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ABSOLUTE RETRACTIVITY OF THE COMMON FIXED POINTS SET OF TWO MULTIFUNCTIONS

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Abstract. In 1970, Schirmer discussed about topological properties of the fixed point set of multifunctions ([4]). Later, some authors continued this study by providing different conditions ([1] and [3]). Recently, Sintamarian proved results on absolute retractivity of the common fixed points set of two multivalued operators ([5] and [6]). We shall present some results on absolute retractivity of the common fixed points set of two multifunctions by using different conditions.

1. Introduction

Let X be a nonempty set, $P(X)$ the set of all nonempty subsets of X, $F_1, F_2\colon X \to P(X)$ two multifunctions, $\mathcal{F}_{F_1}$ the fixed point set of F_1, $(\mathcal{CF})_{F_1,F_2}$ the common fixed point set of F_1 and F_2, that is $(\mathcal{CF})_{F_1,F_2} = \{x \in X : x \in F_1x \cap F_2x\}$. Let X and Y be nonempty sets and $F\colon X \to P(Y)$ a multifunctions. A mapping $\varphi\colon X \to Y$ is called a selection of F whenever $\varphi(x) \in Fx$ for all $x \in X$. Throughout the paper, for a topological space X we denote the set of all nonempty closed subsets of X by $P_{\rm cl}(X)$, the set of all nonempty convex subsets of X by $P_{\rm cv}(X)$ when X is a vector space, the set of all nonempty closed and bounded subsets of X by $P_{\rm b,cl}(X)$ when X is a metric space and $P_{\rm cl,cv}(X) = P_{\rm cl}(X) \cap P_{\rm cv}(X)$ when X is a normed space.

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Let (X,d) be a metric space. For $x \in X$ and $A, B \subseteq X$, set

$$D(x,A) = \inf_{y\in A} d(x,y) \quad \text{and} \quad H(A,B) = \max\left\{\sup_{x\in A} D(x,B), \sup_{y\in B} D(y,A)\right\}.$$

It is known that, H is a metric on closed bounded subsets of X which is called the Hausdorff metric.

We say that a topological space X is an absolute retract for metric spaces whenever for each metric space Y, $A \in P_{\mathrm{cl}}(Y)$ and continuous function $\psi: A \to X$, there exists a continuous function $\varphi: Y \to X$ such that $\varphi|_A = \psi$. Let $\mathcal{M}$ be the set of all metric spaces, $X \in \mathcal{M}$, $\mathcal{D} \in P(\mathcal{M})$ and $F: X \to P_{\mathrm{b,cl}}(X)$ a lower semi-continuous multifunction. We say that F has the selection property with respect to $\mathcal{D}$ if for each $Y \in \mathcal{D}$, continuous function $f: Y \to X$ and continuous functional $g: Y \to (0,\infty)$ such that $G(y) := \overline{F(f(y)) \cap N_{g(y)}(f(y))} \neq \emptyset$ for all $y \in Y$, $A \in P_{\mathrm{cl}}(Y)$, every continuous selection $\psi: A \to X$ of $G|_A$ admits a continuous extension $\varphi: Y \to X$, which is a selection of G. If $\mathcal{D} = \mathcal{M}$, then we say that F has the selection property and we denote this by $F \in \mathrm{SP}(X)$ ([5]).

An interesting problem in fixed point theory of multivalued operators is to investigate under what conditions some properties of the values of a multifunction are inherited by its fixed point set. For some multifunctions, this problem was studied by Schirmer in 1970 ([4]), by Alicu and Mark in 1980 ([1]) and by Ricceri in 1987 ([3]). For example, Schirmer proved that if the values of a contractive multifunction $F: \mathbb{R} \to P(\mathbb{R})$ are closed, bounded and convex, then the fixed point set of F is compact and closed. Recently, Sintamarian proved some results on absolute retractivity of the common fixed points set of two multivalued operators under some conditions ([5] and [6]). In 2008, Lazar, O'Regan and Petrusel obtained fixed points of Ciric-type multifunctions on a set with two metrics ([2]). In this paper, we shall present some results on absolute retractivity of the common fixed points set of two multifunctions by using different conditions.

2. Main results

The following result improves [5; Theorem 2.1] which use arguments similar to those in [5].

Theorem 2.1. *Let (X,d) be a metric space and absolute retract for metric spaces, $F_1, F_2 \in \mathrm{SP}(X)$ and $f: X \to X$ a continuous function such that*

$$\alpha d(x,y) \leq d(f(x), f(y))$$

for some $\alpha > 0$ and all $x, y \in X$, and $f(F_1 x) \subseteq F_1 f(x)$ and $f(F_2 x) \subseteq F_2 f(x)$ for all $x \in X$. Suppose that there exist $a_1, \ldots, a_5 \in (0,\infty)$ such that $a_1 + a_2 + a_3 + 2\max\{a_4, a_5\} < 1$ and

$$H(F_1 x, F_2 y) \leq a_1 d(x,y) + a_2 D(x, F_1 x) + a_3 D(y, F_2 y) + a_4 D(x, F_2 y) + a_5 D(y, F_1 x)$$

for all $x, y \in X$. Then the set $B = \{x \in X : x \in F_1f(x) \cap F_2f(x)\}$ is an absolute retract for metric spaces.

PROOF. Let

$$q \in \left(1, \frac{1}{a_1 + a_2 + a_3 + 2\max\{a_4, a_5\}}\right)$$

and set

$$l := \max\left\{\frac{a_1 + a_2 + a_4}{1 - (a_3 + a_4)}, \frac{a_1 + a_3 + a_5}{1 - (a_2 + a_5)}\right\} < 1.$$

Then we have $ql < 1$. Let $Y \in \mathcal{M}$, $A \in P_{\mathrm{cl}}(Y)$ and $\psi: A \to B$ a continuous function. Since X is an absolute retract for metric spaces, there exists a continuous function $\varphi_0: Y \to X$ such that $\varphi_0|_A = \psi$. Define the functional $g_0: Y \to (0, \infty)$ by

$$g_0(y) = \sup\{d(f(\varphi_0(y)), z) : z \in F_1f(\varphi_0(y))\} + 1$$

for all $y \in Y$. Note that, g_0 is continuous and

$$F_1f(\varphi_0(y)) \cap N_{g_0(y)}(f(\varphi_0(y))) = F_1f(\varphi_0(y))$$

for all $y \in Y$. Also, we observe that the function $\psi: A \to B$ is a continuous selection of the multifunction $A \ni y \vdash F_1f(\varphi_0(y))$. Since $F_1 \in \mathrm{SP}(X)$, there exists a continuous function $\varphi_1: Y \to X$ such that $\varphi_1|_A = \psi$ and $\varphi_1(y) \in F_1f(\varphi_0(y))$ for all $y \in Y$. Thus, $f(\varphi_1(y)) \in f(F_1f(\varphi_0(y))) \subseteq F_1f(\varphi_0(y))$ and

$$\begin{aligned}
D(f(\varphi_1(y)), &F_2f(\varphi_1(y))) \leq H(F_1f(\varphi_0(y)), F_2f(\varphi_1(y))) \\
&\leq a_1d(f(\varphi_0(y)), f(\varphi_1(y))) + a_2D(f(\varphi_0(y)), F_1f(\varphi_0(y))) \\
&\quad + a_3D(f(\varphi_1(y)), F_2f(\varphi_1(y))) + a_4D(f(\varphi_0(y)), F_2f(\varphi_1(y))) \\
&\quad + a_5D(f(\varphi_1(y)), F_1f(\varphi_0(y))) \\
&\leq a_1d(f(\varphi_0(y)), f(\varphi_1(y))) + a_2d(f(\varphi_0(y)), f(\varphi_1(y))) \\
&\quad + a_3D(f(\varphi_1(y)), F_2f(\varphi_1(y))) + a_4d(f(\varphi_0(y)), f(\varphi_1(y))) \\
&\quad + a_4D(f(\varphi_1(y)), F_2f(\varphi_1(y))).
\end{aligned}$$

This implies that

$$\begin{aligned}
D(f(\varphi_1(y)), &F_2f(\varphi_1(y))) \\
&\leq \frac{(a_1 + a_2 + a_4)}{(1 - a_3 - a_4)} d(f(\varphi_0(y)), f(\varphi_1(y))) \leq ld(f(\varphi_0(y)), f(\varphi_1(y))) \\
&< ld(f(\varphi_0(y)), f(\varphi_1(y))) + l < ld(f(\varphi_0(y)), f(\varphi_1(y))) + q^{-1}.
\end{aligned}$$

Hence, $G_2(y) := F_2f(\varphi_1(y)) \cap N_{ld(f(\varphi_0(y)), f(\varphi_1(y))) + q^{-1}}(f(\varphi_1(y))) \neq \emptyset$. Since $F_2 \in \mathrm{SP}(X)$, there exists a continuous function $\varphi_2: Y \to X$ such that $\varphi_2|_A = \psi$ and $\varphi_2(y) \in \overline{G_2(y)}$ for all $y \in Y$. Thus, $\varphi_2|_A = \psi$, $\varphi_2(y) \in F_2f(\varphi_1(y))$ for all

$y \in Y$. Hence, $f(\varphi_2(y)) \in f(F_2 f(\varphi_1(y))) \subseteq F_2 f(\varphi_1(y))$ for all $y \in Y$. It is easy to see that $F_2 f(\varphi_1(y)) \subseteq N_{ld(f(\varphi_0(y)), f(\varphi_1(y)))+q^{-1}}(f(\varphi_1(y)))$. Thus,

$$d(f(\varphi_2(y)), f(\varphi_1(y))) \leq ld(f(\varphi_0(y)), f(\varphi_1(y))) + q^{-1}$$

and so

$$\begin{aligned} D(f(\varphi_2(y)),&F_1 f(\varphi_2(y))) \leq H(F_1 f(\varphi_2(y)), F_2 f(\varphi_1(y))) \\ &\leq a_1 d(f(\varphi_1(y)), f(\varphi_2(y))) + a_2 D(f(\varphi_2(y)), F_1 f(\varphi_2(y))) \\ &\quad + a_3 D(f(\varphi_1(y)), F_2 f(\varphi_1(y))) + a_4 D(f(\varphi_2(y)), F_2 f(\varphi_1(y))) \\ &\quad + a_5 D(f(\varphi_1(y)), F_1 f(\varphi_2(y))) \\ &\leq a_1 d(f(\varphi_1(y)), f(\varphi_2(y))) + a_2 D(f(\varphi_2(y)), F_1 f(\varphi_2(y))) \\ &\quad + a_3 d(f(\varphi_1(y)), f(\varphi_2(y))) + a_3 D(f(\varphi_2(y)), F_2 f(\varphi_1(y))) \\ &\quad + a_4 D(f(\varphi_2(y)), F_2 f(\varphi_1(y))) + a_5 d(f(\varphi_1(y)), f(\varphi_2(y))) \\ &\quad + a_5 D(f(\varphi_2(y)), F_1 f(\varphi_2(y))). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} D(f(\varphi_2(y)),&F_1 f(\varphi_2(y))) \leq \frac{(a_1 + a_3 + a_5)}{(1 - a_2 - a_5)} d(f(\varphi_1(y)), f(\varphi_2(y))) \\ &\leq ld(f(\varphi_1(y)), f(\varphi_2(y))) < ld(f(\varphi_1(y)), f(\varphi_2(y))) + l \\ &< ld(f(\varphi_1(y)), f(\varphi_2(y))) + q^{-1} < l^2 d(f(\varphi_0(y)), f(\varphi_1(y))) + q^{-2}. \end{aligned}$$

Thus, $G_3(y) := F_1 f(\varphi_2(y)) \cap N_{l^2 d(f(\varphi_0(y)), f(\varphi_1(y)))+q^{-2}}(f(\varphi_2(y))) \neq \emptyset$. Since $F_1 \in \mathrm{SP}(X)$, there exists a continuous function $\varphi_3\colon Y \to X$ such that $\varphi_3|_A = \psi$ and $\varphi_3(y) \in F_1 f(\varphi_2(y))$ for all $y \in Y$. Therefore, $\varphi_3|_A = \psi$, $f(\varphi_3(y)) \in F_1 f(\varphi_2(y))$ and $d(f(\varphi_2(y)), f(\varphi_3(y))) \leq l^2 d(f(\varphi_0(y)), f(\varphi_1(y))) + q^{-2}$ for all $y \in Y$. By continuing this process, we obtain a sequence $\{\varphi_n\}_{n \geq 0}$, where $\varphi_n\colon Y \to X$ is a continuous function for all $n \geq 0$, such that $\varphi_n|_A = \psi$, $\varphi_{2n-1}(y), f(\varphi_{2n-1}(y)) \in F_1 f(\varphi_{2n-2}(y))$ and $\varphi_{2n}(y), f(\varphi_{2n}(y)) \in F_2 f(\varphi_{2n-1}(y))$, and

$$d(f(\varphi_{n-1}(y)), f(\varphi_n(y))) \leq l^{n-1} d(f(\varphi_0(y)), f(\varphi_1(y))) + q^{-(n-1)}$$

for all $n \geq 1$ and $y \in Y$. Now, for each $\lambda > 0$, we put

$$Y_\lambda := \{y \in Y : d(f(\varphi_0(y)), f(\varphi_1(y))) < \lambda\}.$$

Since $f(\varphi_1(y)) \in F_1 f(\varphi_0(y))$ and

$$F_1 f(\varphi_0(y)) \cap N_{g_0(y)}(f(\varphi_0(y))) = F_1 f(\varphi_0(y)),$$

$f(\varphi_1(y)) \in N_{g_0(y)}(f(\varphi_0(y)))$. Hence, $d(f(\varphi_0(y)), f(\varphi_1(y))) < \lambda_y := g_0(y)$. Thus, $y \in Y_{\lambda_y}$. Since Y_λ is open for each $\lambda > 0$, the family of sets $\{Y_\lambda | \lambda > 0\}$ is an open covering of Y and we have

$$\begin{aligned}\alpha d(\varphi_{n-1}(y),\varphi_n(y)) &\leq d(f(\varphi_{n-1}(y)),f(\varphi_n(y)))\\ &\leq l^{n-1}d(f(\varphi_0(y)),f(\varphi_1(y)))+q^{-(n-1)}\end{aligned}$$

for all $n\geq 1$ and $y\in Y$. Since $l<1$, $q>1$, $\alpha>0$ and X is complete, the sequence $\{\varphi_n\}_{n\geq 0}$ converges uniformly on Y_λ for all $\lambda>0$. Let $\varphi\colon Y\to X$ be the pointwise limit of $\{\varphi_n\}_{n\geq 0}$ and note that φ is continuous and $\varphi|_A=\psi$ because $\varphi_n|_A=\psi$ for all $n\geq 0$. Since f is continuous, $\varphi_{2n-1}(y)\in F_1f(\varphi_{2n-2}(y))$ and $\varphi_{2n}(y)\in F_2f(\varphi_{2n-1}(y))$ for all $n\geq 1$ and $y\in Y$, we get $\varphi(y)\in F_1f(\varphi(y))\cap F_2f(\varphi(y))$ for all $y\in Y$. Therefore, $\varphi\colon Y\to B$ is a continuous extension of ψ, that is, $B=\{x\in X: x\in F_1f(x)\cap F_2f(x)\}$ is an absolute retract for metric spaces. □

Theorem 2.2. *Let (X,d) be a metric space and absolute retract for metric spaces, $F_1,F_2\in \mathrm{SP}(X)$ and $f\colon X\to X$ a continuous function such that*

$$\alpha d(x,y)\leq d(f(x),f(y))$$

for some $\alpha>0$ and all $x,y\in X$, and $f(F_1x)\subseteq F_1f(x)$ and $f(F_2fx)\subseteq F_2f(x)$ for all $x\in X$. Suppose that there exist $a_1,\dots,a_5\in(0,\infty)$ such that $a_1+a_2+a_3+2\max\{a_4,a_5\}<1$ and

$$H(F_1x,F_2y)\leq a_1d(x,y)+a_2D(x,F_1x)+a_3D(y,F_2y)+a_4D(x,F_2y)+a_5D(y,F_1x)$$

for all $x,y\in X$. Then the set $B_m=\{x\in X: x\in F_1f^m(x)\cap F_2f^m(x)\}$ is an absolute retract for metric spaces for all $m\geq 1$.

Proof. We note that $\alpha^m d(x,y)\leq d(f^m(x),f^m(y))$, $f^m(F_1x)\subseteq F_1f^m(x)$ and $f^m(F_2x)\subseteq F_2f^m(x)$ for all $x,y\in X$ and $m\geq 1$. Now, as before, we can obtain the result. □

If $X=\mathbb{R}$ and $f(x)=2x$ for $x>0$ and $f(x)=3x$ for $x\leq 0$, then $\alpha d(x,y)\leq d(f(x),f(y))$ for some $\alpha=2$. Note that, f is not linear. Also, define $F_1x=[0,x]$ if $x>0$, $F_1x=[x,-x]$ if $x\leq 0$, $F_2x=[x,2x]$ if $x>0$ and $F_2x=[x,0]$ if $x\leq 0$. Then, $fF_1x=F_1f(x)$ and $fF_2x=F_2f(x)$ for all $x\in X$.

Theorem 2.3. *Let (X,d) be a metric space and absolute retract for metric spaces, $F\in\mathrm{SP}(X)$ and $f\colon X\to X$ a continuous function such that $f(Fx)\subseteq Ff(x)$ for all $x\in X$. Suppose that there exist $a_1,\dots,a_5\in(0,\infty)$ such that $a_1+a_2+a_3+2a_4<1$ and*

$$H(Fx,Fy)\leq a_1d(x,y)+a_2D(x,Fx)+a_3D(y,Fy)+a_4D(x,Fy)+a_5D(y,Fx)$$

for all $x,y\in X$. Then the set $B=\{f(x): f(x)\in Ff(x)\}$ is an absolute retract for metric spaces.

Proof. Let $q \in (1, 1/(a_1 + a_2 + a_3 + 2a_4))$ and set $l := (a_1 + a_2 + a_4)/(1 - a_3 - a_4)$. Then we have $ql < 1$. Let $Y \in \mathcal{M}$, $A \in P_{\rm cl}(Y)$ and $\psi\colon A \to B$ a continuous function. Since X is an absolute retract for metric spaces, there exists a continuous function $\varphi_0\colon Y \to X$ such that $\varphi_0|_A = \psi$.

Define the functional $g_0\colon Y \to (0, \infty)$ by

$$g_0(y) = \sup\{d(\varphi_0(y), z) : z \in F\varphi_0(y)\} + 1$$

for all $y \in Y$. Note that, g_0 is continuous and

$$F\varphi_0(y) \cap N_{g_0(y)}(\varphi_0(y)) = F\varphi_0(y)$$

for all $y \in Y$. Also, we observe that the function $\psi\colon A \to B$ is a continuous selection of the multifunction $A \ni y \vdash F\varphi_0(y)$. Since $F \in \mathrm{SP}(X)$, there exists a continuous function $\varphi_1\colon Y \to X$ such that $\varphi_1|_A = \psi$ and $\varphi_1(y) \in F\varphi_0(y)$ for all $y \in Y$. Thus, $f(\varphi_1(y)) \in f(F\varphi_0(y)) \subseteq Ff(\varphi_0(y))$ and

$$\begin{aligned}
D(\varphi_1(y),F\varphi_1(y)) &\leq H(F\varphi_0(y), F\varphi_1(y)) \\
&\leq a_1 d(\varphi_0(y), \varphi_1(y)) + a_2 D(\varphi_0(y), F\varphi_0(y)) \\
&\quad + a_3 D(\varphi_1(y), F\varphi_1(y)) + a_4 D(\varphi_0(y), F\varphi_1(y)) + a_5 D(\varphi_1(y), F\varphi_0(y)) \\
&\leq a_1 d(\varphi_0(y), \varphi_1(y)) + a_2 d(\varphi_0(y), \varphi_1(y)) \\
&\quad + a_3 D(\varphi_1(y), F\varphi_1(y)) + a_4 d(\varphi_0(y), \varphi_1(y)) + a_4 D(\varphi_1(y), F\varphi_1(y)).
\end{aligned}$$

Now, we obtain

$$\begin{aligned}
D(\varphi_1(y), F\varphi_1(y)) &\leq \frac{(a_1 + a_2 + a_4)}{(1 - a_3 - a_4)}\, d(\varphi_0(y), \varphi_1(y)) \leq l d(\varphi_0(y), \varphi_1(y)) \\
&< l d(\varphi_0(y), \varphi_1(y)) + l < l d(\varphi_0(y), \varphi_1(y)) + q^{-1}.
\end{aligned}$$

Hence, $G_2(y) := F\varphi_1(y) \cap N_{ld(\varphi_0(y),\varphi_1(y))+q^{-1}}(\varphi_1(y)) \neq \emptyset$. Since $F \in \mathrm{SP}(X)$, there exists a continuous function $\varphi_2\colon Y \to X$ such that $\varphi_2|_A = \psi$ and $\varphi_2(y) \in \overline{G_2(y)}$ for all $y \in Y$. Thus, $\varphi_2|_A = \psi$, $\varphi_2(y) \in F\varphi_1(y)$ for all $y \in Y$. Hence, $f(\varphi_2(y)) \in f(F\varphi_1(y)) \subseteq Ff(\varphi_1(y))$ for all $y \in Y$. It is easy to see that $F\varphi_1(y) \subseteq N_{ld(\varphi_0(y),\varphi_1(y))+q^{-1}}(\varphi_1(y))$. Thus, $d(\varphi_2(y), \varphi_1(y)) \leq l d(\varphi_0(y), \varphi_1(y)) + q^{-1}$ and so by using an argument similar to that in the proof Theorem 2.1, we obtain

$$D(\varphi_2(y), F\varphi_2(y)) \leq l^2 d(\varphi_0(y), \varphi_1(y)) + q^{-2}.$$

Again, by continuing this process, we obtain a sequence $\{\varphi_n\}_{n\geq 0}$, where $\varphi_n\colon Y \to X$ is a continuous function for all $n \geq 0$, such that $\varphi_n|_A = \psi$, $\varphi_n(y) \in F\varphi_{n-1}(y))$, $f(\varphi_n(y)) \in Ff(\varphi_{n-1}(y))$ and

$$d(\varphi_{n-1}(y), \varphi_n(y)) \leq l^{n-1} d(\varphi_0(y), \varphi_1(y)) + q^{-(n-1)}$$

for all $n \geq 1$ and $y \in Y$. Now, for each $\lambda > 0$ we put

$$Y_\lambda := \{y \in Y : d(f(\varphi_0(y)), f(\varphi_1(y))) < \lambda\}.$$

The family of sets $\{Y_\lambda \mid \lambda > 0\}$ is an open covering of Y. Since $l < 1$, $q > 1$ and X is complete, the sequence $\{\varphi_n\}_{n\geq 0}$ converges uniformly on Y_λ for all $\lambda > 0$. Let $\varphi: Y \to X$ be the pointwise limit of $\{\varphi_n\}_{n\geq 0}$. Note that φ is continuous and $\varphi|_A = \psi$ because $\varphi_n|_A = \psi$ for all $n \geq 0$. Since f is continuous and $f(\varphi_n(y)) \in Ff(\varphi_{n-1}(y))$ for all $n \geq 1$ and $y \in Y$, we get $f(\varphi(y)) \in Ff(\varphi(y))$ for all $y \in Y$. Therefore, $\varphi: Y \to B$ is a continuous extension of ψ, that is, $B = \{f(x) : f(x) \in Ff(x)\}$ is an absolute retract for metric spaces. $\square$

By using similar proofs we can conclude the following results.

Corollary 2.4. *Let (X, d) be a metric space and absolute retract for metric spaces, $F \in \mathrm{SP}(X)$ and $f: X \to X$ a continuous function such that $f(Fx) \subseteq Ff(x)$ for all $x \in X$. Suppose that there exist $a_1, \dots, a_5 \in (0, \infty)$ such that $a_1 + a_2 + a_3 + 2a_4 < 1$ and*

$$H(Fx, Fy) \leq a_1 d(x, y) + a_2 D(x, Fx) + a_3 D(y, Fy) + a_4 D(x, Fy) + a_5 D(y, Fx)$$

for all $x, y \in X$. Then the set $B_m = \{f^m(x) : f^m(x) \in Ff^m(x)\}$ is an absolute retract for metric spaces for all $m \geq 1$.

Corollary 2.5. *Let (X, d) be a metric space and absolute retract for metric spaces, $F_1, F_2 \in \mathrm{SP}(X)$ and $f: X \to X$ a continuous function such that $f(F_1x) \subseteq F_1f(x)$ and $f(F_2x) \subseteq F_2f(x)$ for all $x \in X$. Suppose that there exist $a_1, \dots, a_5 \in (0, \infty)$ such that $a_1 + a_2 + a_3 + 2\max\{a_4, a_5\} < 1$ and*

$$H(F_1x, F_2y) \leq a_1 d(x, y) + a_2 D(x, F_1x) + a_3 D(y, F_2y) + a_4 D(x, F_2y) + a_5 D(y, F_1x)$$

for all $x, y \in X$. Then the set $B_m = \{f^m(x) \in X : f^m(x) \in F_1f^m(x) \cap F_2f^m(x)\}$ is an absolute retract for metric spaces for all $m \geq 1$.

Corollary 2.6. *Let (X, d) be a metric space and absolute retract for metric spaces, $F \in \mathrm{SP}(X)$ and $f: X \to X$ a continuous function such that $\alpha d(x, y) \leq d(f(x), f(y))$ for some $\alpha > 0$ and all $x, y \in X$, and $f(Fx) \subseteq Ff(x)$ for all $x \in X$. Suppose that there exist $a_1, \dots, a_5 \in (0, \infty)$ such that $a_1 + a_2 + a_3 + 2a_4 < 1$ and*

$$H(Fx, Fy) \leq a_1 d(x, y) + a_2 D(x, Fx) + a_3 D(y, Fy) + a_4 D(x, Fy) + a_5 D(y, Fx)$$

for all $x, y \in X$. Then the set $B_m = \{x \in X : x \in Ff^m(x)\}$ is an absolute retract for metric spaces for all $m \geq 1$.

Remark 2.7. Let (X, d) be metric space and F_1 and F_2 two multifunctions on X. We say that F_1 and F_2 are Sintamarian-type multifunctions if there exist $a_1, \dots, a_5 \in (0, \infty)$ such that $a_1 + a_2 + a_3 + 2\max\{a_4, a_5\} < 1$ and

$$H(F_1x, F_2y) \leq a_1 d(x, y) + a_2 D(x, F_1x) + a_3 D(y, F_2y) + a_4 D(x, F_2y) + a_5 D(y, F_1x)$$

for all $x, y \in X$. Also, we say that F_1 and F_2 are Ciric-type multifunctions if there exists $\alpha \in [0, 1)$ such that

$$H(F_1x, F_2y) \leq \alpha \max\left\{d(x, y), D(x, F_1x), D(y, F_2y), \frac{1}{2}[D(x, F_2y) + D(y, F_1x)]\right\}$$

for all $x, y \in X$. Note that, F_1 and F_2 are Sintamarian-type multifunctions if and only if F_1 and F_2 are Ciric-type multifunctions. Thus, the results of Sintamarian (and our results) hold for Ciric-type multifunctions.

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