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EXISTENCE OF SOLUTIONS ON COMPACT AND NON-COMPACT INTERVALS FOR SEMILINEAR IMPULSIVE DIFFERENTIAL INCLUSIONS WITH DELAY

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Abstract. In this paper we deal with impulsive Cauchy problems in Banach spaces governed by a delay semilinear differential inclusion $y' \in A(t)y + F(t, y_t)$. The family $\{A(t)\}_{t\in[0,b]}$ of linear operators is supposed to generate an evolution operator and F is a upper Carathèodory type multifunction. We first provide the existence of mild solutions on a compact interval and the compactness of the solution set. Then we apply this result to obtain the existence of mild solutions for the impulsive Cauchy problem on non-compact intervals.

1. Introduction

Impulsive differential equations and inclusions find wide applicability in several fields of applied science as Biology, Economics, Physics, since they are an appropriate model for describing phenomena where systems instantaneously change their state.

For a bibliography on the theory of impulsive differential equations one can see, for instance, the monographs [1], [2], [19]. About theory and applications of impulsive differential equations or inclusions, see e.g. [6], [7], [14], [16], [20], [21].

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In particular, as recent works on impulsive differential equations or inclusions in presence of delay, we refer to [3], [4], [8], [10], [11], [15], [22].

In the present paper we consider the impulsive Cauchy problem governed by a delay semilinear differential inclusion both for compact and non-compact intervals.

For a fixed $\tau > 0$ and a given piece-wise continuous function $x\colon [-\tau, 0] \to E$, where E is a Banach space, in the first case the problem we deal with is

$$(\mathrm{P}) \qquad \begin{cases} y'(t) \in A(t)y(t) + F(t, y_t) & \text{a.e. } t \in [0, b],\ t \neq t_k,\ k = 1, \ldots, N, \\ y(t) = x(t) & \text{for } t \in [-\tau, 0], \\ y(t_k^+) = y(t_k) + I_k(y_{t_k}) & \text{for } k = 1, \ldots, N \end{cases}$$

where $\{A(t)\}_{t\in[0,b]}$ is a family of linear operators (not necessarily bounded) in E generating an evolution operator; F is a upper Carathèodory type multifunction; $y_t(\theta) = y(t + \theta)$, $\theta \in [-\tau, 0]$; $0 = t_0 < t_1 < \ldots < t_N < t_{N+1} = b$; I_k are impulse functions, $k = 1, \ldots, N$, and $y(t^+) = \lim_{s\to t^+} y(s)$.

In Section 3 we state and prove the existence of mild solutions for problem (P) and the compactness of the solution set (see Theorem 3.7).

We note that our Theorem 3.7 contains the analogous result due to Benedetti ([4, Theorem 3.4]).

Then, in Section 4 we treat the case on non-compact domains. First we yield the existence of mild solutions for problem

$$(\mathrm{P})^\infty \qquad \begin{cases} y'(t) \in A(t)y(t) + F(t, y_t) & \text{a.e. } t \in [0, \infty[,\ t \neq t_k,\ k \in \mathbb{N}, \\ y(t) = x(t) & \text{for } t \in [-\tau, 0], \\ y(t_k^+) = y(t_k) + I_k(y_{t_k}) & \text{for } k \in \mathbb{N}^+, \end{cases}$$

where this time $(t_k)_{k\in\mathbb{N}}$ is an increasing sequence of given points in $[0, \infty[$, without accumulation points.

This result is achieved by applying Theorem 3.7 jointly with a diagonal process. Then, we note that the method of proof carried over in the case of unbounded domains provides also the existence of mild solutions for the impulsive Cauchy problem defined on a non-closed interval $[0, b[$, $0 < b < \infty$, just by suitable adaptation of the assumptions.

The results of Section 4 are new even in the non-impulsive case.

2. Preliminaries

Let X, Y, be two topological vector spaces. We denote by $\mathcal{P}(Y)$ the family of all non-empty subsets of Y and put

$$\mathcal{K}(Y) = \{C \in \mathcal{P}(Y),\ \text{compact}\}, \quad \mathcal{K}v(Y) = \{D \in \mathcal{P}(Y),\ \text{compact and convex}\}.$$

A multivalued map $F: X \to \mathcal{P}(Y)$ is said to be:

- *upper semicontinuous* (for shortness u.s.c.) if $F^{-1}(V) = \{x \in X : F(x) \subset V\}$ is an open subset of X for every open $V \subseteq Y$;
- *closed* if its graph $G_F = \{(x, y) \in X \times Y : y \in F(x)\}$ is a closed subset of $X \times Y$.

For u.s.c. multimaps the following result holds (see, e.g. [9]).

PROPOSITION 2.1. *Let $F: X \to \mathcal{K}(Y)$ be an u.s.c. multimap. If $C \subset X$ is a compact set then its image $F(C)$ is a compact subset of Y.*

Let E be a real Banach space. If $(N, \geq)$ is a partially ordered set, we recall that a map $\beta: \mathcal{P}(E) \to N$ is said to be a *measure of non compactness* (MNC) in E if $\beta(\overline{\mathrm{co}}\,\Omega) = \beta(\Omega)$ for every $\Omega \in \mathcal{P}(E)$ (see, e.g. [12] for details).

A measure of non compactness β is called:

- *monotone* if $\Omega_0, \Omega_1 \in \mathcal{P}(E)$, $\Omega_0 \subseteq \Omega_1$ imply $\beta(\Omega_1) \geq \beta(\Omega_0)$;
- *nonsingular* if $\beta(\{c\} \cup \Omega) = \beta(\Omega)$ for every $c \in E$, $\Omega \in \mathcal{P}(E)$;
- *real* if $N = [0, \infty]$ with the natural ordering and $\beta(\Omega) < \infty$ for every bounded Ω;
- *regular* if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

A well known example of measure of non compactness satisfying all of the above properties is the Hausdorff MNC

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

If X is a subset of E and Λ a space of parameters, a multimap $F: X \to \mathcal{K}(E)$, or a family of multimaps $G: \Lambda \times X \to \mathcal{K}(E)$, is called *condensing* relative to a MNC β, or β-condensing, if for every $\Omega \subseteq X$ that is not relatively compact we have, respectively $\beta(F(\Omega)) \ngeq \beta(\Omega)$ or $\beta(G(\Lambda \times \Omega)) \ngeq \beta(\Omega)$.

The following property of the fixed points set of F will be useful in the sequel.

PROPOSITION 2.2 ([12, Proposition 3.5.1]). *Let M be a closed subset of E, $F: M \to \mathcal{K}(E)$ a closed multimap β-condensing on every bounded subset of M, β a monotone* MNC *defined on E. If* $\mathrm{Fix}\, F = \{x \in M : x \in F(x)\}$ *is bounded, then it is compact.*

Let $[a, b]$ be an interval of the real line. By the symbol $L^1([a, b]; E)$ we denote the space of all Bochner summable functions and, for simplicity of notations, we write $L^1_+([a, b])$ instead of $L^1([a, b]; \mathbb{R}^+)$.

We denote by $\mathcal{C}([a, b]; E)$ the space of all piece-wise continuous functions $c: [a, b] \to E$ with a finite number of discontinuity points $\{t_*\}$ such that $t_* \neq b$ and all values

$$c(t_*^+) = \lim_{h \to 0^+} c(t_* + h)$$

are finite. Of course, the space $\mathcal{C}([a,b];E)$ is a normed space with the norm:

$$\|c\|_{\mathcal{C}} = \sup_{a\le t\le b} \|c(t)\|.$$

A multifunction $\mathcal{G}\colon [a,b] \to \mathcal{K}(E)$ is said to be

- *integrable* if it has a summable selection $g \in L^1([a,b];E)$;
- *integrably bounded* if there exists a summable function $\omega(\,\cdot\,) \in L^1_+([a,b])$ such that

$$\|\mathcal{G}(t)\| := \sup\{\|g\| : g \in \mathcal{G}(t)\} \le \omega(t), \quad \text{a.e. } t \in [a,b].$$

Finally, a countable set $\{f_n\}_{n=1}^{\infty} \subset L^1([a,b];E)$ is said to be *semicompact* if:

- it is integrably bounded: $\|f_n(t)\| \le \omega(t)$ for a.e. $t \in [a,b]$ and every $n \ge 1$, where $\omega(\,\cdot\,) \in L^1_+([a,b])$;
- the set $\{f_n(t)\}_{n=1}^{\infty}$ is relatively compact for a.e. $t \in [a,b]$.

3. Existence of solutions on compact intervals

Let $[0,b]$ be a fixed interval of the real line. Put $\Delta = \{(t,s) \in [0,b]\times[0,b] : 0 \le s \le t \le b\}$, we recall (see, e.g. [18]) that a two parameter family $\{T(t,s)\}_{(t,s)\in\Delta}$, $T(t,s)\colon E \to E$ bounded linear operator, $(t,s) \in \Delta$, is called an *evolution system* if the following conditions are satisfied:

- $T(s,s) = I$, $0 \le s \le b$; $T(t,r)T(r,s) = T(t,s)$, $0 \le s \le r \le t \le b$;
- $(t,s) \mapsto T(t,s)$ is strongly continuous on Δ (see, e.g. [13]).

For every evolution system, we can consider the correspondent *evolution operator* $T\colon \Delta \to \mathcal{L}(E)$, where $\mathcal{L}(E)$ is the space of all bounded linear operators in E.

We observe that, since the evolution operator T is strongly continuous on the compact set Δ, there exists a constant $D = D_\Delta > 0$ such that

$$\|T(t,s)\|_{\mathcal{L}(E)} \le D, \quad (t,s) \in \Delta. \tag{3.1}$$

In this section we consider the impulsive Cauchy problem (P).

On the linear part of the differential inclusion we assume the following hypothesis:

(A) $\{A(t)\}_{t\in[0,b]}$ is a family of linear not necessarily bounded operators ($A(t)\colon D(A) \subset E \to E$, $t \in [0,b]$, $D(A)$ a dense subset of E not depending on t) generating an evolution operator $T\colon \Delta \to \mathcal{L}(E)$.

On the multimap $F\colon [0,b] \times \mathcal{C}([-\tau,0];E) \to \mathcal{K}v(E)$ we consider the following upper Carathèodory type hypotheses:

(F1) the multimap $F(\,\cdot\,,c)\colon [0,b] \to \mathcal{K}v(E)$ has a strongly measurable selection for every $c \in \mathcal{C}([-\tau,0];E)$, i.e. there exists a strongly measurable function $f\colon [0,b] \to E$ such that $f(t) \in F(t,c)$ for almost every $t \in [0,b]$;

(F2) the multimap $F(t,\,\cdot\,)\colon \mathcal{C}([-\tau,0];E)\to \mathcal{K}v(E)$ is u.s.c. for almost every $t\in[0,b]$.

Moreover, we require on F also the following assumptions

(F3) there exists a function $\alpha\in L^1_+([0,b])$ such that for every $c\in\mathcal{C}([-\tau,0];E)$ is

$$\|F(t,c)\|\le \alpha(t)(1+\|c\|_{\mathcal{C}}),\quad \text{a.e. } t\in[0,b];$$

(F4) there exists a function $\mu\in L^1_+([0,b])$ such that, for every bounded $D\subset\mathcal{C}([-\tau,0];E)$,

$$\chi(F(t,D))\le \mu(t)\sup_{-\tau\le\theta\le 0}\chi(D(\theta)),\quad \text{a.e. } t\in[0,b],$$

where χ is the Hausdorff MNC in E.

Definition 3.1. A function $y\in\mathcal{C}([-\tau,b];E)$ is a *mild solution* for the impulsive Cauchy problem (P) if

(a) $y(t)=T(t,0)x(0)+\sum_{0<t_k<t}T(t,t_k)I_k(y_{t_k})+\int_0^t T(t,s)f(s)\,ds,\ t\in[0,b]$, where $f\in L^1([0,b];E)$, $f(s)\in F(s,y_s)$ for almost every $s\in[0,b]$

(b) $y(t)=x(t)$, $t\in[-\tau,0]$,

(c) $y(t_k^+)=y(t_k)+I_k(y_{t_k})$, $k=1,\dots,N$.

Let $k\in\{1,\dots,N\}$ be fixed. First of all, in order to obtain the main result of this section, we consider the generalized Cauchy operator $G^k\colon L^1([t_{k-1},t_k];E)\to C([t_{k-1},t_k];E)$ defined by

$$G^kf(t)=\int_{t_{k-1}}^t T(t,s)f(s)\,ds,\quad t\in[t_{k-1},t_k]$$

(see [5, Definition 1]). We recall that for G^k the following result holds.

Proposition 3.2 ([5, Theorem 2]). *The generalized Cauchy operator G^k satisfies the properties:*

(G1) *there exists $\zeta_k\ge 0$ such that*

$$\|G^kf(t)-G^kg(t)\|\le\zeta_k\int_{t_{k-1}}^t\|f(s)-g(s)\|\,ds,\quad t\in[t_{k-1},t_k]$$

for every $f,g\in L^1([t_{k-1},t_k];E)$;

(G2) *for any compact $K\subset E$ and sequence $(f_n)_{n=1}^\infty$, $f_n\in L^1([t_{k-1},t_k];E)$, such that $\{f_n(t)\}_{n=1}^\infty\subset K$ for almost every $t\in[t_{k-1},t_k]$, the weak convergence $f_n\rightharpoonup f_0$ implies the convergence $G^kf_n\to G^kf_0$.*

Remark 3.3. Let us note that we may assume $\zeta_k=D$, where D is from equation (3.1).

Furthermore, we also need the following properties of operators satisfying conditions (G1) and (G2).

Proposition 3.4 (cf. [12, Theorem 5.1.1]). *Let operator* $S: L^1([t_{k-1}, t_k]; E) \to C([t_{k-1}, t_k]; E)$ *satisfy condition* (G2) *and the Lipschitz condition* (*weaker than* (G1))

(G1') $\|Sf - Sg\|_C \leq D\|f - g\|_{L^1([t_{k-1},t_k];E)}$ (*where* $\|\cdot\|_C$ *is the usual sup-norm*).

Then for every semicompact set $\{f_n\}_{n=1}^{\infty} \subset L^1([t_{k-1}, t_k]; E)$ *the set* $\{Sf_n\}_{n=1}^{\infty}$ *is relatively compact in* $C([t_{k-1}, t_k]; E)$ *and, moreover, if* $f_n \rightharpoonup f_0$ *then* $Sf_n \to Sf_0$.

Proposition 3.5 (cf. [12, Theorem 4.2.2]). *Let the operator* S *satisfy conditions* (G1) *and* (G2) *and let the set* $\{f_n\}_{n=1}^{\infty}$ *be integrably bounded with the property* $\chi(\{f_n(t)\}_{n=1}^{\infty}) \leq \eta(t)$ *for almost every* $t \in [t_{k-1}, t_k]$ *where* $\eta(\cdot) \in L^1_+([t_{k-1}, t_k])$ *and* χ *is the Hausdorff* MNC. *Then*

$$\chi(\{Sf_n(t)\}_{n=1}^{\infty}) \leq 2D \int_{t_{k-1}}^{t} \eta(s)\, ds, \quad t \in [t_{k-1}, t_k]$$

where $D \geq 0$ *is from* (3.1) (*see also Remark* 3.3).

Moreover, we consider the multivalued superposition operator

$$P_F^{k,\xi}: C([t_{k-1}, t_k]; E) \to \mathcal{P}(L^1([t_{k-1}, t_k]; E))$$

defined as

$$P_F^{k,\xi}(z) = \{f \in L^1([t_{k-1}, t_k]; E) : f(s) \in F(s, z[\xi]_s) \text{ a.e. } s \in [t_{k-1}, t_k]\},$$

where

$$z[\xi](t) = \begin{cases} \xi(t) & \text{for } t \in [-\tau, t_{k-1}[, \\ z(t) & \text{for } t \in [t_{k-1}, t_k], \end{cases} \tag{3.2}$$

and $\xi \in \mathcal{C}([-\tau, t_{k-1}]; E)$ is a fixed function.

In the next theorem we will use the following

Lemma 3.6 ([12, Lemma 5.1.1]). *Assume that multimap the* F *satisfies hypotheses* (F1)–(F3). *If the sequences*

$$\{x_n\}_{n=1}^{\infty} \subset C([t_{k-1}, t_k]; E), \quad \{f_n\}_{n=1}^{\infty} \subset L^1([t_{k-1}, t_k]; E)$$

$f_n \in P_F^{k,\xi}(x_n)$, $n \geq 1$, *are such that* $x_n \to x_0$, $f_n \rightharpoonup f_0$, *then* $f_0 \in P_F^{k,\xi}(x_0)$.

Now we state and prove the main result of this section. In order to prove the first part of the following theorem, we need the existence of global mild solutions for an associated non-impulsive Cauchy problem. In a slight different setting, this existence has been obtained by Obukhovskiĭ in [17]; here, we provide the result by following another proof's outline.

Theorem 3.7. *Under assumptions* (A) *and* (F1)–(F4) *the problem* (P) *has at least one mild solution on* $[-\tau, b]$. *Moreover, if the impulse functions*

$$I_k: \mathcal{C}([-\tau,0];E) \to E, \quad k = 1,\dots,N$$

are continuous, then the solutions set is a compact subset of $\mathcal{C}([-\tau,b];E)$.

Proof. We divide the proof in several steps.

Step 1. Given the non impulsive Cauchy problem

$$(\mathrm{P})_1 \qquad \begin{cases} y'(t) \in A(t)y(t) + F(t,y_t) & \text{a.e. } t \in [0,t_1], \\ y(t) = x(t) & \text{for } t \in [-\tau,0], \end{cases}$$

we search for the existence of a mild solution, i.e. a function $y \in \mathcal{C}([-\tau,t_1];E)$ such that

(a) $y(t) = T(t,0)x(0) + \int_0^t T(t,s)f(s)\,ds$, $t \in [0,t_1]$, where $f \in L^1([0,t_1];E)$, $f(s) \in F(s,y_s)$ for almost every $s \in [0,t_1]$,

(b) $y(t) = x(t)$, $t \in [-\tau,0]$.

To this aim, we first consider the integral multioperator $\Gamma^1: C([0,t_1];E) \to \mathcal{P}(C([0,t_1];E))$ defined as:

$$\Gamma^1(z) = \bigg\{ y \in C([0,t_1];E) : y(t) = T(t,0)x(0) + \int_0^t T(t,s)f(s)\,ds \\ f \in L^1([0,t_1];E),\ f(s) \in F(s,z[x]_s) \text{ a.e. } s \bigg\}.$$

It is clear that if $z \in \operatorname{Fix}\Gamma^1$ then $z[x]$, where $z[x]$ is from (3.2), is a mild solution of $(\mathrm{P})_1$ on the interval $[-\tau,t_1]$.

The multioperator Γ^1 is a closed multioperator with compact, convex values.

Let $\{y_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty \subset C([0,t_1];E)$ with $z_n \to z_0$, $y_n \in \Gamma^1(z_n)$, $n \geq 1$ and $y_n \to y_0$. Take a sequence $\{f_n\}_{n=1}^\infty \subset L^1([0,t_1];E)$ such that $f_n \in P_F^{1,x}(z_n)$, $n \geq 1$. From assumption (F3) it follows that the sequence $\{f_n\}_{n=1}^\infty$ is integrably bounded. Moreover, hypothesis (F4) implies that:

$$\chi(\{f_n(t)\}_{n=1}^\infty) \leq \mu(t) \sup_{-\tau \leq t \leq 0} \chi(\{(z_n[x])_t\}_{n=1}^\infty) \leq \mu(t) \sup_{0 \leq \sigma \leq t} \chi(\{z_n(\sigma)\}_{n=1}^\infty) = 0,$$

for almost every $t \in [0,t_1]$, so the set $\{f_n(t)\}_{n=1}^\infty$ is relatively compact for almost every $t \in [0,t_1]$. Hence the set $\{f_n\}_{n=1}^\infty$ is semicompact and then it is also weakly compact in $L^1([0,t_1];E)$ (cf. [12, Proposition 4.2.1]). So we can assume, without loss of generality, that $f_n \rightharpoonup f_0$ in $L^1([0,t_1];E)$.

Applying Proposition 3.4 and the uniqueness of the limit algorithm, we conclude that $y_n = G^1 f_n \to G^1 f_0 = y_0$, where for $f \in L^1([0,t_1];E)$,

$$G^1(f) = T(t,0)x(0) + \int_0^t T(t,s)f(s)\,ds.$$

Moreover, by Lemma 3.6, we have $f_0 \in P_F^{1,x}(z_0)$, therefore $y_0 \in G^1 \circ P_F^{1,x}(z_0) = \Gamma^1(z_0)$, demonstrating that the multioperator Γ^1 is closed.

Let us prove now that Γ^1 has compact values. To this aim, we consider an arbitrary $z \in C([0,t_1];E)$. Then, every sequence $\{f_n\}_{n=1}^\infty \in P_F^{1,x}(z)$ is semicompact. Therefore, with the same arguments as above, the set $\{Gf_n\}_{n=1}^\infty$ is relatively compact. So the compactness of $\Gamma^1(z)$ follows from its closeness.

Further, the convexity of $\Gamma^1(z)$ is the consequence of the convexity of values of the multimap F and the linearity of G.

Consider now the function ν defined on bounded sets $\Omega \subset C([0,t_1];E)$ with values in $(\mathbb{R}^2, \geq)$ as:

$$\nu(\Omega) = \max_{D \in \mathcal{D}(\Omega)} (\gamma(D), \delta(D))$$

where $\mathcal{D}(\Omega)$ is the collection of all denumerable subsets of Ω and, for a given constant $L > 0$,

$$\gamma(D) = \sup_{0 \leq t \leq t_1} e^{-Lt} \chi(D(t)), \quad \delta(D) = \mathrm{mod}_C(D).$$

Example 2.1.4 in [12] shows that ν is a monotone, non singular, regular MNC.

Let us prove that the multioperator Γ^1 is condensing on bounded subsets of $C([0,t_1];E)$ with respect to the MNC ν.

Let $\Omega \subset C([0,t_1];E)$ be a bounded set such that

$$\nu(\Gamma^1(\Omega)) \geq \nu(\Omega). \tag{3.3}$$

Let the maximum of $\nu(\Gamma^1(\Omega))$ be achieved for the countable set $D' = \{g_n\}_{n=1}^\infty$, where $g_n = G^1(f_n)$, $f_n \in P_F^{1,x}(z_n)$, $n \geq 1$ and $\{z_n\}_{n=1}^\infty \subset \Omega$. From (3.3) we have:

$$\gamma(\{g_n\}_{n=1}^\infty) \geq \gamma(\{z_n\}_{n=1}^\infty). \tag{3.4}$$

From (F4) we have for $s \in [0,t_1]$:

$$\begin{aligned}\chi(\{f_n(s)\}_{n=1}^\infty) &\leq \mu(s) \sup_{0 \leq \sigma \leq s} \chi(\{z_n(\sigma)\}_{n=1}^\infty) = e^{Ls}\mu(s)e^{-Ls} \sup_{0 \leq \sigma \leq s} \chi(\{z_n(\sigma)\}_{n=1}^\infty) \\ &\leq e^{Ls}\mu(s) \sup_{0 \leq \sigma \leq t_1} e^{-L\sigma}\chi(\{z_n(\sigma)\}_{n=1}^\infty) = e^{Ls}\mu(s)\gamma(\{z_n\}_{n=1}^\infty).\end{aligned}$$

Moreover, from properties of MNC χ we have for $t \in [0,t_1]$:

$$\begin{aligned}\chi(\{g_n(t)\}_{n=1}^\infty) &= \chi\bigg(T(t,0)x(0) + \int_0^t T(t,s)\{f_n(s)\}_{n=1}^\infty\,ds\bigg) \\ &\leq \chi(T(t,0)x(0)) + \chi\bigg(\int_0^t T(t,s)\{f_n(s)\}_{n=1}^\infty\,ds\bigg) \\ &= \chi\bigg(\int_0^t T(t,s)\{f_n(s)\}_{n=1}^\infty\,ds\bigg).\end{aligned}$$

Applying Proposition 3.5 we have

$$\begin{aligned} e^{-Lt}\chi(\{g_n(t)\}_{n=1}^\infty) &\le e^{-Lt}2D\int_0^t e^{Ls}\mu(s)\,ds\,\gamma(\{z_n\}_{n=1}^\infty) \\ &\le 2D\sup_{t\in[0,t_1]} e^{-Lt}\int_0^t e^{Ls}\mu(s)\,ds\gamma(\{z_n\}_{n=1}^\infty), \end{aligned}$$

where $D>0$ is from (3.1). Then

$$\begin{aligned} \gamma(\{g_n\}_{n=1}^\infty) &\le \sup_{t\in[0,t_1]} e^{-Lt}2D\int_0^t e^{Ls}\mu(s)\,ds\gamma(\{z_n\}_{n=1}^\infty) \\ &= 2D\gamma(\{z_n\}_{n=1}^\infty)\sup_{t\in[0,t_1]}\int_0^t e^{-L(t-s)}\mu(s)\,ds. \end{aligned}$$

We can choose the constant $L>0$ so that

$$\sup_{t\in[0,t_1]}\bigg[2D\int_0^t e^{-L(t-s)}\mu(s)\,ds\bigg]<1,$$

so

$$\gamma(\{g_n\}_{n=1}^\infty)<\gamma(\{z_n\}_{n=1}^\infty). \tag{3.5}$$

Then from (3.4) and (3.5) we have $\gamma(\{g_n\}_{n=1}^\infty)=\gamma(\{z_n\}_{n=1}^\infty)=0$ and hence $\chi(\{z_n(t)\}_{n=1}^\infty)=0$, for all $t\in(0,t_1]$.

With the same arguments used before we obtain that $\{g_n\}_{n=1}^\infty$ is a relatively compact sequence, therefore $\delta(\{g_n\}_{n=1}^\infty)=0$, i.e. $\nu(\Gamma^1(\Omega))=(0,0)$ and from (3.3) $\nu(\Omega)=(0,0)$, then Ω is a relatively compact set.

Let the function $\widetilde{y}^0\in C([0,t_1];E)$ be defined by $\widetilde{y}^0(t)\equiv T(t,0)\,x(0)$, $t\in[0,t_1]$. Consider the following family of multimaps $\Phi\colon C([0,t_1];E)\times[0,1]\to \mathcal{K}v(C([0,t_1];E))$ given by:

$$\Phi(z,\lambda)=\bigg\{\widetilde{y}^0(t)+\lambda\int_0^t T(t,s)f(s)\,ds,\ t\in[0,t_1],\ f\in P_F^{1,x}(z)\bigg\}. \tag{3.6}$$

We will show that the set of fixed points of Φ, i.e. $\operatorname{Fix}\Phi=\{z\in\Phi(z,\lambda)$ for some $\lambda\in[0,1]\}$, is a priori bounded.

Let $z\in\operatorname{Fix}\Phi$. Then there exists $f\in P_F^{1,x}(z)$ such that, by using (3.1) and (F3), for any $t\in[0,t_1]$ we have:

$$\begin{aligned} \|z(t)\| &= \bigg\|T(t,0)x(0)+\lambda\int_0^t T(t,s)f(s)\,ds\bigg\| \le D\|x(0)\|+D\int_0^t\|f(s)\|\,ds \\ &\le D\|x(0)\|+D\int_0^t\alpha(s)(1+\|z[x]_s\|_{\mathcal{C}})\,ds \\ &\le D\|x(0)\|+D\|\alpha\|_{L^1_+[0,t_1]} \\ &\quad + D\int_0^t\alpha(s)\bigg(\sup_{-\tau\le\sigma\le0}\|x(\sigma)\|+\sup_{0<\sigma\le s}\|z(\sigma)\|\bigg)\,ds \end{aligned}$$

$$\leq D\|x(0)\| + D\|\alpha\|_{L^1_+[0,t_1]} + D\|\alpha\|_{L^1_+[0,t_1]} \sup_{-\tau\leq\sigma\leq 0}\|x(\sigma)\|$$
$$+ D\int_0^t \alpha(s) \sup_{0<\sigma\leq s}\|z(\sigma)\|\,ds$$
$$\leq D\|x(0)\| + D\|\alpha\|_{L^1_+[0,t_1]}(1+N) + D\int_0^t \alpha(s)\sup_{0<\sigma\leq s}\|z(\sigma)\|\,ds$$

where $N = \sup_{-\tau\leq t\leq 0}\|x(t)\|$. The right hand side is an increasing function in t, so we have the same estimate for all $0 < r \leq t$, i.e.

$$\sup_{0<r\leq t}\|z(r)\| \leq R + D\int_0^t \alpha(s)\sup_{0<\sigma\leq s}\|z(\sigma)\|\,ds,$$

where $R = D\|x(0)\| + D\|\alpha\|_{L^1_+[0,t_1]}(1+N)$. Since z is continuous on $[0,t_1]$, the function $\psi(t) = \sup\limits_{0<r\leq t}\|z(r)\|$ is also continuous, so

$$\psi(t) \leq R + D\int_0^t \alpha(s)\psi(s)\,ds$$

by Gronwall–Bellmann inequality:

$$\psi(t) \leq R\exp\left\{D\int_0^t \alpha(s)\,ds\right\} \leq R\exp\{D\,\|\alpha\|_{L^1_+[0,t_1]}\} := H.$$

Using the same arguments as before we may verify that the family Φ defined in (3.6) is ν-condensing on every bounded set $\Omega \subset C([0,t_1];E)$.

Now we take an open ball $U \subset C([0,t_1];E)$ of radius greater than H and with center $\widetilde{y}^0$ (then containing the set $\operatorname{Fix}\Phi$). The family Φ is fixed point free on the boundary ∂U and hence it determines an homotopy between the multifield $i - \Gamma^1$ and the multifield $i - \widetilde{y}^0$. In this framework it is possible to apply the relative topological degree theory for condensing multifields developed in [12]. In this case we evaluate the degree with respect to the whole space E. Taking into account that $\widetilde{y}^0 \in U$ and using the homotopy and normalization properties of the degree, we obtain that $\deg(i - \Gamma^1, \overline{U}) = \deg(i - \widetilde{y}^0, \overline{U}) = 1$ and therefore (see [12, Theorem 3.3.1])

$$\emptyset \neq \operatorname{Fix}\Gamma^1 \subset U.$$

Then also the set Σ^1_x of all mild solutions of problem $(\mathrm{P})_1$ is nonempty.

Now, we prove that it is compact. First of all, by applying Proposition 2.2 we can claim that the fixed points set of Γ^1 is compact. Let us take the function $\kappa^1\colon C([0,t_1];E) \to \mathcal{C}([-\tau,t_1];E)$ defined by

$$\kappa^1(z) = z[x]$$

where $z[x]$ is from (3.2). Since κ^1 is a continuous map, then the set $\kappa^1(\operatorname{Fix}\Gamma^1)$ is compact. The equality $\kappa^1(\operatorname{Fix}\Gamma^1) = \Sigma^1_x$ concludes the proof of the step.

Step 2. Let us fix $z^1 \in \Sigma^1_x$ and let us consider the non impulsive Cauchy problem:

$$(\mathrm{P})_{2;z^1} \qquad \begin{cases} y'(t) \in A(t)y(t) + F(t, y_t) & \text{a.e. } t \in [t_1, t_2], \\ y(t) = z^1(t) & \text{for } t \in [-\tau, t_1], \\ y(t_1) = z^1(t_1) + I_1(z^1_{t_1}). \end{cases}$$

Of course, for this problem a mild solution is a function $y \in \mathcal{C}([-\tau, t_2]; E)$ such that

(a') $y(t) = T(t, t_1)[z^1(t_1) + I_1(z^1_{t_1})] + \int_{t_1}^t T(t, s)f(s)\, ds$, $t \in [t_1, t_2]$ where $f \in L^1([t_1, t_2]; E)$, $f(s) \in F(s, y_s)$ for almost every $s \in [t_1, t_2]$

(b') $y(t) = z^1(t)$, $t \in [-\tau, t_1]$.

To provide such a solution for problem $(\mathrm{P})_{2;z^1}$, in analogy with the previous step we consider the integral multioperator $\Gamma^2 \colon C([t_1, t_2]; E) \to \mathcal{P}(C([t_1, t_2]; E))$ defined as:

$$\Gamma^2(z) = \bigg\{ y \in C([t_1, t_2]; E) : y(t) = T(t, t_1)[z^1(t_1) + I_1(z^1_{t_1})] + \int_{t_1}^t T(t, s)f(s)\, ds, \\ f \in L^1([t_1, t_2]; E),\ f(s) \in F(s, z[z^1]_s) \text{ a.e. } s \bigg\}$$

where $z[z^1]$ is defined by (3.2).

Also here, if $z \in \operatorname{Fix} \Gamma^2$ then $z[z^1]$ is a mild solution of $(\mathrm{P})_{2;z^1}$ on the interval $[-\tau, t_2]$.

Moreover, by proceeding in the same way as in Step 1, we can claim that this problem has at least one mild solution and the solution set is a compact set, say $\Sigma^2_{z^1}$.

Of course, we can iterate this process till a problem $(\mathrm{P})_{N;z^1,\dots,z^{N-1}}$ and obtain that also this problem has solutions and that these solutions form a compact set $\Sigma^N_{z^1,\dots,z^{N-1}}$.

Now, every solution of $(\mathrm{P})_{N;z^1,\dots,z^{N-1}}$ is a solution of (P) and the first part of the theorem is proved.

Step 3. It remains to prove that the set of all solutions of (P), i.e.

$$\Sigma = \bigcup \{\Sigma^N_{z^1,\dots,z^{N-1}} : z^1 \in \Sigma^1_x; \dots ; z^{N-1} \in \Sigma^{N-1}_{z^1,\dots,z^{N-2}}\} \tag{3.7}$$

is compact.

To this aim, from now on we assume that the impulse functions I_k are continuous. First of all, we define the multifunction $H^1 \colon \Sigma^1_x \to \mathcal{P}(C([-\tau, t_2]; E))$ as

$$H^1(z^1) = \Sigma^2_{z^1}.$$

From Step 2 we know both that Σ^1_x is compact and that H^1 has compact values.

Now, we prove that it is u.s.c. so that, by applying Proposition 2.1, we get the range $\bigcup_{z^1\in\Sigma^1_x}\Sigma^2_{z^1}$ to be compact.

Note that defining the multifunction $Q^1\colon\Sigma^1_x\to\mathcal{K}(C([t_1,t_2];E))$ as

$$Q^1(z^1)=\Sigma^2_{z^1|[t_1,t_2]},$$

the multifunction H^1 can be written as the composition of the multimap $P^1\colon\Sigma^1_x\to\mathcal{K}(\Sigma^1_x\times C([t_1,t_2];E))$ defined by

$$P^1(z^1)=\{z^1\}\times Q^1(z^1)$$

with the continuous map $\eta^1\colon P^1(\Sigma^1_x)\to\mathcal{C}([-\tau,t_2];E)$ defined by

$$\eta^1(z^1,z)=z[z^1]$$

(see (3.2)).

So, we first prove that the multifunction Q^1 is u.s.c. We assume to the contrary that there exists $\overline{z}^1\in\Sigma^1_x$ such that Q^1 is not u.s.c. in $\overline{z}^1$. Therefore there exist $\overline{\varepsilon}>0$ and two sequences $\{z^1_n\}_{n=1}^\infty$, $z^1_n\to\overline{z}^1$ in $\mathcal{C}([-\tau,t_1];E)$, and $\{z^2_n\}_{n=1}^\infty$, $z^2_n\in\Sigma^2_{z^1_n|[t_1,t_2]}$, such that

$$z^2_n\notin B(\Sigma^2_{\overline{z}^1|[t_1,t_2]},\overline{\varepsilon}),\quad n\geq 1. \tag{3.8}$$

Since $\{z^2_n\}_{n=1}^\infty$ is a sequence of solutions, we have:

$$z^2_n(t)=T(t,t_1)[z^1_n(t_1)+I_1(z^1_{n\,t_1})]+\int_{t_1}^t T(t,s)f^2_n(s)\,ds,\quad t\in[t_1,t_2] \tag{3.9}$$

where $f^2_n\in L^1([t_1,t_2];E)$, $f^2_n(s)\in F(s,z^2_{n\,s})$ for almost every $s\in[t_1,t_2]$. Then, for $s\in[t_1,t]$, (F4) yields

$$\begin{aligned}\chi(\{f^2_n(s)\}_{n=1}^\infty)&\leq\mu(s)\sup_{-\tau\leq\theta\leq 0}\chi(\{z^2_n(s+\theta)\}_{n=1}^\infty)\\&\leq\mu(s)\bigg(\sup_{-\tau\leq\eta\leq 0}\chi(x(\eta))+\sup_{0\leq\eta\leq t_1}\chi(\{z^1_n(\eta)\}_{n=1}^\infty)+\sup_{t_1\leq\eta\leq s}\chi(\{z^2_n(\eta)\}_{n=1}^\infty)\bigg).\end{aligned}$$

The MNC $\chi(\{z^1_n(\eta)\}_{n=1}^\infty)=0$, since $\{z^1_n\}_{n=1}^\infty$ is a converging sequence, then

$$\chi(\{f^2_n(s)\}_{n=1}^\infty)\leq e^{Ls}\mu(s)\sup_{t_1\leq\eta\leq t_2}e^{-L\eta}\chi(\{z^2_n(\eta)\}_{n=1}^\infty).$$

Now, using similar arguments as in the proof of condensivity in Step 1, it is possible to prove that the set $\{z^2_n\}_{n=1}^\infty$ is relatively compact in $C([t_1,t_2];E)$. Therefore without loss of generality we can assume that there exists $\overline{z}^2\in C([t_1,t_2];E)$ such that $z^2_n\to\overline{z}^2$ in $C([t_1,t_2];E)$.

Now we prove that $\overline{z}^2\in Q^1(\overline{z}^1)$. For every $n\geq 1$, we consider $z^2_n\in\Sigma^2_{z^1|[t_1,t_2]}$ and the corresponding function f^2_n from (3.9).

As in Step 1 it is possible to prove that there exists $\overline{f}^2 \in L^1([t_1, t_2]; E)$ such that $f_n^2 \rightharpoonup \overline{f}^2 \in L^1([t_1, t_2]; E)$. Now, by using Lemma 3.6, we have

$$\overline{f}^2(t) \in F(t, \overline{z}_t^2), \quad \text{a.e. } t \in [t_1, t_2].$$

From Proposition 3.4 and the fact that the function I_1 is continuous, by considering the limit in both sides of (3.9) we get

$$\overline{z}^2(t) = T(t, t_1)[z^1(t_1) + I_1(z_{t_1}^1)] + \int_{t_1}^{t} T(t, s)\overline{f}^2(s)\, ds, \quad t \in [t_1, t_2]$$

where $\overline{f}^2 \in L^1([t_1, t_2]; E), \overline{f}^2(s) \in F(s, z_s^2)$ for almost every $s \in [t_1, t_2]$, that is $\overline{z}^2 \in \Sigma^2_{z^1\,|[t_1,t_2]} = Q^1(\overline{z}^1)$.

The fact that $z_n^2 \to \overline{z}^2 \in Q^1(\overline{z}^1)$ leads a contradiction with (3.8). Therefore, by applying well known results on composition and cartesian product of multimaps (see e.g. [12, Theorems 1.2.12 and 1.2.8]), we can conclude that H^1 is u.s.c.

By iterating this process we obtain the compactness of the solution set Σ (cf. (3.7)) on the whole interval $[-\tau, b]$. □

Remark 3.8. We point out that our Theorem 3.7 contains the analogous result due to Benedetti ([4, Theorem 3.4]) for Cauchy problems governed by semilinear differential inclusions with linear part given by a constant operator A.

In fact, if A is the infinitesimal generator of a C_0-semigroup, then A satisfies assumption (A) just by defining every term of the family $\{A(t)\}_{t\in[0,b]}$ as $A(t) = A$ for every $t \in [0, b]$ and by recalling that a C_0-semigroup $\{U(t)\}_{t\in[0,b]}$ leads to an evolution system by means of relation $T(t, s) = U(t - s)$.

Moreover, every mild solution of (P) can be rewritten as a mild solution of the Cauchy problem with constant operator.

Remark 3.9. Note that if $I_k \equiv 0$ for every $k = 1, \ldots, N$, then Theorem 3.7 provides both the existence of mild solutions on the whole interval $[-\tau, b]$ for the non-impulsive delay evolution Cauchy problem

$$\begin{cases} y'(t) \in A(t)y(t) + F(t, y_t) & \text{for a.e. } t \in [0, b], \\ y(t) = x(t) & \text{for } t \in [-\tau, 0], \end{cases}$$

and the compactness of the solution set for this problem in the space $C([-\tau, b]; E)$.

4. Existence of solutions on non compact intervals

Using the results in the previous section we can give an existence result for the problem $(\mathrm{P})^\infty$. Let us consider the set $\Delta_\infty = \{(t, s) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : 0 \le s \le t\}$ and the evolution system $\{T(t, s)\}_{(t,s)\in\Delta_\infty}$.

In this framework, for every natural number $k \geq 1$ there exists a constant $D_k = D_{\Delta_k} > 0$ such that

$$\|T(t,s)\|_{\mathcal{L}(E)} \leq D_k, \quad (t,s) \in \Delta_k = \{(t,s) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : 0 \leq s \leq t \leq t_{k+1}\}.$$

On the linear part of the differential inclusion of problem $(P)^\infty$ we assume a condition that we call $(\mathrm{A})_\infty$, which is the same as condition (A) but on the interval $[0,\infty[$.

Let us denote by $L^1_{\mathrm{loc}}([0,\infty[)$ the set of all Bochner summable functions on the compact subsets of $[0,\infty[$.

The multivalued map $F\colon [0,\infty[\times \mathcal{C}([-\tau,0];E) \to \mathcal{K}v(E)$ is such that:

- $(\mathrm{F1})^\infty$ The multifunction $F(\,\cdot\,,c)\colon [0,\infty[\to \mathcal{K}v(E)$ has a strongly measurable selection for every $c \in \mathcal{C}([-\tau,0];E)$, i.e. there exists a strongly measurable function $f\colon [0,\infty[\to E$ such that $f(t) \in F(t,c)$ for almost every $t \in [0,\infty[$;
- $(\mathrm{F2})^\infty$ The multimap $F(t,\,\cdot\,)\colon \mathcal{C}([-\tau,0];E) \to \mathcal{K}v(E)$ is u.s.c. for almost every $t \in [0,\infty[$;
- $(\mathrm{F3})^\infty$ There exists a function $\alpha \in L^1_{\mathrm{loc}}([0,\infty[)$ such that:
$$\|F(t,c)\| \leq \alpha(t)(1+\|c\|_{\mathcal{C}}), \quad \text{a.e. } t \in [0,\infty[;$$
- $(\mathrm{F4})^\infty$ There exists a function $\mu \in L^1_{\mathrm{loc}}([0,\infty[)$ such that:
$$\chi(F(t,D)) \leq \mu(t) \sup_{-\tau\leq t\leq 0} \chi(D(t)), \quad \text{a.e. } t \in [0,\infty[,$$
for every bounded $D \subset \mathcal{C}([-\tau,0];E)$.

Definition 4.1. A function $y \in \mathcal{C}([-\tau,\infty[;E)$ is said to be a *mild solution* of the problem $(\mathrm{P})^\infty$ if

(a) $y(t) = T(t,0)x(0) + \sum_{0<t_k<t} T(t,t_k)I_k(y_{t_k}) + \int_0^t T(t,s)f(s)\,ds$, for $t \in [0,\infty[$, where $f \in L^1_{\mathrm{loc}}([0,\infty[;E)$ and $f(s) \in F(s,y_s)$ for almost every $s \in [0,\infty[$,
(b) $y(t) = x(t)$, $t \in [-\tau,0]$,
(c) $y(t_k^+) = y(t_k) + I_k(y_{t_k})$, $k \in \mathbb{N}^+$.

Theorem 4.2. *Suppose that hypotheses* $(\mathrm{A})^\infty$ *and* $(\mathrm{F1})^\infty$–$(\mathrm{F4})^\infty$ *hold and assume that maps* $I_k\colon \mathcal{C}([-\tau,0];E) \to E$, $k \in \mathbb{N}^+$, *are continuous. Then problem* $(\mathrm{P})^\infty$ *has at least one mild solution on* $[-\tau,\infty[$.

Proof. Let $\{T_k\}_{k\in\mathbb{N}^+}$ be the monotone family of compact intervals $T_k = [0,t_k]$. Of course $\bigcup_{k\in\mathbb{N}^+} T_k = [0,\infty[$. For every $k \in \mathbb{N}^+$, we consider the problem

$$(\mathrm{P})^k \qquad \begin{cases} y'(t) \in A(t)y(t) + F(t,y_t) & \text{a.e. } t \in T_k,\ t \neq t_j,\ j<k, \\ y(t) = x(t) & \text{for } t \in [-\tau,0], \\ y(t_j^+) = y(t_j) + I_j(y_{t_j}) & \text{for } j<k. \end{cases}$$

From Theorem 3.7 there exists at least one mild solution $\varphi_k \in \mathcal{C}([-\tau, t_k]; E)$ for this problem.

Let us consider the sequence of the restrictions to the interval $[-\tau, t_1]$ of the mild solutions fixed above, i.e. $(\varphi_{k|[-\tau,t_1]})_{k\in\mathbb{N}^+}$. Since this sequence is contained in the solution set of problem $(\mathrm{P})^1$, say Σ^1, and this set is a compact subset of the space $\mathcal{C}([-\tau, t_1]; E)$ (see again Theorem 3.7), we can claim that there exists a subsequence $(\varphi_{k_n|[-\tau,t_1]})_{n\in\mathbb{N}}$ converging to a function $\psi^1 \in \mathcal{C}([-\tau, t_1]; E)$ mild solution of the problem $(\mathrm{P})^1$; so

$$\psi^1(t) = T(t,0)x(0) + \int_0^t T(t,s)f^1(s)\,ds, \quad t \in T_1 \tag{4.1}$$

where $f^1 \in L^1(T_1; E)$, $f^1(s) \in F(s, \psi^1_s)$ for almost every $s \in T_1$.

Now, let us consider in $\mathcal{C}([-\tau, t_2]; E)$ the sequence $(\varphi_{k_n|[-\tau,t_2]})_{n\in\mathbb{N}}$ of mild solutions for the impulsive Cauchy problem $(\mathrm{P})^2$. Again, we know that the set of all mild solutions of $(\mathrm{P})^2$ is compact. Then, there exists a subsequence $(\varphi_{k_{n_h}|[-\tau,t_2]})_{h\in\mathbb{N}^+}$ converging in the space $\mathcal{C}([-\tau, t_2]; E)$ to a function $\psi^2 \in \mathcal{C}([-\tau, t_2]; E)$ which is a mild solution for the impulsive Cauchy problem $(\mathrm{P})^2$, so that

$$\psi^2(t) = T(t,0)x(0) + \sum_{0<t_j<t} T(t,t_j)I_j(\psi^2_{t_j}) + \int_0^t T(t,s)f^2(s)\,ds, \quad t \in T_2 \tag{4.2}$$

for $f^2 \in L^1(T_2; E)$, $f^2(s) \in F(s, \psi^2_s)$ for almost every $s \in T_2$. Of course,

$$\psi^2_{|T_1} = \psi^1.$$

Iterating this process we obtain, for every $k \in \mathbb{N}^+$, a mild solution $\psi^k\colon [-\tau, t_k] \to E$ for problem $(\mathrm{P})^k$ with

$$\psi^k(t) = T(t,0)x(0) + \sum_{0<t_j<t} T(t,t_j)I_j(\psi^k_{t_j}) + \int_0^t T(t,s)f^k(s)\,ds, \quad t \in T_k \tag{4.3}$$

where $f^k \in L^1(T_k; E), f^k(s) \in F(s, \psi^k_s)$ for almost every $s \in T_k$. Of course, for every integer $k \geq 2$, we have

$$\psi^k_{|T_{k-1}} = \psi^{k-1}. \tag{4.4}$$

Consider now the function $\psi\colon [-\tau, \infty[\to E$ defined by

$$\psi(t) = \begin{cases} \mathcal{X}_{T_1}(t)\psi^1(t) + \displaystyle\sum_{k=2}^{\infty} \mathcal{X}_{T_k \setminus T_{k-1}}(t)\psi^k(t) & \text{for } t \geq 0, \\ x(t) & \text{for } t \in [-\tau, 0], \end{cases} \tag{4.5}$$

where $\mathcal{X}_I$ is the characteristic function of a set I.

We claim that function ψ is a mild solution of problem $(\mathrm{P})^\infty$. Of course, it is enough to prove condition (a) of Definition 4.1. To this aim, let us note that if we put

$$f_{T_2}(t) = \begin{cases} f^1(t) & \text{for } t \in T_1, \\ f^2(t) & \text{for } t \in T_2 \setminus T_1, \end{cases}$$

then $f_{T_2} \in L^1(T_2;E)$ and, by (4.4), $f_{T_2}(t) \in F(t,\psi^2_t)$ for almost every $t \in T_2$. Moreover, it is also true that

$$\psi^2(t) = T(t,0)x(0) + \sum_{0<t_j<t} T(t,t_j)I_j(\psi^2_{t_j}) + \int_0^t T(t,s)f_{T_2}(s)\,ds, \quad t \in T_2.$$

In fact by (4.1), (4.2), (4.4) we have

$$\int_0^{t_1} T(t_1,s)f^1(s)\,ds = \int_0^{t_1} T(t_1,s)f^2(s)\,ds \tag{4.6}$$

then

$$\begin{aligned}\psi^2(t) =& T(t,0)x(0) + \sum_{0<t_j<t} T(t,t_j)I_j(\psi^2_{t_j}) \\ &+ T(t,t_1)\int_0^{t_1} T(t_1,s)f^2(s)\,ds + \int_{t_1}^t T(t,s)f^2(s)\,ds \\ =& T(t,0)x(0) + \sum_{0<t_j<t} T(t,t_j)I_j(\psi^2_{t_j}) \\ &+ T(t,t_1)\int_0^{t_1} T(t_1,s)f^1(s)\,ds + \int_{t_1}^t T(t,s)f^2(s)\,ds \\ =& T(t,0)x(0) + \sum_{0<t_j<t} T(t,t_j)I_j(\psi^2_{t_j}) + \int_0^t T(t,s)f_{T_2}(s)\,ds.\end{aligned}$$

Analogously, put

$$f_{T_3}(t) = \begin{cases} f^1(t) & \text{for } t \in T_1, \\ f^2(t) & \text{for } t \in T_2 \setminus T_1, \\ f^3(t) & \text{for } t \in T_3 \setminus T_2, \end{cases}$$

then $f_{T_3} \in L^1(T_3;E)$ and $f_{T_3}(t) \in F(t,\psi^3_t)$ for almost every $t \in T_3$. Further, for $t \in T_3$, relation (4.3) gives

$$\begin{aligned}\psi^3(t) =& T(t,0)x(0) + \sum_{0<t_j<t} T(t,t_j)I_j(\psi^3_{t_j}) + T(t,t_1)\int_0^{t_1} T(t_1,s)f^3(s)\,ds \\ &+ T(t,t_2)\int_{t_1}^{t_2} T(t_2,s)f^3(s)\,ds + \int_{t_2}^t T(t,s)f^3(s)\,ds.\end{aligned}$$

Now, from the value of ψ^3 in t_1, (4.4) and (4.6), we have that

$$\int_0^{t_1} T(t_1,s)f^3(s)\,ds = \int_0^{t_1} T(t_1,s)f^2(s)\,ds = \int_0^{t_1} T(t_1,s)f^1(s)\,ds. \tag{4.7}$$

Moreover, obviously it is

$$\int_{t_1}^{t_2} T(t_2,s)f^3(s)\,ds = \psi^3(t_2) - T(t_2,0)x(0) - T(t_2,t_1)I_1(\psi^3_{t_1}) - \int_0^{t_1} T(t_2,s)f^3(s)\,ds;$$

by using the first equality of (4.7), the last term can be written as

$$\begin{aligned}\int_0^{t_1} T(t_2,s)f^3(s)\,ds &= T(t_2,t_1)\int_0^{t_1} T(t_1,s)f^3(s)\,ds \\ &= T(t_2,t_1)\int_0^{t_1} T(t_1,s)f^2(s)\,ds.\end{aligned}$$

Then from the value of $\psi^2(t_2)$ we get the identity

$$\int_{t_1}^{t_2} T(t_2,s)f^3(s)\,ds = \int_{t_1}^{t_2} T(t_2,s)f^2(s)\,ds.$$

So we can conclude that

$$\psi^3(t) = T(t,0)x(0) + \sum_{0<t_j<t} T(t,t_j)I_j(\psi^3_{t_j}) + \int_0^t T(t,s)f_{T_3}(s)\,ds, \quad t \in T_3.$$

By the iteration of this process we have

$$\psi^k(t) = T(t,0)x(0) + \sum_{0<t_j<t} T(t,t_j)I_j(\psi^k_{t_j}) + \int_0^t T(t,s)f_{T_k}(s)\,ds, \quad t \in T_k \tag{4.8}$$

where

$$f_{T_k}(s) = \mathcal{X}_{T_1}(s)f^1(s) + \sum_{j=2}^{k} \mathcal{X}_{T_j\setminus T_{j-1}}(s)f^k(s), \quad s \in T_k$$

(and $f_{T_k} \in L^1(T_k;E)$, $f_{T_k}(s) \in F(s,\psi^k_s)$ for almost every $s \in T_k$).

Now, we show that the locally summable function $f\colon [0,\infty[\to E$ defined by

$$f(t) = \mathcal{X}_{T_1}(t)f^1(t) + \sum_{k=2}^{\infty} \mathcal{X}_{T_k\setminus T_{k-1}}(t)f^k(t), \quad t \geq 0 \tag{4.9}$$

is suitable for (a) of Definition 4.1.

In fact, fixed $t \geq 0$, there exists a unique $k \in \mathbb{N}^+$ such that $t \in T_k \setminus T_{k-1}$ and therefore, bearing in mind (4.5), (4.8) and (4.9), we get

$$\begin{aligned}\psi(t) = \psi^k(t) &= T(t,0)x(0) + \sum_{0<t_j<t} T(t,t_j)I_j(\psi^k_{t_j}) + \int_0^t T(t,s)f_{T_k}(s)\,ds \\ &= T(t,0)x(0) + \sum_{0<t_j<t} T(t,t_j)I_j(\psi_{t_j}) + \int_0^t T(t,s)f(s)\,ds.\end{aligned}$$

Of course, by (4.9) and (4.5) we also obtain $f(s) = f^k(s) \in F(s,\psi^k_s) = F(s,\psi_s)$ for almost every $s \geq 0$, which concludes the proof. □

Remark 4.3. In analogy with Remarks 3.8 and 3.9, we observe that Theorem 4.2 provides the existence of mild solutions for problem $(\mathrm{P})^\infty$ also in the autonomuos case (i.e. A not depending on t) and, if $I_k \equiv 0$ for every $k \in \mathbb{N}^+$, it furnishes the existence of mild solutions on $[-\tau, \infty[$ for the non-impulsive delay evolution Cauchy problem

$$\begin{cases} y'(t) \in A(t)y(t) + F(t, y_t) & \text{for a.e. } t \in [0, \infty[, \\ y(t) = x(t) & \text{for } t \in [-\tau, 0], \end{cases}$$

Remark 4.4. We note that the method of proof used for Theorem 4.2 also provides the existence of mild solutions for the impulsive Cauchy problem defined on the non closed interval $[0, b[$ $(0 < b < \infty)$

$$\begin{cases} y'(t) \in A(t)y(t) + F(t, y_t) & \text{a.e. } t \in [0, b[,\ t \neq t_k,\ k \in \mathbb{N}, \\ y(t) = x(t) & \text{for } t \in [-\tau, 0], \\ y(t_k^+) = y(t_k) + I_k(y_{t_k}) & \text{for } k \in \mathbb{N}^+, \end{cases}$$

where the jump points are an increasing sequence of times $t_k \in [0, b[$ converging to b. Of course, assumptions $(\mathrm{A})^\infty$ and $(F)^\infty$ must be suitably adapted to the present case.

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