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HE'S HOMOTOPY PERTURBATION METHOD FOR THE TEMPERATURE DISTRIBUTION IN CONVECTIVE STRAIGHT FINS WITH TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITY

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Abstract. This paper applies J. H. He's homotopy perturbation method (HPM) to calculate the temperature distribution in convective straight fins with temperature-dependent thermal conductivity. The temperature distribution of straight fins is obtained as a function of thermo-geometric fin parameter. Comparison with the exact solution shows that the method is very effective and convenient, only one iteration leads to an accurate solution.

1. Introduction

With the rapid development of nonlinear science, there has appeared ever-increasing interest of scientists and engineers in the analytical asymptotic techniques for nonlinear problems. Perturbation method provides the most versatile tools available in nonlinear analysis of engineering problems, but the small parameter assumption greatly restricts its application. As is well known, an overwhelming majority of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all. To overcome the difficulty arising in perturbation methods, many new analytical methods are proposed in the last

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few decades, for example, Adomian decomposition method [1], [2], [4], modified Lindstedt–Poincaré method [13]–[15], [25], variational iteration method [3], [21], [23], [24], [26], [27], [34], homotopy perturbation method [5]–[12], [18], [20], [28], [29], parameter-expanding methods [31], and variational approaches [22], [30], [32], [33], a complete review can be found in [16], [19] or Ji-Huan He's monograph [17]. In this paper we will apply the Homotopy Perturbation Method [6], [16], [17], [19] to calculate analytically the temperature distribution of straight fins with temperature-dependent thermal conductivity. The basic idea of homotopy method is to deform continuously a simple problem which easy to be solved into the difficult problem under study. Comparison with exact solution shows it is a very promising method.

2. Formulation

Consider a straight fin as illustrated in Figure 1, where A_c is an arbitrary constant cross-sectional area, P is perimeter, b is length, h is the local heat transfer coefficient along the fin surface, T_b is the fin base temperature and $T_a$is the temperature of a convective environment. The fin extends into the convective environment and its tip is insulated. The one-dimensional energy balance equation is given as follows [1], [4], [17]

$$A_c \frac{d}{dx}\left[k(T)\frac{dT}{dx}\right] - ph(T_b - T_a) = 0, \quad k(T) = k_a[1 + \lambda(T - T_a)], \tag{2.1}$$

in which k_a is the thermal conductivity at the ambient fluid temperature of the fin and λ is the parameter describing the variation of the thermal conductivity.

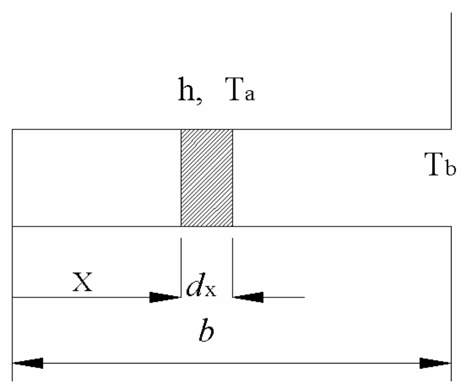


FIGURE 1. Geometry of a straight fin

When employing the following dimensionless parameters

$$\theta = \frac{T - T_a}{T_b - T_a}, \quad \xi = \frac{x}{b}, \quad \beta = \lambda(T_b - T_a), \quad \psi = \left(\frac{hpb^2}{k_a A_c}\right)^{1/2}$$

equation (2.1) can be written in the following dimensionless form

$$\begin{aligned} &\frac{d^2\theta}{d\xi^2} + \beta\theta\frac{d^2\theta}{d\xi^2} + \beta\left(\frac{d\theta}{d\xi}\right)^2 - \psi^2\theta = 0, \\ &\frac{d\theta}{d\xi} = 0 \quad \text{at } \xi = 0, \\ &\theta = 1 \quad \text{at } \xi = 1. \end{aligned} \tag{2.2}$$

3. Homotopy perturbation method

Recently, some rather extraordinary virtues of the homotopy perturbation method have been exploited [5]–12, [16]–[20], [28], [29]. The method has eliminated limitations of the traditional perturbation methods, on the other hand it can take full advantage of the traditional perturbation techniques [16], [17], [19].

According to the homotopy perturbation method [6], [16], [17], [19], we construct a homotopy in the form

$$\left[\frac{d^2\theta}{d\xi^2} + a\right] + P\left[\frac{d^2\theta}{d\xi^2} + \beta\theta\frac{d^2\theta}{d\xi^2} + \beta\left(\frac{d\theta}{d\xi}\right)^2 - \psi^2\theta - a\right] = 0, \quad P \in [0,1], \tag{3.1}$$

with the initial conditions

$$\theta(1) = 1, \quad \theta'(0) = 0.$$

When $p = 0$, equation (3.1) becomes a linearized equation, $\theta'' + a = 0$, where a is an unknown parameter to be further determined; when $p = 1$, it turns out to be the original one. The embedding parameter p monotonically increases from zero to unit as the trivial problem, $\theta'' + a = 0$, is continuously deformed to the original problem, equation (2.2), as illustrated in Figure 2.

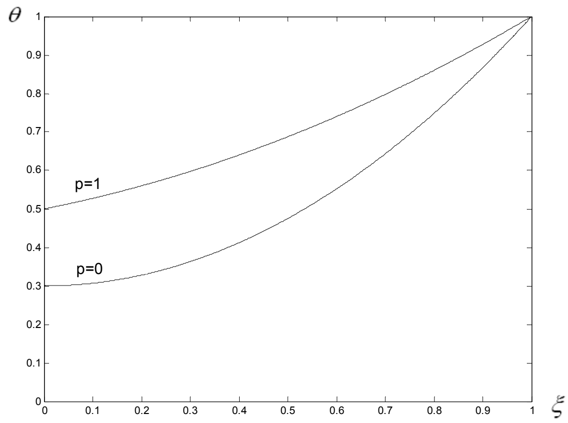


Figure 2. Homotopy perturbation method is to deform continuously from a simple problem ($p = 0$) to the original nonlinear problem ($p = 14$)

According to the homotopy perturbation method, we assume that the solution to (3.1) may be written as a power series in p:

$$\theta = \theta_0 + p\theta_1 + p^2\theta_2 + \dots . \tag{3.2}$$

Substituting (3.2) into (3.1) and equating the terms with the identical powers of p, we have

$$p^0\colon \theta_0'' + a = 0, \qquad \theta_0(1) = 1,\ \theta'(0) = 0, \tag{3.3}$$

$$p^1\colon \theta_1'' + \beta\theta_0\theta_0'' + \beta(\theta_0')^2 - \psi^2\theta_0 - a = 0, \qquad \theta_1(1) = 0,\ \theta_1'(0) = 0, \tag{3.4}$$

$$p^2\colon \theta_2'' + \theta_1'' + \beta\theta_1\theta_0'' + \beta\theta_0\theta_1'' + 2\beta\theta_0'\theta_1' - \psi^2\theta_1 = 0, \qquad \theta_2(1) = 0,\ \theta_2'(0) = 0. \tag{3.5}$$

The solution of equation (3.3) can be readily obtained, which reads

$$\theta_0(\xi) = \left(1 + \frac{a}{2}\right) - \frac{a}{2}\xi^2. \tag{3.6}$$

Substituting (3.6) into (3.4), considering the initial conditions $\theta_1(1) = 0$ and $\theta_1'(0) = 0$, we can easily solve θ_1 which reads

$$\theta_1 = \frac{1}{2}\left(\beta a + \frac{\beta}{2}a^2 + \psi^2 + \frac{\psi^2}{2}a + a\right)\xi^2 - \frac{1}{12}\left(\frac{3\beta}{2}a^2 + \frac{\psi^2}{2}a\right)\xi^4,$$

where

$$a = \begin{cases} \dfrac{-(12\beta + 5\psi^2 + 12) + \sqrt{(12\beta + 5\psi^2 + 12)^2 - 144\beta\psi^2}}{6\beta}, & \beta \neq 0, \\ a = -\dfrac{12\psi^2}{5\psi^2 + 12}, & \beta = 0. \end{cases}$$

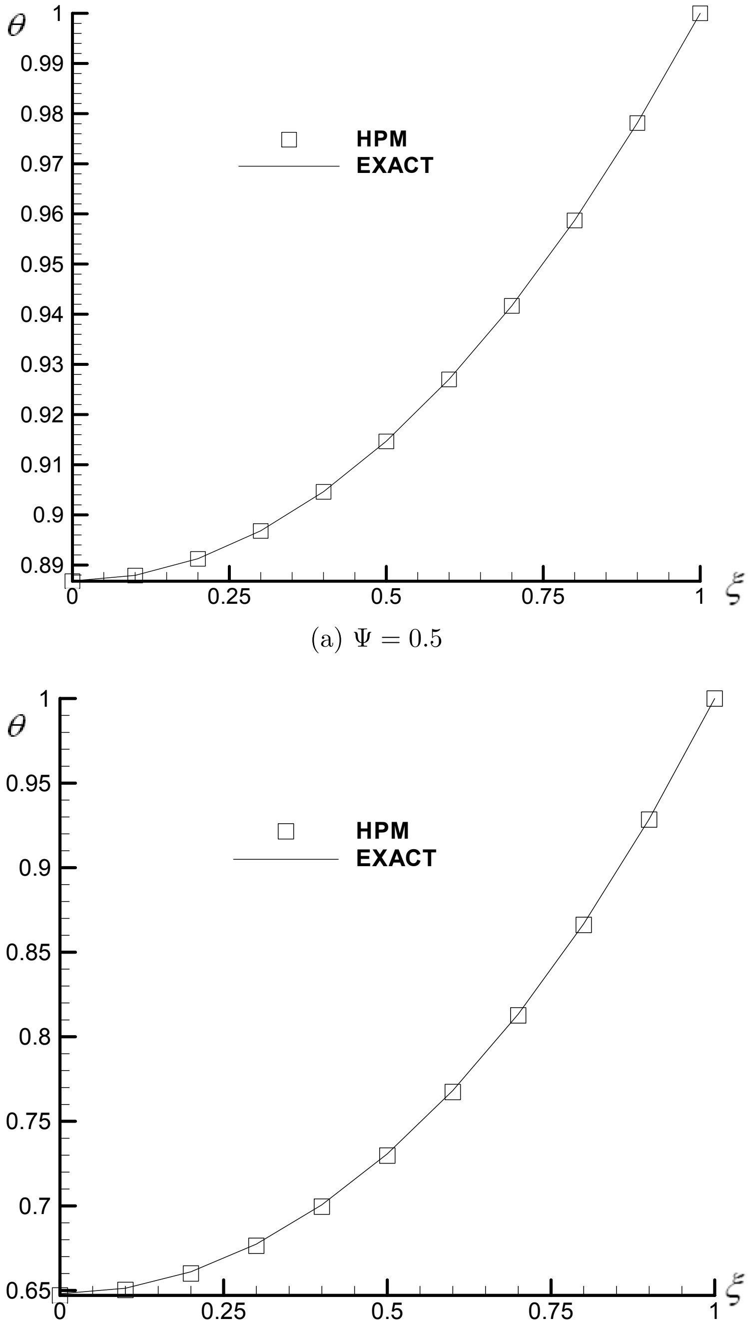
We, therefore, obtain the first-order approximate solution, which reads

$$\theta = \theta_0 + \theta_1 = \left(1 + \frac{a}{2}\right) + \frac{1}{2}\left(\beta a + \frac{\beta}{2}a^2 + \psi^2 + \frac{\psi^2}{2}a\right)\xi^2 - \frac{1}{12}\left(\frac{3\beta}{2}a^2 + \frac{\psi^2}{2}a\right)\xi^4. \tag{3.7}$$

The main merit of the homotopy perturbation method is that only one iteration leads to a high accurate solution. In case $\beta = 0$, e.g. constant thermal conductivity, comparison of the first-order approximate solution, equation (3.7), with exact solution [1] is tabulated in Table 1 and illustrated in Figure 3, showing an remarkable agreement. Of course, we can obtain even higher accurate solutions without any difficulty.

The dimensionless temperature distribution along the fin surface with β varying from -0.5 to 0.5 are shown in Figure 4 for different values of $\psi = 0.5$ and $\psi = 1.0$, respectively.

From Figure 4, it can be seen that the mean temperature increases with the increase of the thermal conductivity of the fin's material.



(a) $\Psi = 0.5$

(b) $\Psi = 1.0$

FIGURE 3. The temperature distribution for the case of constant thermal conductivity $\beta = 0$.

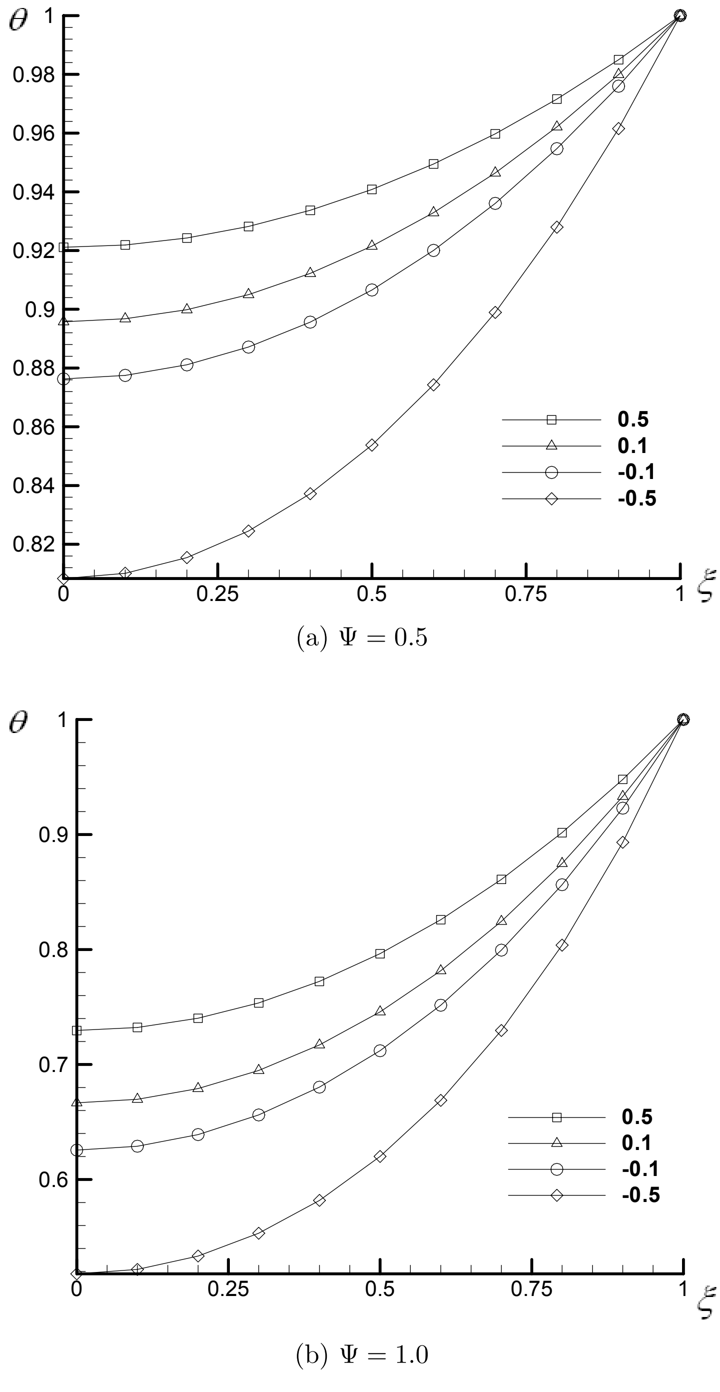


FIGURE 4. The temperature distribution in convective fins with variable thermal conductivity β varying from -0.5 to 0.5

	$\Psi = 0.5$			$\Psi = 1.0$		
ξ	HPM	Exact	Relative Error	HPM	Exact	Relative Error
0.0	0.886792453	0.886819	2.99351E−005	0.647058824	0.648054	1.53564E−003
0.2	0.891230189	0.891257	3.00822E−005	0.660047059	0.661059	1.53079E−003
0.4	0.904588679	0.904614	2.79909E−005	0.699576471	0.700594	1.45238E−003
0.6	0.927003774	0.927026	2.39756E−005	0.767341176	0.768246	1.17778E−003
0.8	0.958701887	0.958715	1.36777E−005	0.866164706	0.866731	6.53368E−004
1.0	1.000000	1.000000	0.00000E+000	1.000000	1.000000	0.00000E+000

Table 1. The dimensionless temperature distribution for the case of constant thermal conductivity, i.e. $\beta = 0$

4. Conclusions

In this paper, we have applied He's homotopy perturbation method in obtaining the temperature distribution of straight fins with temperature-dependent thermal conductivity. The obtained solutions are in good agreement with exact values. The results show that He's homotopy perturbation method does not require small parameters in the equations, so the limitations of the traditional perturbation methods can be eliminated completely. The reliability of the method and reduction in the size of computational domain give a wider applicability to this method.

References

[1] C. Arslanturk, *A decomposition method for fin efficiency of convective straight fins with temperature-dependent thermal conductivity*, Internat. Commun. Heat Mass Transfer **32** (2005), 831–841.

[2] same, *Optimum design of space radiators with temperature-dependent thermal conductivity*, Appl. Thermal Engineering **26** (2006), 1149–1157.

[3] N. Bildik and A. Konuralp, *The use of variational iteration method, differential transform method and Adomian decomposition method for solving different types of nonlinear partial differential equations*, Internat. J. Nonlinear Sci. Numer. Simul. **7** (2006), 65–70.

[4] M. H. Chang, *A decomposition solution for fins with temperature dependent surface heat flux*, Internat. J. Heat Mass Transfer. **48** (2005), 1819–1824.

[5] M. El-Shahed, *Application of He's homotopy perturbation method to Volterra's integro-differential equation*, Internat. J. Nonlinear Sci. Numer. Simul. **6** (2005), 163–168.

[6] J. H. He, *A coupling method of a homotopy technique and a perturbation technique for non-linear problems*, Internat. J. Non-Linear Mech. **35** (2000), 37–43.

[7] ______, *Application of homotopy perturbation method to nonlinear wave equations*, Chaos Solitons Fractals **26** (2005), 695–700.

[8] ______, *Homotopy perturbation method for bifurcation of nonlinear problems*, Internat. J. Nonlinear Sci. Numer. Simul. **6** (2007), 207–208.

[9] ———, *Homotopy perturbation method for solving boundary value problems*, Phys. Lett. A **350** (2006), 87–88.

[10] ———, *Homotopy perturbation method: a new nonlinear analytical technique*, Appl. Math. Comput. **135** (2003), 73–79.

[11] ———, *Homotopy perturbation technique*, Comput. Methods Appl. Mech. Engrg. **178** (1999), 257–262.

[12] ———, *Limit cycle and bifurcation of nonlinear problems*, Chaos Solitons Fractals **26** (2005), 827–833.

[13] ———, *Modified Lindsted–Poincaré methods for some strongly nonlinear oscillations, Part* III: *Double series expansion*, Internat. J. Nonlinear Sci. Numer. Simul. **2** (2001), 317–320.

[14] ———, *Modified Lindstedt–Poincaré methods for some strongly non-linear oscillations, Part* I: *expansion of a constant*, Internat. J. Nonlinear Mech. **37** (2002), 309–314.

[15] ———, *Modified Lindstedt–Poincaré methods for some strongly non-linear oscillations, Part* II: *a new transformation*, Internat. J. Nonlinear Mech. **37** (2002), 315–320.

[16] ———, *New interpretation of homotopy perturbation method*, Internat. J. Modern Phys. B. **20** (2006), 2561–2568.

[17] ———, *Non-perturbative methods for strongly nonlinear problems* (2006), dissertation. de–Verlag im Internet GmbH, Berlin.

[18] ———, *Periodic solutions and bifurcations of delay-differential equations*, Phys. Lett. A **347** (2005), 228–230.

[19] ———, *Some asymptotic methods for strongly nonlinear equations*, Internat. J. Modern Phys. B **20** (2006), 1141–1199.

[20] ———, *homotopy perturbation method for nonlinear oscillators with discontinuities*, Appl. Math. Comput. **151** (2004), 287–292.

[21] ———, *Variational iteration method — a kind of non-linear analytical technique: Some examples*, Internat. J. Nonlinear Mech. **34** (1999), 699–708.

[22] ———, *Variational principles for some nonlinear partial differential equations with variable coefficients*, Chaos Solitons Fractals **19** (2004), 847–851.

[23] ———, *Variational iteration method — some recent results and new interpretations*, J. Comput. Appl. Math. **207** (2007), 3–17.

[24] J. H. He and Wu XH, *Construction of solitary solution and compacton-like solution by variational iteration method*, Chaos Solitons Fractals **29** (2006), 108–113.

[25] H. M. Liu, *Approximate period of nonlinear oscillators with discontinuities by modified Lindstedt–Poincaré method*, Chaos Solitons Fractals **23** (2005), 577–579.

[26] S. Momani and S. Abuasad, *Application of He's variational iteration method to Helmholtz equation*, Chaos Solitons Fractals **27** (2006), 1119–1123.

[27] Z. M. Odibat and S. Momani, *Application of variational iteration method to nonlinear differential equations of fractional order*, Internat. J. Nonlinear Sci. Numer. Simul. **7** (2006), 27–34.

[28] A. M. Siddiqui, M. Ahmed and Q. K. Ghori, *Couette and Poiseuille flows for non-Newtonian fluids*, Internat. J. Nonlinear Sci. Numer. Simul. **7** (2006), 15–26.

[29] A. M. Siddiqui, R. Mahmood and Q. K. Ghori, *Thin film flow of a third grade fluid on a moving belt by He's homotopy perturbation method*, Internat. J. Nonlinear Sci. Numer. Simul. **7** (2006), 7–14.

[30] Y. Wu, *Variational approach to higher-order water-wave equations*, Chaos Solitons Fractals **32** (2007), 195–198.

[31] L. Xu, *He's parameter-expanding methods for strongly nonlinear oscillators*, J. Comput. Appl. Math. **207** (2007), 148–154.

[32] ———, *Variational approach to solitons of nonlinear $K(m,n)$ equations*, Chaos Solitons Fractals (to appear).

[33] ———, *Variational principles for coupled nonlinear Schrödinger equations*, Phys. Lett. A **359** (2006), 627–629.

[34] ———, *Variational iteration method-reality, potential, and challenges*, J. Comput. Appl. Math. **207** (2007), 1–2.

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