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DETERMINATION OF LIMIT CYCLES BY ITERATED HOMOTOPY PERTURBATION METHOD FOR NONLINEAR OSCILLATORS WITH STRONG NONLINEARITY

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Abstract. He's Homotopy Perturbation Method which reduced to an Iterative Scheme is applied to nonlinear oscillators with strong nonlinearity. With the method, the iteration scheme provides excellent approximations to the solutions even though the iteration can only be done to the first stage.

1. Introduction

In this study, we consider the following type of nonlinear oscillation:

$$u'' + \varepsilon f(u, u') + u = 0$$

The study of nonlinear oscillators is of interest to many researchers and there are a variety of techniques for constructing analytical approximations to the solutions to the oscillatory systems. The perturbation methods are well established tools to study diverse aspects of nonlinear problems. The Lindstedt–Poincaré method, multi-time expansions, harmonic balance method, and the averaging technique are among those of the methods commonly used in nonlinear analysis [1]–[3]. However, the use of perturbation theory in many important practical problems is

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invalid, or it simply breaks down for parameters beyond a certain specified range. To overcome the limitations, for example, He [4]–[9] proposed a perturbation technique, so called He's homotopy perturbation method (HPM), which does not require a small parameter in the equation and takes the full advantage of the traditional perturbation methods and the homotopy techniques. Relatively recent survey on the method and its applications can be found in [10]–[43]. There also exists a wide range of literature dealing with the approximate determination of periodic solutions for nonlinear problems by using a mixture of methodologies [44]–[68].

The main purpose of this paper is to propose a new approach coupling iteration method and He's HPM for the periodic solutions to nonlinear oscillators with strong nonlinearity.

2. Solution method

As it is well known, in He's homotopy perturbation method, the solution of functional equation is considered as a sum of an infinite series usually converging to the solution. To be more specific, consider nonlinear differential equation:

$$N(u) = f \tag{2.1}$$

where N is a general differential operator and f is a known analytic function. We can define a homotopy $H(u,p)$ by

$$H(u,0) = L(u) - L(v_0) = 0, \quad H(u,1) = N(u) - f = 0$$

where $L(u)$ is a functional operator with a known solution v_0, which can be easily obtained. Classically, we may choose a convex homotopy by

$$H(u,p) = (1-p)L(u) + p[N(u) - f] = 0 \tag{2.2}$$

and continuously trace an implicitly defined curve from a starting point$H(v_0, 0)$ to a solution function $H(g,1)$ where g is a solution of (2.1). The embedding parameter p monotonically changes from zero to unity as the trivial problem $L(u) - L(v_0)$ is continuously deformed to the original problem$N(u) - f$. If the embedding parameter p is considered as a "small parameter", applying the classical perturbation technique, we can assume that the solution of equation (2.2), can be given as a power series in p, i.e.

$$v = v_0 + pv_1 + p^2 v_2 + \ldots \tag{2.3}$$

and setting $p = 1$ results in the approximate solution of (2.1) as

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \tag{2.4}$$

In HPM, the deformation process validated by substituting (2.3) into (2.2) and equating the coefficients of like powers p results in a series of nonhomogeneous linear differential equations which are recursively ordered to solve. i.e.

$$\begin{aligned} &p^0\colon L(u_0) - L(v_0) = 0, \\ &p^1\colon L(u_1) + L(v_0) + N(u_0) - f = 0, \\ &p^2\colon L(u_2) + N(u_1) = 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned} \tag{2.5}$$

Hence, the approximate solution can be readily obtained as in(2.4).

But, the deformation process mentioned above and given by (2.5) can also be expressed as an iterative procedure. Alternatively, we first linearize the original nonlinear equation, and apply the perturbation to find a correction to the linearized solution v_0 iteratively which may give an equivalent recursive process.

To illustrate the idea, consider a nonlinear oscillator modeled by the equation

$$u'' + f(u) = 0, \quad u(0) = A, \quad u'(0) = 0 \tag{2.6}$$

Now, we suppose that the natural frequency of the system (2.6) is ω, which is unknown to be further determined. Hence, the system (2.6) can be rewritten as

$$u'' + \omega^2 u = \omega^2 u - f(u) =: g(u), \quad u(0) = A, \quad u'(0) = 0 \tag{2.7}$$

The linearized form of the equation (2.7) is

$$u'' + \omega^2 u = 0, \quad u(0) = A, \quad u'(0) = 0 \tag{2.8}$$

Remember that in equation (2.3) the second term pv_1 is a correction term to the leading term v_0 and so on. Due to the fact that any initial approximation is obtained by using (2.8), means that initial approximation can be considered as an approximated solution to the original problem (2.7).

Also, comparing equation (2.6) with (2.8), it is easily seen that even though $f(u)$ is not "small", the function $g(u) = \omega^2 u - f(u)$ is "small". Then the left-hand side of equation (2.6) is linear and the term $g(u)$ on the right-hand side is a "small" function, namely, $g(u)$ does not have for small u a dominant term proportional to u. Hence, we equivalently solve equation (2.7) instead of (2.6) for convenience. This process can also be named as linearization of the perturbation process (see [66] and [67]).

We construct an iterative formula for the above equation:

$$u''_{k+1} + \omega^2 u_{k+1} = g(u_k), \quad u_k(0) = A, \quad u'_k(0) = 0, \quad k = 0, 1, \dots \tag{2.9}$$

where the starting function is

$$u_0(t) = A\cos\omega t \tag{2.10}$$

In this way, the deformation of the perturbed linear problem to the original nonlinear problem can be performed monotonically step by step until the desired accuracy is obtained.

This iteration can be performed to any value k; but for most of the cases, the iteration can be stopped at $k = 2$. Because, even termination at $k = 2$ is capable of providing very high accuracy for approximate analytical solution to the exact one.

In the next section, the operations of this procedure will be illustrated by applying it to two examples.

3. Examples

Example 3.1. Consider the following oscillation

$$u'' + u - \varepsilon u^3 = 0, \quad u(0) = A, \quad u'(0) = 0$$

For this example, iteration scheme (2.9) gives

$$u''_{k+1} + \omega^2 u_{k+1} = \omega^2 u_k - u_k + \varepsilon u_k^3, \quad u_k(0) = A, \quad u'_k(0) = 0.$$

The first iteration function (2.10) leads to

$$u''_1 + \omega^2 u_1 = (\omega^2 - 1)A\cos\omega t + \varepsilon(A\cos\omega t)^3$$

or

$$u''_1 + \omega^2 u_1 = \left[\omega^2 - 1 + \frac{3}{4}\varepsilon A^2\right] A\cos\omega t + \frac{1}{4}\varepsilon A^3\cos 3\omega t \tag{3.1}$$

The requirement of no secular terms in $u_1(t)$ implies

$$\omega = \sqrt{1 - \frac{3}{4}\varepsilon A^2} \tag{3.2}$$

Equation (3.1) reduces to

$$u''_1 + \omega^2 u_1 = \frac{1}{4}\varepsilon A^3\cos 3\omega t$$

with the initial conditions

$$u_1(0) = A, \quad u_1^1(0) = 0.$$

Hence, the first-order approximate solution reads

$$u_1 = A\cos\omega t + \frac{\varepsilon A^3}{32\omega^2}(\cos\omega t - \cos 3\omega t).$$

From (3.2) the approximated value of the period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{1 - \frac{3}{4}\varepsilon A^2}} = 2\pi\left(1 + \frac{3}{8}\varepsilon A^2 + \frac{27}{128}\varepsilon A^4 + \dots\right) + O(A^6).$$

Observe that the present method gives exactly the same results as the modified straightforward expansion solution obtained by present authors [45].

For comparison, the exact period reads:

$$T = \frac{4\sqrt{2}}{\sqrt{(2-\varepsilon A^2)}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1-\beta \sin^2 \phi}}, \quad \beta = \frac{\varepsilon A^2}{2-\varepsilon A^2}.$$

Deduce that small amplitudes yield

$$T = 2\pi\left(1 + \frac{3}{8}\varepsilon A^2 + \frac{57}{256}\varepsilon^2 A^4 + \dots\right) + O(A^6), \quad \varepsilon A^2 < 2.$$

Hence, we can clearly see that its first-order approximation is of high accuracy. It can be easily shown that the maximal relative error is less than 5.3%.

EXAMPLE 3.2. Consider the following oscillation

$$u'' + \sin u = 0, \quad u(0) = A, \quad u'(0) = 0.$$

Iteration scheme (2.9) gives

$$u''_{k+1} + \omega^2 u_{k+1} = \omega^2 u_k - \sin(u_k), \quad u_k(0) = A, \quad u_k^{'}(0) = 0$$

The first iteration function (2.10) leads to

$$u''_1 + \omega^2 u_1 = \omega^2 u_0 - \sin(u_0)$$

or

$$u''_1 + \omega^2 u_1 = \omega^2 A \cos \omega t - \sin(A \cos \omega t)$$

The requirement of no secular term in $u_1(t)$ implies that

$$\int_0^T \sin \omega(t-s)\{A\omega^2 \cos \omega t - \sin(A \cos \omega t)\}\, dt = 0$$

with $T = 2\pi/\omega$. Thus, we have

$$\omega^2 = \frac{\int_0^T \sin \omega t.\sin(A\cos \omega t) dt}{\int_0^T A \sin \omega t.\cos \omega t dt} = \frac{2J_1(A)}{A}$$

where $J_1(A)$ is the first-order Bessel function of the first kind;

$$J_1(A) = \frac{1}{2}A - \frac{1}{16}A^3 + \frac{1}{384}A^5 + \dots$$

The period then can be calculated as follows:

$$T = \frac{2\pi}{\sqrt{2J_1(A)/A}} = \frac{2\pi}{\sqrt{1 - A^2/8 + A^4/192 + \cdots}} = 2\pi\left(1 + \frac{1}{16}A^2 + \frac{5}{1536}A^4 + \dots\right) + O(A^6). \tag{3.3}$$

For comparison, the hyper geometric function approach of T_{ex} reads that the period of the pendulum as $4K(\beta)$, where

$$K(\beta) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\beta\sin^2\phi}}, \quad \beta = \sin^2\frac{1}{2}A$$

and is the complete elliptic integral of the first kind. The power series representation of $K(\beta)$ can be given by

$$K(\beta) = \frac{1}{2}\pi\left[1 + \left(\frac{1}{2}\right)^2\beta + \left(\frac{1.3}{2.4}\right)^2\beta^2 + \ldots\right], \quad |\beta| < 1.$$

Thus, deduce that the period of the oscillation of the pendulum for small amplitudes is given by

$$T_{ex} = 2\pi\left[1 + \frac{1}{16}A^2 + \frac{11}{3072}A^4 + \ldots\right] + O(A^6) \tag{3.4}$$

It can be observed that our solution (3.3) is in harmony with the perturbed series of T_{ex} given in (3.4).

3. Conclusion

In summary, iterated HPM method for calculating analytical approximations to the periodic solutions of nonlinear oscillators with strong nonlinearities has been proposed. Its applicability has been demonstrated by means of two examples. The major conclusion is that the iteration scheme provides exceptional approximations to the solutions even though the iteration can only be done to the first stage.

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