) *and* (3.3) *hold. Then there exists a solution of $x'' = f(t,x,x')$ satisfying the Dirichlet conditions $x(i) = x_i$.*

PROOF. Let N be the bound provided by (3.3) (see Lemma 3.5 in previous section) and define

$$\mathcal{D} = \{x \in C^1 : \operatorname{Im}(x) \subset D \text{ and } \|x'\|_\infty < N + 1\}.$$

We shall construct a homotopy starting from the functional $u \mapsto F(f,u,x_0,x_1)$, so we may calculate its degree in the open set $\mathcal{D}$, provided that it does not vanish on $\partial\mathcal{D}$ along the homotopy.

The homotopy shall be constructed in three steps:

Step 1. Let $F^1_\lambda(u) := F(f,u,x_0,\gamma(\lambda))$ where $\gamma: [0,1] \to D$ is a path joining x_0, x_1. It is obvious from Corollary 2.2 and from the choice of N that F^1_λ has no zeros on the boundary of $\mathcal{D}$. The problem is now homotopic to the same problem with boundary conditions $x(0) = x(1) = x_0$.

Step 2. Let $\lambda_0 > 0$ be such that $\overline{B(x_0, \lambda_0 N)} \subset D$ and set

$$f^2_\lambda(t,x,y) := \lambda^2 f(t,x,\lambda^{-1}y).$$

The function $\mathbb{R} \to T$, defined by $\lambda \mapsto f^2_\lambda$ is continuous for $\lambda \in [\lambda_0, 1]$.

Next, define $F^2_\lambda(u) := F(f^2_\lambda, u, x_0, x_0)$ so $F^2_1 = F^1_0$.

REMARK 4.3. The function f^2_λ satisfies (1.6), because $\mathbb{I}_x(y)$ is quadratic in y. This means (from Corollary 2.2) that there are no solutions tangent to ∂D from inside. Also, as $x_0 \in D$ we know that solutions do not touch the boundary at $t = 0, 1$.

Now we have to estimate the derivative of the solutions x_λ of the equation $F^2_\lambda(x) = 0$ satisfying $x_\lambda(t) \in \overline{D}$.

Let $y(t) = x_\lambda(\lambda^{-1}t)$ for $t \in [0,\lambda]$, then

$$\begin{aligned} y''(t) &= \lambda^{-2} x''_\lambda(\lambda^{-1}t) = \lambda^{-2} f^2_\lambda(\lambda^{-1}t, x_\lambda(\lambda^{-1}t), x'_\lambda(\lambda^{-1}t)) \\ &= f(\lambda^{-1}t, x_\lambda(\lambda^{-1}t), \lambda^{-1}x'_\lambda(\lambda^{-1}t)) = f(\lambda^{-1}t, y(t), y'(t)). \end{aligned}$$

So y is a solution of $y''(t) = f(\lambda^{-1}t, y(t), y'(t))$ for $t \in [0, \lambda]$ and (3.3) applies (here we use the fact that N in Lemma 3.5 does not depend on the interval of definition of y). Then we get

$$N > |y'(t)| = |\lambda^{-1}x(\lambda^{-1}t)| \quad \text{for } t \in [0, \lambda],$$

which implies

$$\|x'_\lambda\|_\infty < \lambda N. \tag{4.1}$$

Step 3. For $\lambda \in [0, \lambda_0]$, let us define

$$f^3_\lambda(t, x, y) := \lambda^2 f(t, x, y\lambda_0^{-1}), \qquad F^3_\lambda(u) := F(f^3_\lambda, u, x_0, x_0).$$

Let $x_\lambda \in \overline{\mathcal{D}}$ be a solution of $F^3_\lambda(x) = 0$. If $\lambda > 0$ it is clear that $|f^3_\lambda| \le |f^2_{\lambda_0}|$ so $|x'_\lambda| \le \lambda_0 N$ still holds. Hence $x_\lambda \in B(x_0, \lambda_0 N) \subset\subset D$ so $x_\lambda \notin \partial\mathcal{D}$.

Finally, $F^3_0(u) = F(0, u, x_0, x_0) = u - x_0$ so $deg(F^3_0, \mathcal{D}, 0) = 1$ and the proof is complete. □

4.2. Periodic conditions. In order to state our existence result for periodic conditions, now we shall consider:

$$\begin{aligned} T &= \{f \in C(I \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n) : f(0, x, y) = f(1, x, y)\}, \\ C^i_{\text{per}} &= \{x \in C^i : x^{(j)}(0) = x^{(j)}(1),\ j < i\}, \\ \widetilde{C}^i &= \{x \in C^i : \overline{x} = 0\}. \end{aligned}$$

The map $L: C^2_{\text{per}} \cap \widetilde{C} \to \widetilde{C}$ given by $Lx = x''$ is an isomorphism, denote by $K: \widetilde{C} \to C^2_{\text{per}} \cap \widetilde{C}$ its inverse and $\iota: C^2 \to C^1$ the compact inclusion.

Let $P: C^1 \to \widetilde{C}^1$ be the projection associated with the decomposition $C^1 = \mathbb{R}^n \oplus \widetilde{C}^1$ and let $\mathcal{N}: T \times C^1 \to C$ as before. Following [9], define $F: T \times C^1 \to C^1$ as the operator given by

$$F(f, u) = u - \overline{u} - \overline{\mathcal{N}(f, u)} + \iota K P \mathcal{N} u$$

and $G: T \times C^1 \times \mathbb{R} \to C^1$ by

$$G(f, u, \mu) = u - \overline{u} - \overline{\mathcal{N}(f, u)} + \mu \iota K P \updownarrow N u.$$

The following result is easily verified as in [9]:

Lemma 4.4. *$F(f, u) = 0$ if and only if u is a solution of the nonlinear problem:*

$$\begin{cases} u'' = f(t, u, u'), \\ u(0) = u(1), \\ u'(0) = u'(1). \end{cases}$$

Hence we may establish our main result for periodic conditions:

Theorem 4.5. *Let* $f\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ *be such that* (1.6) *holds, and either* (3.2) *or* (3.3) *is satisfied. Then there exists a periodic solution of* $x'' = f(t,x,x')$.

Proof. The proof follows the same outline of Theorem 4.2, conveniently modified for this situation. In first place, it is obvious that we do not need to move the boundary conditions.

Let us set again $\mathcal{D} = \{x \in C^1 : \mathrm{Im}(x) \subset D \text{ and } \|x'\|_\infty < N+1\}$ with $N = N(C+1, 2\phi)$ where C and ϕ are as in (3.2) or (3.3),

The problem of adapting the previous proof to this new context relies in the fact that, due to the resonance of the operator u'', the bound for x'_λ does not force solutions to be far from the boundary. To overcome this difficulty we need f to point outwards the open set D, for every t and y.

Step 1. As M is a C^1 manifold, we may suppose n_x is a continuous and bounded function defined in $\mathbb{R}^n$. Furthermore, we may suppose that $|n_x| \le 1$ (for example using Dugundji's extension theorem). In fact, as we are assuming that $D = \eta^{-1}(-\infty,0)$ for some smooth η, we may choose the vector field $\nabla\eta(x)$, properly normalized in a neighbourhood of M.

For $\lambda \in [0,2]$, set

$$f^1_\lambda(t,x,y) = f(t,x,y) + n_x\lambda\bigg(\max\{0, -\langle f(t,x,y), n_x\rangle\} + \frac{1}{2}\min\{1,\phi(y)\}\bigg)$$

where, when (3.3) is assumed, $\phi(y) := \gamma|y|^2 + C$, and

$$F^1(\lambda,u) = F(f^1_\lambda, u)$$

so that $F^1(0,u) = F(f,u)$.

By Lemma 4.6 below, we know that $|f^1_\lambda| \le |f| + \min\{1,\phi\}$. Thus, in both cases (3.2) and (3.3) it is easy to see that f_λ also satisfies it as well, with $\widehat{C} = C+1$ and $\widehat{\phi} = 2\phi$. Indeed, for condition (3.2)

$$\begin{aligned}
|f^1_\lambda| &\le |f| + \min\{1,\phi\}\\
&\le d^2_x\eta(y,y) + d_x\eta(f(t,x,y)) + C + 1\\
&\le d^2_x\eta(y,y) + d_x\eta(f^1_\lambda(t,x,y)) + \widehat{C},\\
|f^1_\lambda| &\le |f| + \phi \le 2\phi = \widehat{\phi}
\end{aligned}$$

and, for condition (3.3),

$$|f^1_\lambda| \le |f| + 1 \le \gamma|y|^2 + C + 1.$$

Moreover, condition (1.6) is trivially satisfied and it is easy to prove that there are no solutions of $x'' = f^1_\lambda(t,x,x')$ on the boundary of $\mathcal{D}$, for any $\lambda \in [0,2]$.

Furthermore,

$$\langle f_\lambda^1(t,x,y), n_x\rangle = \begin{cases} \langle f(t,x,y), n_x\rangle(1-\lambda) + \dfrac{\lambda}{2}\min\{1,\phi(y)\} & \text{if } \langle f(t,x,y), n_x\rangle < 0, \\ \langle f(t,x,y), n_x\rangle + \dfrac{\lambda}{2}\min\{1,\phi(y)\} & \text{otherwise,} \end{cases}$$

and hence

$$\langle f_2^1(t,x,y), n_x\rangle > 0 \tag{4.2}$$

for all t, y and $x \in M$. Thus,

$$\deg(\overline{f_2^1(t,x,0)}, D, 0) = \deg(n_x, D, 0).$$

Now the problem is homotopic to $x''(t) = f_2^1(t,x,x')$, where f_2^1 points outwards over the boundary (namely, it satisfies (4.2)).

Step 2. Following the idea from Step 2 in Theorem 4.2, let

$$f_\lambda^2(t,x,y) = \lambda^2 f_2^1(t,x,\lambda^{-1}y), \qquad F^2(\lambda,u) = F(f_\lambda^2, u).$$

Both Remark 4.3 and the bound obtained in (4.1) apply here (in contrast with the Dirichlet case, Lemma 3.5 is not needed here since now we may extend solutions periodically) so we deduce that solutions of $x_\lambda'' = f_\lambda^2(t, x_\lambda, x_\lambda')$ are not on $\partial\mathcal{D}$.

Now we claim there exists $\lambda_0 > 0$ such that there are no solutions of $x''(t) = \mu f_{\lambda_0}^2(t,x,y)$ in $\partial\mathcal{D}$ for $\mu \in (0,1]$. Suppose again, by contradiction, that there exists a sequence $x_\lambda \in \partial\mathcal{D}$ of solutions of $x_\lambda''(t) = \mu_\lambda f_\lambda^2(t,x,y)$ with $\mu_\lambda \in (0,1]$ and $\lambda \to 0$. By (4.1), from Theorem 4.2 we know that $|x_\lambda'| \leq \lambda N$ and by compactness we can suppose that $x_\lambda \to p$ uniformly for some $p \in \partial D$. By periodicity, $\int_0^T x_\lambda'' = 0$, so

$$0 = \int_0^T \langle \mu_\lambda^{-1}\lambda^{-2}x_\lambda'', n_p\rangle = \int_0^T \langle f_2^1(t, x_\lambda, \lambda^{-1}x_\lambda'), n_p)\rangle\, dt.$$

Passing to a subsequence, we may assume that

$$f_2^1(t, x_\lambda, \lambda^{-1}x_\lambda') - f_2^1(t, p, \lambda^{-1}x_\lambda') \to 0$$

uniformly, and we deduce:

$$\int_0^T \langle f_2^1(t, p, \lambda^{-1}x_\lambda'), n_p)\rangle\, dt \to 0.$$

This is a contradiction, because (4.2) implies that $\langle f_2^1(t,p,y), n_p\rangle$ has a positive minimum over the compact set $I \times \{p\} \times \overline{B(0,N)}$, which contains $(t, p, \lambda^{-1}x_\lambda')$ for all t.

Step 3. Now we set

$$F^3_\mu(u) = G(f^2_{\lambda_0}, u, \mu)$$

where G is defined as before, and observe that a zero of F^3_μ with $\mu \in (0,1]$ is a solution of the equation $x''(t) = \mu\lambda_0^2 f^1_2(t, x, \lambda_0^{-1} y)$, so it does not belong to $\partial\mathcal{D}$.

Finally $F^3_0(u) = u - \overline{u} - \overline{\mathcal{N}(f^2_{\lambda_0}, u)}$ and its Leray–Schauder degree is equal to the Brouwer degree on D of the function $-\psi$, where $\psi\colon \overline{D} \longrightarrow \mathbb{R}^n$ is given by

$$\psi(p) := \int_0^T f^1_2(t, p, 0)\, dt.$$

Using (4.2), we deduce that $\int_0^T \langle f^1_2(t, x, 0), n_x\rangle\, dt > 0$ when $x \in \partial D$, so the function ψ is linearly homotopic to the normal unit vector field n. Due to a theorem by Hopf [6], the degree of n is equal to $\chi(\overline{D})$, where χ denotes the Euler characteristic. In Section 5 we shall prove that $\overline{D}$ is contractible, so $\chi(\overline{D}) = 1$ and the proof is complete. □

Lemma 4.6. *Let $v, n \in \mathbb{R}^d$ with $|n| \le 1$. Then for $\lambda \in [0,2]$ we have:*

$$|v - \lambda n\langle v, n\rangle| \le |v|.$$

Proof. We compute

$$\begin{aligned} |v - \lambda n\langle v, n\rangle|^2 &= |v|^2 - 2\lambda\langle v, n\rangle^2 + \lambda^2\langle v, n\rangle^2 |n|^2 \\ &\le |v|^2 - 2\lambda\langle v, n\rangle^2 + \lambda^2\langle v, n\rangle^2 = |v|^2 + (\lambda^2 - 2\lambda)\langle v, n\rangle^2 \le |v|^2 \end{aligned}$$

for $\lambda \in [0,2]$. □

5. Topology of the domain

In this section we will show that the conditions in the preceding results imply that D must be contractible. This is proved in Theorems 5.4 and 5.6 using two preliminary lemmas. Our main tool shall be Morse theory for manifolds with boundary.

Definition 5.1. Let $M, N \subseteq \mathbb{R}^n$ be oriented C^2 manifolds with normal unit vector fields n_N and n_M. We shall say that M and N are outside tangent at $p \in N \cap M$ if $T_pM = T_pN$ and $n_N(p) = -n_M(p)$.

Lemma 5.2. *Let $\eta\colon \mathbb{R}^n \to \mathbb{R}$ be C^2 and suppose $S = \{\eta = \eta_0\}$ is a C^2 manifold oriented in the direction of $\nabla\eta$. Then $d^2_x\eta(y,y) = -|\nabla\eta| . \mathbb{I}^S_x(y)$.*

Proof. Let x be a curve such that $x(0) = x$ and $x'(0) = y$. As

$$\left\langle \frac{d}{dt}(\nabla\eta(x(t))), x'(t)\right\rangle = d^2_x\eta(x', x'),$$

and writing $\nabla_{x(t)}\eta = \delta(t).n_S(x(t))$ where $\delta = |\nabla\eta|$, we obtain:

$$\frac{d}{dt}(\nabla\eta(x(t))) = \delta'(t)n_S + \delta.g(x')$$

and hence

$$d_x^2\eta(x', x') = \delta\langle g(x'), x'\rangle = -|\nabla\eta|\mathbb{I}_x(x'). \qquad \square$$

Lemma 5.3. *Let x be a curve in M, S as before and suppose M and N are outside tangent at $x(0)$. If $\mathbb{I}^M < -\mathbb{I}^S$ then $(\eta\circ x)''(0) > 0$.*

Proof. Using the previous lemma, we deduce

$$\begin{aligned}(\eta\circ x)'' = d_x^2\eta(x', x') + d_x\eta(x'') &= -|\nabla\eta|\mathbb{I}_x^S(x') + |\nabla\eta|\langle n_S, x''\rangle,\\ (\eta\circ x)''(0) &= -(\mathbb{I}^S + \mathbb{I}^M)|\nabla\eta| > 0. \qquad \square\end{aligned}$$

Theorem 5.4. *Let D satisfy $R_D\mathrm{curv}(D, p) > -1$ for all $p \in M = \partial D$. Then D is contractible.*

Proof. Without loss of generality, we may assume that D is centered at 0, that is: $\overline{D} \subseteq \overline{B}(R_D, 0)$. Let $R > R_D$ be such that $\mathrm{curv}(D, p) > -1/R$ for every $p \in M$. Let $\eta\colon \overline{D} \to \mathbb{R}$,

$$\eta(x, y) = x - \sqrt{R^2 - |y|^2}$$

where $x \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$. Clearly η is a C^∞ function defined also in a neighbourhood of $\overline{D}$ which is an n-dimensional manifold with boundary ∂D. Consider $S(t) = \{\eta = t\} = \partial B((t, 0), R) \cap \{x \geq t\}$ with the orientation given by $\nabla\eta$.

Let us prove that η is a Morse function in $\overline{D}$. First of all, there are no critical points inside D. A critical point of $\eta|_M$ is a point p such that $S(t)$ is tangent to M in p for some t. A critical point of η as a Morse function in a manifold with boundary is a critical point of $\eta|_M$ such that $\nabla\eta$ points inwards D. In such a point p, $S(t)$ and M are outside tangent. Also

$$\mathbb{I}_p^{S(t)}(v) = -|v|^2\frac{1}{R} < |v|^2\mathrm{curv}(D, p) \leq -\mathbb{I}_p^M(v)$$

for every $t \in \mathbb{R}, v \in \mathbb{R}^n$.

The previous lemma applies and we get that p is a nondegenerate local minimum.

Morse theory implies now that $\overline{D}$ has the homotopy type of a disjoint union of points, one per each critical point. As D is connected, we deduce that $\overline{D}$ is contractible. $\square$

Remark 5.5. As we saw in Remark 2.4(c), $\mathrm{curv}(D, p)$ can be calculated using tangent balls. This might suggest a generalization of the notion of curvature for arbitrary open sets if one defines the concept of exterior tangent ball in the following way: B is an exterior tangent ball at $p \in \partial D$ if $p \in \overline{B}$ and there exists a neighbourhood V of p such that $D \cap V \subset \mathbb{R}^n \setminus B$.

Then it is natural to ask if Theorem 5.4 is still valid in this context. The answer is negative:

Consider for example $n = 3$,

$$D = B(0,1) \setminus (\overline{B}((0,1,0), 1+\varepsilon) \cup \overline{B}((0,-1,0), 1+\varepsilon))$$

for small $\varepsilon > 0$. This set obviously satisfies $R_D curv(D,p) > -1$ because for every point in the boundary there is an external tangent ball of radius $1+\varepsilon$, but it has the homotopy type of S^1.

However, if we approximate D by smooth domains, it is clear that the condition fails. This shows that the previous definition of curvature for arbitrary domains is not accurate.

Theorem 5.6. *Let D satisfy* (3.2) *for some η and some f. Then D is contractible.*

Proof. Using (3.2), let us firstly notice that if $|\nabla\eta| < 1$, then $0 \leq d_x^2\eta(y,y) + C$ for arbitrary y, so $d_x^2\eta$ must be positive semidefinite.

Let $K_a = \{x \in D : d_x\eta = 0,\ \eta(x) = a\}$ be the critical set of level a and $\eta^a = \{x \in D : \eta(x) < a\}$ the level set.

As $\nabla\eta$ is continuous in $\overline{D}$ there is an $\varepsilon > 0$ such that for all a, $d^2\eta$ is positive semidefinite in $O_a = B(K_a, \varepsilon)$. Since $\eta \geq a$ in O_a, we deduce that if $b > a$ then $O_b \cap K_a = \emptyset$. This implies that there are only finite critical values. Also it is clear that $O_a \cap \eta^a = \emptyset$, so the Morse deformation Lemma shows that D has the homotopy type of the finite disjoint union of the level sets K_a. Again, since D is connected there is only one critical set K_a which is also the minimum set of η. Now let $\delta > 0$ be small enough such that $\eta^{a+\delta} \subseteq O_a$ (where a is the minimum). The set $\eta^{a+\delta}$ is a level set of a (non strict) convex function so it is a convex set. Then the critical set K_a is the intersection of all such convex sets so it is again convex. $\square$

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