Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 31, 2008, 243–252

APPLICATION OF HOMOTOPY PERTURBATION METHOD TO THE BRATU-TYPE EQUATIONS

Xinlong Feng — Yinnian He — Jixiang Meng

Abstract. A new algorithm is presented for solving the Bratu-type equations. The numerical scheme based on the homotopy perturbation method is deduced. Two boundary value problems and an initial value problem are given to illustrate effectiveness and convenience of the proposed scheme. Our results agree very well with the numerical solutions showing that the homotopy perturbation method is a promising method.

1. Introduction

It is well known that the Bratu boundary value problem in one-dimensional planar coordinates is of the form

$$-u'' = \lambda e^u, \tag{1.1}$$

$$u(0) = u(1) = 0. \tag{1.2}$$

where $\lambda > 0$ is called the eigenvalue [3], [8], [9], [19], [21]. This problem comes originally from a simplification of the solid fuel ignition model in thermal combustion theory. The exact solution of (1.1) and (1.2) is given by

$$u(x) = -2\ln\left[\cosh\left(\frac{0.5(x-0.5)\theta}{\cosh(\theta/4)}\right)\right]$$

2000 *Mathematics Subject Classification.* Primary 34E10, 65C20; Secondary 34E05.
Key words and phrases. Bratu problem, homotopy perturbation method, Adomian decomposition method, boundary value problems, initial value problems.
This work is supported in parts by the NSF of China 10671154, 10671165 and 10726006.

provided θ is the solution of $\theta = \sqrt{2\lambda}\cosh(\theta/4)$. The problem has zero, one or two solutions when $\lambda > \lambda_0$, $\lambda = \lambda_0$ and $\lambda < \lambda_0$ respectively, where λ_0 satisfies the equation $\sqrt{2\lambda}\sinh(\theta/4) = 4$. It was evaluated [6], [9], [14] that the critical value λ_0 is equal to 3.513830719.

Due to its importance, the Bratu problem has been studied extensively. Firstly, it arises in a wide variety of physical applications, ranging from chemical reaction theory, radiative heat transfer and nanotechnology [29] to the expansion of universe [19]. Secondly, because of its simplicity, the equation is widely used as a benchmarking tool for numerical methods. Several numerical techniques, such as the finite difference method, finite element approximation, the shooting method [21] and Adomian decomposition method [1], [8], [19], [27], have been implemented independently to handle the Bratu model numerically. In addition, Boyd [6], [7] employed Chebyshev polynomial expansions and the Gegenbauer polynomials as base functions. He [17], [18] applied the variational method to study this problem. Wazwaz [30] applied the Adomian decomposition method to study the Bratu-type equations.

In 1998, He [15], [16] employed the basic ideas of the homopoty in topology to propose a general analytic method — homotopy perturbation method (HPM) for nonlinear problems. This method has been successfully applied to solve many types of nonlinear problems by others [1], [2], [4], [5], [10]–[14], [22]–[26], [28], [29], [31]. For example, the Duffing-harmonic oscillator [4], the strongly nonlinear equations [18], Helmholtz equation and KdV equation [22], [24], [28], the generalized nonlinear Boussinesq equation [25]. In reference [1], a homotopy technique and a perturbation technique is proposed to solve non-linear problems by Abd El-Latif. As a matter of fact, the proposed method is namely HPM. In practice, HPM is a powerful and easy-to-use analytic tool and does not need small parameters in the equations. HPM yields rapidly convergent series solutions. Recently, the applicability of HPM was extended to partial differential equations [5]. We introduce a small parameter and use Taylor expansion, based on the homotopy perturbation method [15], to study the Bratu-type equations in this paper. The new algorithm is implemented for two boundary value problems of the Bratu-type models given by

$$-u'' = -\lambda e^{u}, \tag{1.3}$$

and

$$-u'' = \lambda e^{-u}, \tag{1.4}$$

which satisfy the boundary value conditions (1.2), respectively. And an initial value problem of the Bratu-type model is given by

$$-u'' = -\lambda e^u, \tag{1.5}$$

$$u(0) = u'(0) = 0. \tag{1.6}$$

where $0 \leq x \leq 1$. Compared with the Adomian decomposition method, the numerical results show that the scheme approximates the exact solution with a high degree of accuracy using only few terms of the algorithm. And the procedure is more simple.

2. Formal deduction of the new algorithm

We apply homotopy perturbation method [12]–[18], [20] and directly introduce a parameter $p \in [0, 1]$ in the Bratu equation in the form

$$-u'' = \lambda e^{pu}, \tag{2.1}$$

$$u(0) = u(1) = 0. \tag{2.2}$$

It is obvious that when $p = 0$, equation (2.1)–(2.2) become a linear equation; when $p = 1$, it becomes the original nonlinear one.

Due to the fact that $p \in [0, 1]$, the imbedding parameter can be considered as a "small parameter". Applying the perturbation technique [21], we can assume that the solution of equation (2.1)–(2.2) can be expressed as a power series in p:

$$u = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \dots \tag{2.3}$$

Setting $p = 1$ results in the approximate solution of (1.1) and (1.2):

$$u^* = \lim_{p \to 1} u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + u_3 + \dots \tag{2.4}$$

To obtain its approximate solution of (2.1) and (2.2), we expand e^u into a Taylor series

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \dots \tag{2.5}$$

Substituting (2.2)–(2.5) into (2.1)–(2.2), and equating the coefficients of the same powers of p, we obtain

$$\begin{aligned}
p^0: \quad -u_0'' &= \lambda, & u_0(0) &= 0, \quad u_0(1) = 0,\\
p^1: \quad -u_1'' &= \lambda u_0, & u_1(0) &= 0, \quad u_1(1) = 0,\\
p^2: \quad -u_2'' &= \lambda\left(u_1 + \frac{1}{2}u_0^2\right), & u_2(0) &= 0, \quad u_2(1) = 0,\\
p^3: \quad -u_3'' &= \lambda\left(u_2 + u_0 u_1 + \frac{1}{6}u_0^3\right), & u_3(0) &= 0, \quad u_3(1) = 0.
\end{aligned}$$

From the above equations, we can obtain

$$\begin{aligned} u_0 &= \frac{\lambda}{2}(x - x^2), \\ u_1 &= \frac{\lambda^2}{24}(x^4 - 2x^3 + x), \\ u_2 &= \frac{\lambda^3}{1440}(-8x^6 + 24x^5 - 15x^4 - 10x^3 + 9x), \\ u_3 &= \frac{\lambda^4}{20160}(17x^8 - 68x^7 + 77x^6 + 7x^5 - 35x^4 - 21x^3 + 23x). \end{aligned}$$

In principle, it is possible to calculate more components in the expansion series to enhance the approximation. Therefore, we get the third-order approximation

$$u \approx u_0 + u_1 + u_2 + u_3.$$

From the above results, we concluded that the n-order approximation is a polynomial of degree $2(n+1)$. Hence, it is easy to compute the approximate solution.

3. First Bratu-type problem

Firstly, we consider the Bratu-type model (1.3). As indicated before, (1.3) differs from the standard Bratu problem by the sign of λ. According to the same technique, we can get

$$\begin{aligned} u_0 &= \frac{\lambda}{2}(x^2 - x), \\ u_1 &= \frac{\lambda^2}{24}(x^4 - 2x^3 + x), \\ u_2 &= \frac{\lambda^3}{1440}(8x^6 - 24x^5 + 15x^4 + 10x^3 - 9x) \\ u_3 &= \frac{\lambda^4}{20160}(17x^8 - 68x^7 + 77x^6 + 7x^5 - 35x^4 - 21x^3 + 23x). \end{aligned} \tag{3.1}$$

In view of (3.1), if $\lambda = \pi^2$, then the solution $u(x)$ is readily obtained in a series form by

$$\begin{aligned} u(x) = {}&\frac{\pi^2}{2}(x^2 - x) + \frac{\pi^4}{24}(x^4 - 2x^3 + x) + \frac{\pi^6}{1440}(8x^6 - 24x^5 + 15x^4 + 10x^3 - 9x) \\ &+ \frac{\pi^8}{20160}(17x^8 - 68x^7 + 77x^6 + 7x^5 - 35x^4 - 21x^3 + 23x) + \ldots \end{aligned}$$

This result differs from the one obtained by Adomian decomposition method [30] given by

$$u(x) = \pi x + \frac{\pi^2}{2!}x^2 + \frac{\pi^3}{3!}x^3 + \frac{2\pi^4}{4!}x^4 + \frac{\pi^5}{4!}x^5 + \frac{16\pi^6}{6!}x^6 + \frac{61\pi^7}{6!}x^7 + \ldots \tag{3.2}$$

Here, the exact solution of (1.3) is $u(x) = -\ln(1 + \cos((0.5 + x)\pi))$. It is interesting to point out that unlike the solution of the standard Bratu equation where $u(x) < \infty$ for all values of x of its domain, the exact solution blows up at $x = 0.5$ and the approximate solution (3.2) deviates from the exact solution when $x > 0.5$ as shown by Figures 1 and 2, where the dashed line denotes the exact solution and the solid line denotes the approximate solution, respectively.

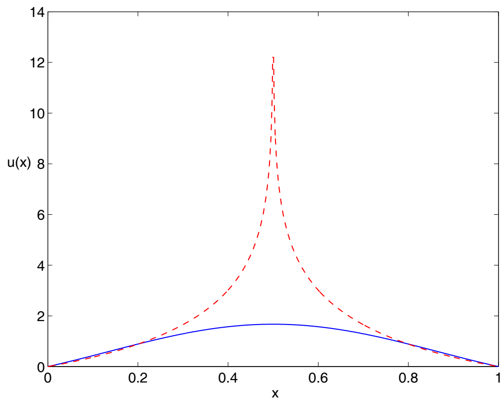

FIGURE 1. The approximate solution (3.1) of the first Bratu-type

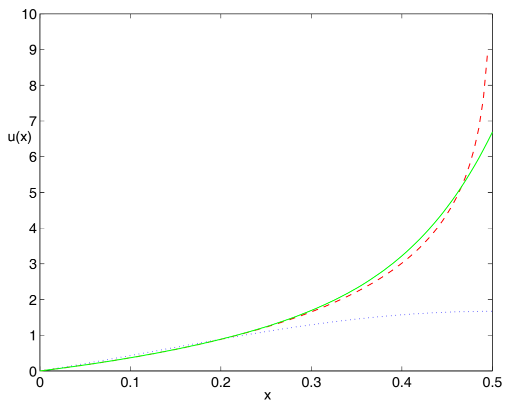

FIGURE 2. The approximate solution (3.2) of the first Bratu-type

4. Second Bratu-type problem

Secondly, we consider the Bratu-type model (1.4). (1.4) differs from the standard Bratu problem by the term e^{-u}. According to the same technique in

Section 2, we can obtain

$$
\begin{aligned}
u_0 &= \frac{\lambda}{2}(-x^2 + x), \\
u_1 &= \frac{\lambda^2}{24}(-x^4 + 2x^3 - x), \\
u_2 &= \frac{\lambda^3}{1440}(-8x^6 + 24x^5 - 15x^4 - 10x^3 + 9x), \\
u_3 &= \frac{\lambda^4}{20160}(-17x^8 + 68x^7 - 77x^6 - 7x^5 + 35x^4 + 21x^3 - 23x).
\end{aligned}
\tag{4.1}
$$

In view of (4.1), if $\lambda = \pi^2$, then the solution $u(x)$ is readily obtained in a series form by

$$
\begin{aligned}
u(x) = \frac{\pi^2}{2}(x^2 - x) - \frac{\pi^4}{24}(x^4 - 2x^3 + x) + \frac{\pi^6}{1440}(8x^6 - 24x^5 + 15x^4 + 10x^3 - 9x) \\
- \frac{\pi^8}{20160}(17x^8 - 68x^7 + 77x^6 + 7x^5 - 35x^4 - 21x^3 + 23x) + \ldots
\end{aligned}
$$

This result differs from the one by Adomian decomposition method [30] given by

$$
u(x) = \pi x - \frac{\pi^2}{2!}x^2 + \frac{\pi^3}{3!}x^3 - \frac{2\pi^4}{4!}x^4 + \frac{\pi^5}{4!}x^5 - \frac{16\pi^6}{6!}x^6 + \frac{61\pi^7}{6!}x^7 + \ldots. \tag{4.2}
$$

Here, the exact solution of (1.4) is $u(x) = \ln(1 + \cos((0.5 + x)\pi))$. It is interesting to point out that the exact solution blows up at x = 0.5 and the approximate solution (4.2) deviates from the exact solution in the whole domain as shown by Figures 3 and 4, where the dashed line denotes the exact solution.

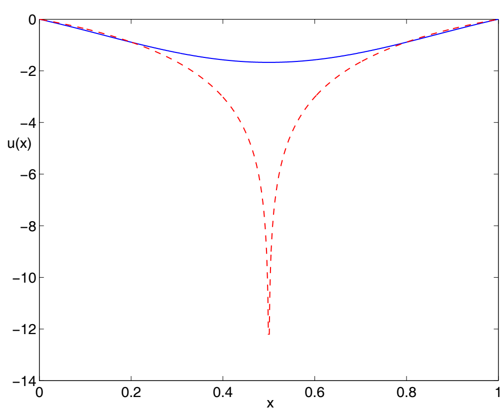

FIGURE 3. The approximate solution (4.1) of the second Bratu-type

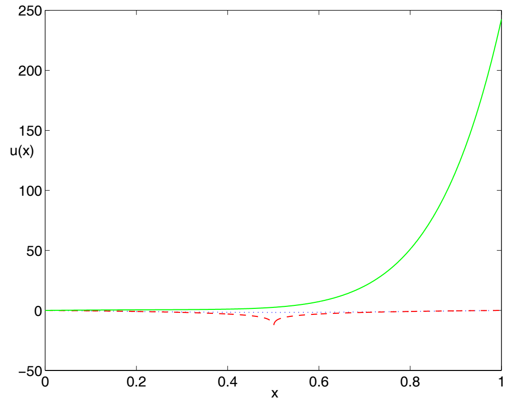

Figure 4. The approximate solution (4.2) of the second Bratu-type

5. Initial value problem of the Bratu-type

Finally, we consider the initial value problem of Bratu-type (1.5). Unlike the Bratu problem where boundary conditions are used, (1.5) is an initial value problem. According the same technique in Section 2, we can obtain

$$\begin{aligned} u_0 &= \frac{\lambda}{2}x^2, & u_1 &= \frac{\lambda^2}{24}x^4, & u_2 &= \frac{\lambda^3}{180}x^6, \\ u_3 &= \frac{17\lambda^4}{20160}x^8, & u_4 &= \frac{31\lambda^5}{226800}x^{10}. \end{aligned} \tag{5.1}$$

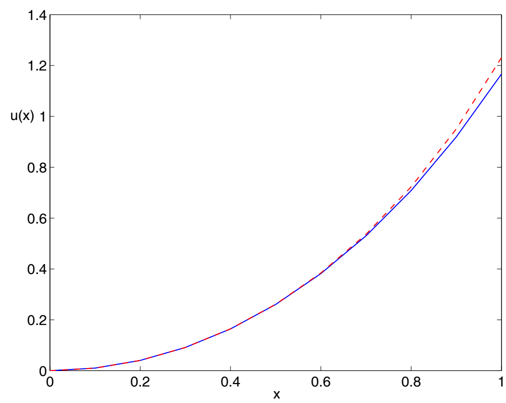

Figure 5. The first-order approximation (5.2) of the third Bratu-type

More components in the expansion series can be computed to enhance the approximation. In view of (5.1), if $\lambda = 2$, then the solution $u(x)$ in a series form is readily obtained by

$$
\begin{aligned}
(5.2)\qquad u(x) &= x^2 + \frac{1}{6}x^4 + \frac{2}{45}x^6 + \frac{17}{1260}x^8 + \frac{62}{14175}x^{10} + \dots \\
&= -2\Big(-\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6 - \frac{17}{2520}x^8 - \frac{31}{14175}x^{10} - \dots \Big).
\end{aligned}
$$

We have the same result with the reference [30] by Adomian decomposition method, the exact solution of (1.5) is given by $u(x) = -2\ln(\cos(x))$. It is noticed that the exact solution is bounded in the domain $0 \le x \le 1$ as shown by Figures 5 and 6, where the dashed line denotes the exact solution. With the second-order approximation $u_0+u_1+u_2$, we can obtain the high accuracy in the whole domain.

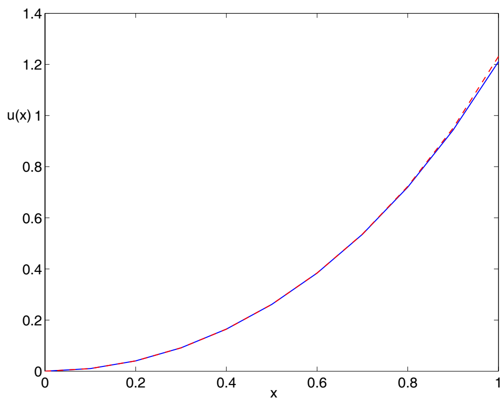

Figure 6. The second-order approximation (5.2) of the third Bratu-type

6. Discussions

The goal to obtain exact solutions for Bratu-type problems by using the new algorithm based on the homotopy perturbation method has been achieved. This algorithm has been applied directly without using linearization or any restrictive assumptions. The solutions for the first and the second Bratu-type problem blow up at the middle of the domain and remain bounded elsewhere. However, the solution for the initial value problem is bounded in the whole domain. The computational size is reasonable when compared to other techniques. This confirms the new algorithm is effective and reliable. Furthermore, this paper shows the validity and great potential of the HPM for nonlinear problems in science and engineering.

Acknowledgements. The authors would like to thank the editor and referees for their valuable comments and suggestions which helped us to improve the results of this paper.

References

[1] G. M. Abd El-Latif, *A homotopy and a perturbation technique for non-linear problems*, Appl. Math. Comput. **169** (2005), 576–588.

[2] P. D. Ariel, T. Hayat and S. Asghar, *Homotopy perturbation method and axisymmetric flow over a stretching sheet*, Internat. J. Nonlinear Sci. Numer. Simul. **7** (2006), 399–406.

[3] R. Aris, *The Mathematical Theory of Diffusion and Reaction in Permeable Catalyst*, vol. I, Clarendon Press, Oxford, 1975.

[4] A. Belendez, A. Hernandez, T. Belendez, et al., *Application of He's homotopy perturbation method to the Duffing-harmonic oscillator*, Internat. J. Nonlinear Sci. Numer. Simul. **8** (2007), 79–88.

[5] J. Biazar, M. Eslami and H. Ghazvini, *Homotopy perturbation method for systems of partial differential equations*, Internat. J. Nonlinear Sci. Numer. Simul. **8** (2007), 413–418.

[6] J. P. Boyd, *An analytical and numerical study of the two-dimensional Bratu equation*, J. Sci. Comput. **1** (1986), 183–206.

[7] ———, *Chebyshev polynomial expansions for simultaneous approximation of two branches of a function with application to the one-dimensional Bratu equation*, Appl. Math. Comput. **142** (2003), 189–200.

[8] E. Y. Deeba and S. A. Khuri, *A decomposition method for solving the nonlinear Klein–Gordon equation*, J. Comput. Phys. **124** (1996), 442–448.

[9] E. Deeba, S. A. Khuri and S. Xie, *An algorithm for solving boundary value problems*, J. Comput. Phys. **159** (2000), 125–138.

[10] D. D. Ganji and A. Sadighi, *Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations*, Internat. J. Nonlinear Sci. Numer. Simul. **7** (2006), 411–418.

[11] Q. K. Ghori, M. Ahmed and A. M. Siddiqui, *Application of homotopy perturbation method to squeezing flow of a Newtonian fluid*, Internat. J. Nonlinear Sci. Numer. Simul. **8** (2007), 179–184.

[12] J. H. He, *A coupling method of homotopy technique and perturbation technique for nonlinear problems*, Internat. J. Non-Linear Mech. **35** (2000), 37–43.

[13] ———, *Homotopy perturbation method for bifurcation of nonlinear problems*, Inernat. J. Nonlinear. Sci. Num. Simul. **6** (2005), 207–208.

[14] ———, *Homotopy perturbation method for solving boundary value problems*, Phys. Lett. A **350** (2006), 87–88.

[15] ———, *Homotopy perturbation technique*, Comput. Methods Appl. Mech. Engrg. **178** (1999), 257–262.

[16] ———, *Newton-like iteration method for solving algebraic equation*, Comm. Nonlinear Sci. Num. Simul. **3** (1998), 106–109.

[17] ———, *Variational iteration method — a kind of non-linear analytical technique: Some examples*, Internat. J. Non-Linear Mech. **34** (1999), 699–708.

[18] ———, *Some asymptotic methods for strongly nonlinear equations*, Internat. J. Modern Phys. B **20** (2006), 1141–1199.

[19] S. A. Khuri, *A new approach to Bratu's problem*, Appl. Math. Comput. **147** (2004), 131–136.

[20] S. J. Liao and A. T. Chwang, *Application of homotopy analysis method in nonlinear oscillations*, ASME J. Appl. Mech. **65** (1998), 914–915.

[21] A. H. Nayfeh, *Perturbation Methods*, Wiley, New York, 2000.

[22] T. OZIS AND A. YILDIRIM, *Traveling wave solution of Korteweg–de Vries equation using He's homotopy perturbation method*, Internat. J. Nonlinear Sci. Numer. Simul. **8** (2007), 239–242.

[23] ———, *A comparative study of He's homotopy perturbation method for determining frequency-amplitude relation of a nonlinear oscillator with discontinuities*, Internat. J. Nonlinear Sci. Numer. Simul. **8**, 243–248.

[24] M. RAFEI AND D. D. GANJI, *Explicit solutions of Helmholtz equation and fifth-order KdV equation using homotopy perturbation method*, Internat. J. Nonlinear Sci. Numer. Simul. **7**, 321–328.

[25] A. SADIGHI AND D. D. GANJI, *Solution of the generalized nonlinear Boussinesq equation using homotopy perturbation and variational iteration methods*, Internat. J. Nonlinear Sci. Numer. Simul. **8** (2007), 319–328.

[26] D. H. SHOU, *Application of parameter-expanding method to strongly nonlinear oscillators*, Internat. J. Nonlinear Sci. Numer. Simul. **8** (2007), 121–124.

[27] M. I. SYAM AND A. HAMDAN, *An efficient method for solving Bratu equations*, Appl. Math. Comput. **176** (2006), 704–713.

[28] H. TARI, D. D. GANJI AND M. ROSTAMIAN, *Approximate solutions of $K(2,2)$, KdV and modified KdV equations by variational iteration method, homotopy perturbation method and homotopy analysis method*, Internat. J. Nonlinear Sci. Numer. Simul. **8** (2007), 203–210.

[29] Y.Q. WAN, Q. GUO AND N. PAN, *Thermo-electro-hydrodynamic model for electrospinning process*, Internat. J. Nonlinear Sci. Numer. Simul. **5** (2004), 5–8.

[30] A. M. WAZWAZ, *Adomian decomposition method for a reliable treatment of the Bratu-type equations*, Appl. Math. Comput. **166** (2005), 152–633.

[31] E. YUSUFOGLU, *Homotopy perturbation method for solving a nonlinear system of second order boundary value problems*, Internat. J. Nonlinear Sci. Numer. Simul. **8** (2007), 353–358.

Manuscript received September 6, 2007

XINLONG FENG
College of Mathematics and System Science
Xinjiang University
Urmuqi 830046, P. R. CHINA

E-mail address: fxlmath@xju.edu.cn

YINNIAN HE
Faculty of Science
Xi'an Jiaotong University
Xi'an 710049, P. R. CHINA

E-mail address: heyn@mail.xjtu.edu.cn

JIXIANG MENG
College of Mathematics and System Science
Xinjiang University
Urmuqi 830046, P. R. CHINA

E-mail address: mjx@xju.edu.cn