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A MULTIPLICITY RESULT FOR A SEMILINEAR MAXWELL TYPE EQUATION

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Abstract. In this paper we look for solutions of the equation

$$\delta d\mathbf{A} = f'(\langle \mathbf{A}, \mathbf{A}\rangle)\mathbf{A} \quad \text{in } \mathbb{R}^{2k},$$

where $\mathbf{A}$ is a 1-differential form and $k \geq 2$. These solutions are critical points of a functional which is strongly indefinite because of the presence of the differential operator δd. We prove that, assuming a suitable convexity condition on the nonlinearity, the equation possesses infinitely many finite energy solutions.

1. Introduction

It is well known that the Maxwell equations in the empty space, written by the differential forms language, are the Euler–Lagrange equations of the following action functional

$$\mathcal{S} = \int_{\mathbb{R}^4} \langle d\eta, d\eta\rangle \ \sigma. \tag{1.1}$$

Here

$$\eta = \sum_{i=1}^{3} A_i dx^i + \varphi dt, \quad A_i, \varphi\colon \mathbb{R}^4 \to \mathbb{R},$$

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is the gauge potential 1-form in the space-time $\mathbb{R}^4$, $d\eta$ denotes the exterior derivative of η, σ is the volume form, and for any differential form γ

$$\langle \gamma, \gamma \rangle := *(*\gamma \wedge \gamma)$$

where $*$ is the Hodge operator with respect to the Minkowski metric in $\mathbb{R}^4$.

According to the classical theory of the electrodynamics, when the electromagnetic field is generated by an assigned source j (e.g. a particle matter), then the action functional becomes

$$\mathcal{S} = \int_{\mathbb{R}^4} (\langle d\eta, d\eta \rangle - \langle j, \eta \rangle)\, \sigma.$$

When instead the source of the field is not assigned but it is an unknown of the problem, then there are two opposite mathematical models describing the interaction between the electromagnetic field and its source: the dualistic model and the unitarian model.

The dualistic model consists in coupling the Maxwell equation with another field equation describing the dynamics of the source that is represented by a travelling solitary wave (i.e. a solution of a field equation whose energy density travels as a localized packet). This approach has been analyzed in many papers and several existence and multiplicity results have been obtained (see e.g. [8], [12]–[14]).

More recently, an unitarian field theory has been introduced by Benci and Fortunato [6], following an idea from Born and Infeld (see [11]). According to this theory (we refer to [6] and [7] for more details), electromagnetic field and matter field are both expression of only one physical entity, and the interaction between them is described by introducing a nonlinear Poincaré invariant perturbation in the Maxwell Lagrangian in the empty space.

Following this new unitarian theory, we perturb the Lagrangian in (1.1) adding a nonlinear term and obtaining the modified action functional

$$\mathcal{S} = \int_{\mathbb{R}^4} (\langle d\eta, d\eta \rangle - f(\langle \eta, \eta \rangle))\, \sigma$$

where $f\colon \mathbb{R} \to \mathbb{R}$. The Euler–Lagrange equation is the following nonhomogeneous Maxwell equation

$$\delta d\eta = j(\eta) \tag{1.2}$$

where $j(\eta) = f'(\langle \eta, \eta \rangle)\eta$ and $\delta = *d*$. The 1-form j representing the source depends itself on the gauge 1-form η, so the equation (1.2) describes the dynamics of the electromagnetic field in presence of an auto-induction phenomenon.

From now on, we will refer to (1.2) as the semilinear Maxwell equation (SME).

In [3] the equation (1.2) has been considered in the magnetostatic case, namely when it has the form

$$\delta d\mathbf{A} = f'(\langle \mathbf{A}, \mathbf{A}\rangle)\mathbf{A} \tag{1.3}$$

where

$$\mathbf{A} = \sum_{i=1}^{3} A_i \, dx^i, \quad A_i\colon \mathbb{R}^3 \to \mathbb{R},$$

and the metric on $\mathbb{R}^3$ is the euclidean one. In that paper a solution $\mathbf{A}$, with the property $\delta\mathbf{A} = 0$, has been found. In [2], ignoring the physical origin of the problem, the equation (1.3) has been studied in the more general context of the k-forms on a n-Riemannian manifold M, and a multiplicity result has been proved when M is compact.

In the same spirit of that paper, here we consider the problem just from a mathematical point of view, looking for solution of

$$\begin{cases} \delta d\mathbf{A} = f'(\langle \mathbf{A}, \mathbf{A}\rangle)\mathbf{A}, \\ \mathbf{A} = \displaystyle\sum_{i=1}^{n} A_i dx^i, \qquad\qquad A_i\colon \mathbb{R}^n \to \mathbb{R} \end{cases} \tag{1.4}$$

where we consider $\mathbb{R}^n$ endowed with the euclidean metric. In the sequel we often will use the notation $\mathbf{A}$ to denote also the vector field $(A_1, \ldots , A_n)$.

Actually, equation (1.4) is the natural extension to the 1-forms of the well-known scalar field equation with

$$-\Delta u = f'(u^2)u.$$

In fact, if we denote by $\Lambda^0(\mathbb{R}^n)$ the set of the scalar fields on $\mathbb{R}^n$, we have

$$\ker(\delta_{|\Lambda^0}) = \Lambda^0(\mathbb{R}^n)$$

and then the operator δd coincides with the Laplace–Beltrami operator.

Now we are going to introduce the main result of this paper. Consider $n \geq 1$ even and denote by $\Lambda^1(\mathbb{R}^n)$ the set of the 1-forms on $\mathbb{R}^n$ with compact support and by T the group of transformations on $\mathbb{R}^n$ so defined:

(1.5) $g \in T$ if and only if $g \in O(n)$ and there exists $(g_i)_{1\leq i\leq n/2}$ in $O(2)$ such that

$$g = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_{n/2} \end{pmatrix}$$

where $O(n)$ and $O(2)$ are respectively the orthogonal groups in $\mathbb{R}^n$ and $\mathbb{R}^2$.

Moreover, denote by $(\,\cdot\,|\,\cdot\,)$ the scalar product on $\mathbb{R}^n$ and assume that

(f_1) $f \in C^1(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, for all $t \geq 0$ such that $f'(t) \geq 0$, and for $2 < p < 2^* < q$, with $2^* = 2n/(n-2)$,

(f$_2$) there exists $c_1 > 0$ such that for all $x, y \in \mathbb{R}^n$

$$f((x|x)) - f((y|y)) - 2f'((y|y))(y|x-y) \geq c_1 \min((x-y|x-y)^{p/2}, (x-y|x-y)^{q/2}),$$

(f$_3$) there exists $c_2 > 0$ such that $|f'(t)| \leq c_2 \min(t^{p/2-1}, t^{q/2-1})$, for all $t \geq 0$,

(f$_4$) there exists $R > 0$ and $\alpha > 2$ such that $0 < (\alpha/2)f(t) \leq f'(t)t$, for all $t \geq R$.

The main result of this paper is the following

THEOREM 1.1. *Let $n \geq 4$ be even and assume that f satisfies* (f$_1$)–(f$_4$). *Then there exist infinitely many nontrivial weak solutions of* (1.4). *Moreover, these solutions have the following particular symmetry:*

$$\mathbf{A}(x) = g^{-1}\mathbf{A}(gx), \quad \textit{for all } g \in T.$$

In the sequel we will assume (f$_1$)–(f$_4$) holding.

REMARK 1.2. Set $g(x) = f(x^2)$ and suppose $f \in C^2(\mathbb{R}, \mathbb{R})$. Observe that by (f$_1$) and (f$_3$) we deduce $g''(0) = 0$, and then we find the so called "zero mass" case. This case has been dealt with by Berestycki and Lions [9], [10] and more recently by Pisani [17], for the scalar version of the equation (1.2)

$$-\Delta u = f'(u^2)u.$$

REMARK 1.3. For every $x \in \mathbb{R}^n$ we can define the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_x$ on the vector space $\Lambda^1(\mathbb{R}^n)$. The assumption (f$_2$) is a condition on the convexity of the functional

$$I_x(\xi) = f(\langle \xi, \xi\rangle_x).$$

In fact, if we take $\xi, \psi \in \Lambda^1(\mathbb{R}^n)$, $\lambda \in]0,1[$ and set $\eta = \lambda\xi + (1-\lambda)\psi$, by (f$_2$) we have

$$\begin{aligned}(1.6)\quad &\lambda(f(\langle\xi,\xi\rangle_x) - f(\langle\eta,\eta\rangle_x) - 2f'(\langle\eta,\eta\rangle_x)\langle\eta,\xi-\eta\rangle_x)\\ &\qquad\geq \lambda\overline{c}\,\min(\langle\xi-\eta,\xi-\eta\rangle_x^{p/2}, \langle\xi-\eta,\xi-\eta\rangle_x^{q/2}) > 0\end{aligned}$$

and

$$\begin{aligned}(1.7)\quad &(1-\lambda)(f(\langle\psi,\psi\rangle_x) - f(\langle\eta,\eta\rangle_x) - 2f'(\langle\eta,\eta\rangle_x)\langle\eta,\psi-\eta\rangle_x)\\ &\qquad\geq (1-\lambda)\overline{c}\,\min(\langle\psi-\eta,\psi-\eta\rangle_x^{p/2}, \langle\psi-\eta,\psi-\eta\rangle_x^{q/2}) > 0\end{aligned}$$

Since

$$\lambda f'(\langle\eta,\eta\rangle_x)\langle\eta,\xi-\eta\rangle_x + (1-\lambda)f'(\langle\eta,\eta\rangle_x)\langle\eta,\psi-\eta\rangle_x = 0,$$

adding (1.6) to (1.7) we obtain

$$\lambda f(\langle\xi,\xi\rangle_x) + (1-\lambda)f(\langle\psi,\psi\rangle_x) - f(\langle\eta,\eta\rangle_x) > 0$$

and then for every $x \in \mathbb{R}^n$ the functional I_x is strictly convex.

The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} a|x|^{p/2} + b & \text{if } |x| > 1, \\ c|x|^{q/2} & \text{if } |x| \le 1, \end{cases}$$

where $2 < p < 2^* < q$ and $(a, b, c) \in \mathbb{R}^2 \times]0, \infty[$ is any solution of the system

$$\begin{cases} a + b = c, \\ ap = cq, \end{cases}$$

is an example of function satisfying (f_2) (see the Appendix for details).

The paper is organized as follows: in Section 1, following [6], we will use a new functional framework related to the Hodge decomposition of the vector field $\mathbf{A}$. We will be led to study the problem in the space

$$\mathcal{D}(\mathbb{R}^n) := \left\{ u \in L^6(\mathbb{R}^n) : \int |\nabla u|^2 \, dx < \infty \right\}$$

and in the Orlicz space $L^p + L^q$ $(2 < p < 6 < q)$. We will recall some basic theorems, obtained in [6], [17], describing the relations between these spaces, and two results, proved respectively in [17] and [4], which will be necessary to get regularity and compactness.

In Section 2, we will give a proof of Theorem 1.1, using a well known multiplicity abstract result (see [1], [5]). Assumption (f_2) will play a key role in order to get regularity.

Finally, in the appendix we will show an example of function satisfying the assumptions of Theorem 1.1.

2. The functional setting

From now on, taken $\mathbf{A} = \sum_{i=1}^n A_i \, dx^i$ a 1-form, by $\nabla \mathbf{A}$ we mean the Jacobian matrix of the field $(A_1, \ldots, A_n)$ and if $\mathbf{B}$ is another 1-form we will use the notation $(\nabla \mathbf{A} | \nabla \mathbf{B})$ to mean the product

$$(\nabla \mathbf{A} | \nabla \mathbf{B}) = \mathrm{Tr}[(\nabla \mathbf{A})(\nabla \mathbf{B})^T]$$

where $(\nabla \mathbf{B})^T$ is transposed of $\nabla \mathbf{B}$ and Tr denotes the trace. Moreover in the sequel we will write $(\mathbf{A}|\mathbf{B})$ to mean the scalar product between $\mathbf{A}$ and $\mathbf{B}$ and we will use $|\mathbf{A}|^2$ and $|\nabla \mathbf{A}|^2$ in the place of $(\mathbf{A}|\mathbf{A})$ and $(\nabla \mathbf{A} | \nabla \mathbf{A})$.

The functional of the action associated to (1.3) is

$$J(\mathbf{A}) = \frac{1}{2} \int_{\mathbb{R}} \langle d\mathbf{A}, d\mathbf{A} \rangle \, dx - \frac{1}{2} \int_{\mathbb{R}} f(|\mathbf{A}|^2) \, dx \tag{2.1}$$

where $dx = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$, being $\{dx^1, \ldots, dx^n\}$ the canonical basis of $\Lambda^1(\mathbb{R}^n)$. The strongly indefinite nature of the functional J doesn't allow us to approach this problem in a standard way. In other words, the functional

J doesn't present the geometry of the mountain pass in any space with finite codimension. This strongly indefiniteness of the functional depends on the fact that, in general,

$$\int_{\mathbb{R}} \langle d\mathbf{A}, d\mathbf{A}\rangle \, dx \neq \int_{\mathbb{R}} |\nabla \mathbf{A}|^2 \, dx$$

since the equality holds only if $\delta\mathbf{A} = 0$. As a consequence, we don't have an a priori bound on the norm $\|\nabla \mathbf{A}\|_{L^2}$. To overcome this difficulty, we look to the Hodge decomposition theorem of the differential forms in order to split

$$\mathbf{A} = u + dw = u + \nabla w \tag{2.2}$$

where u is a 1-form such that

$$\delta u = 0 \tag{2.3}$$

and w is a 0-form, i.e. $w\colon \mathbb{R}^n \to \mathbb{R}$.

Substituting the splitting (2.2) in (2.1), we obtain

$$J(u, w) := J(u + dw) = \frac{1}{2}\int_{\mathbb{R}} |\nabla u|^2 \, dx - \frac{1}{2}\int_{\mathbb{R}} f(|u + \nabla w|^2) \, dx. \tag{2.4}$$

Now we introduce the spaces where the functional J is defined.

For $2 < p < 2n/(n-2) < q$, denote by $(L^p(\mathbb{R}^n), |\cdot|_p)$ and $(L^q(\mathbb{R}^n), |\cdot|_q)$ the Lebesgue spaces defined as the closure of $\Lambda^1(\mathbb{R}^n)$ with respect to the norm

$$|\xi|_h = \left(\int_{\mathbb{R}} |\xi|^h \, dx\right)^{1/h}, \quad h = p, q.$$

Consider the space

$$L^p + L^q := \{\xi \mid \text{there exists } \xi_1 \in L^p(\mathbb{R}^n) \text{ and } \xi_2 \in L^q(\mathbb{R}^n) \text{ such that } \xi = \xi_1 + \xi_2\}.$$

It is well known that $L^p + L^q$ is a Banach space with the norm

$$\|\xi\|_{L^p+L^q} = \inf\{\|\xi_1\|_{L^p} + \|\xi_2\|_{L^q} : (\xi_1, \xi_2) \in L^p \times L^q,\ \xi_1 + \xi_2 = \xi\} \tag{2.5}$$

and its dual space is $L^{p'} \cap L^{q'}$, where $p' = p/(p-1)$ and $q' = q/(q-1)$, endowed with the norm

$$\|\xi\|_{L^{p'} \cap L^{q'}} := \|\xi\|_{L^{p'}} + \|\xi\|_{L^{q'}}.$$

Denote by $C_0^\infty(\mathbb{R}^n)$ the space of the smooth functions with compact support, and set

$$\mathcal{D}(\mathbb{R}^n) := \overline{\Lambda^1(\mathbb{R}^n)}^{\|\cdot\|} \quad \text{and} \quad \mathcal{D}^{p,q}(\mathbb{R}^n) := \overline{C_0^\infty(\mathbb{R}^n)}^{\|\cdot\|_{p,q}}$$

where, for every $\xi \in \Lambda^1(\mathbb{R}^n)$,

$$\|\xi\|^2 := \int_{\mathbb{R}^n} \langle d\xi, d\xi\rangle \, dx + \int_{\mathbb{R}} \langle \delta\xi, \delta\xi\rangle \, dx$$

and for every $g \in C_0^\infty(\mathbb{R}^n)$

$$\|g\|_{\mathcal{D}^{p,q}} := \|\nabla g\|_{L^p+L^q}.$$

We recall some results on the space $L^p + L^q$.

Theorem 2.1.

(a) $\Lambda^1(\mathbb{R}^n)$ *is dense in* $L^p + L^q$.

(b) *Let* $\xi \in L^p + L^q$ *and set*

$$\Omega_\xi := \{x \in \mathbb{R}^n : |\xi(x)| > 1\}. \tag{2.6}$$

Then

$$\max\left(\|\xi\|_{L^q(\mathbb{R}^n - \Omega_\xi)} - 1, \frac{1}{1 + |\Omega_\xi|^{1/r}}\|\xi\|_{L^p(\Omega_\xi)}\right) \le \|\xi\|_{L^p + L^q} \le \max(\|\xi\|_{L^q(\mathbb{R}^n - \Omega_\xi)}, \|\xi\|_{L^p(\Omega_\xi)}) \tag{2.7}$$

where $r = p\,q/(q-p)$.

(c) *For every* $r \in [p,q]$: $L^r \hookrightarrow L^p + L^q$ *continuously.*

(d) *The embedding*

$$\mathcal{D}(\mathbb{R}^n) \hookrightarrow L^p + L^q \tag{2.8}$$

is continuous.

(e) *Set* $\mathcal{F} := \{\xi\colon \mathbb{R}^n \to \mathbb{R}^n : \xi(gx) = g\xi(x)$ *for all* $g \in T$ *and for almost every* $x \in \mathbb{R}^n\}$ *where* T *is defined by* (1.5), *and define the space* $\mathcal{D}_r(\mathbb{R}^n)$ *as follows*

$$\mathcal{D}_r(\mathbb{R}^n) := \mathcal{D}(\mathbb{R}^n) \cap \mathcal{F}. \tag{2.9}$$

Then $\mathcal{D}_r(\mathbb{R}^n) \hookrightarrow L^p + L^q$ *compactly.*

Proof. (a) It can be easily showed using the definition of the $L^p + L^q$-norm and the density of $\Lambda^1(\mathbb{R}^n)$ in the spaces $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$.

(b) See Lemma 1 in [6].

(c) See Corollary 9 in [17].

(d) It follows from (c) and the Sobolev continuous embedding $\mathcal{D}(\mathbb{R}^n) \hookrightarrow L^{2n/(n-2)}$.

(e) The proof follows combining a compactness theorem presented in [4] (see Theorem A.1 in the Appendix) and Lemma 14 in [6]. □

For all $\mathbf{A} \in L^p + L^q$, consider the functional F defined as follows

$$F(\mathbf{A}) := \int_{\mathbb{R}^n} f(|\mathbf{A}|^2)\,dx. \tag{2.10}$$

The following results have been proved in [17].

Theorem 2.2. *If* f_3) *holds, then the functional* F *is continuously differentiable, and its Frechet differential is the continuous and bounded map*

$$DF\colon \mathbf{A} \in L^p + L^q \mapsto 2\int_{\mathbb{R}} f'(|\mathbf{A}|^2)(\mathbf{A}|\cdot)\,dx \in (L^p + L^q)'. \tag{2.11}$$

Using the fact that $f(0) = 0$, from (f_2) we deduce that for every $\xi \in L^p + L^q$

$$f(\langle \xi, \xi\rangle) \geq c_1 \min(\langle \xi, \xi\rangle^{p/2}, \langle \xi, \xi\rangle^{q/2})$$

pointwise almost everywhere in $\mathbb{R}^n$. On the other hand, from (f_1) and (f_3) it follows that

$$f(\langle \xi, \xi\rangle) \leq c_2 \min(\langle \xi, \xi\rangle^{p/2}, \langle \xi, \xi\rangle^{q/2}),$$

pointwise almost everywhere in $\mathbb{R}^n$. So, for every $\xi \in L^p + L^q$

$$c_1 \min(\langle \xi, \xi\rangle^{p/2}, \langle \xi, \xi\rangle^{q/2}) \leq f(\langle \xi, \xi\rangle) \leq c_2 \min(\langle \xi, \xi\rangle^{p/2}, \langle \xi, \xi\rangle^{q/2}),$$

and then we deduce that for any $\xi \in L^p + L^q$:

$$\begin{aligned} c_1\bigg(\int_{\Omega_\xi} |\xi|^p\,dx + \int_{\mathbb{R}^n - \Omega_\xi} |\xi|^q\,dx\bigg) \\ \leq \int_{\mathbb{R}} f(\xi) \leq c_2\bigg(\int_{\Omega_\xi} |\xi|^p\,dx + \int_{\mathbb{R}^n - \Omega_\xi} |\xi|^q\,dx\bigg) \end{aligned} \tag{2.12}$$

By (2.7) and (2.8),

$$J(u, w) < \infty \quad \text{for all } u \in \mathcal{D}(\mathbb{R}^n),\ w \in \mathcal{D}^{p,q}(\mathbb{R}^n). \tag{2.13}$$

In order to have compactness for J, we are going to restrict the domain of the functional to a subspace $H \subset \mathcal{D}(\mathbb{R}^n) \times \mathcal{D}^{p,q}(\mathbb{R}^n)$ such that for all $(u, w) \in H$ we have that $u + \nabla w \in \mathcal{F}$.

It is easy to see that, if we set

$$\mathcal{F}' := \{w\colon \mathbb{R}^n \to \mathbb{R} : w(gx) = w(x) \text{ for all } g \in T \text{ and for almost every } x \in \mathbb{R}^n\},$$

then, for $w\colon \mathbb{R}^n \to \mathbb{R}$ sufficiently smooth, we have

$$\text{if } w \in \mathcal{F}' \text{ then } \nabla w \in \mathcal{F}.$$

So, taking (2.3) into account, we set

$$\mathcal{V} := \{u \in \mathcal{D}_r(\mathbb{R}^n) : \delta u = 0\} \quad \text{and} \quad \mathcal{W} := D^{p,q}(\mathbb{R}^n) \cap \mathcal{F}',$$

and we take $H = \mathcal{V} \times \mathcal{W}$.

Observe that H is nonempty. In fact, $\mathcal{W} \neq \emptyset$ and for any $(a_i)_{1 \leq i \leq n/2}$ in $C_0^\infty(\mathbb{R}^n) \cap \mathcal{F}'$ the 1-form

$$\xi = \sum_{i=1}^{n/2} a_i(x_{2i-1}dx_{2i} - x_{2i}dx_{2i-1})$$

belongs to $\mathcal{V}$.

Now, for every $u \in \mathcal{V}$ and $w \in \mathcal{W}$, set

$$
\begin{aligned}
F_u &\colon w \in \mathcal{W} \mapsto F(u + \nabla w) \in \mathbb{R}, \\
F_w &\colon u \in \mathcal{V} \mapsto F(u + \nabla w) \in \mathbb{R}, \\
J_u &\colon w \in \mathcal{W} \mapsto J(u, w) \in \mathbb{R}, \\
J_w &\colon u \in \mathcal{V} \mapsto J(u, w) \in \mathbb{R}.
\end{aligned}
\tag{2.14}
$$

Remark 2.3. Observe that, by Theorem 2.2, for every $u \in \mathcal{V}$ and $w \in \mathcal{W}$ the functionals J, J_u, J_wm F_u and F_w are continuously differentiable, and the respective Frechet differentials are:

$$
\begin{aligned}
&DJ\colon \mathcal{V} \times \mathcal{W} \to (\mathcal{V} \times \mathcal{W})', \\
&\begin{matrix} DJ_u \\ DF_u \end{matrix} \colon \mathcal{W} \to \mathcal{W}', \\
&\begin{matrix} DJ_w \\ DF_w \end{matrix} \colon \mathcal{V} \to \mathcal{V}'.
\end{aligned}
$$

Moreover, if we set

$$
\frac{\partial J}{\partial w}(u, w) := DJ_u(w) \in \mathcal{W}' \tag{2.15}
$$

$$
\frac{\partial J}{\partial u}(u, w) := DJ_w(u) \in \mathcal{V}', \tag{2.16}
$$

by some computations we can see that, for every $u, \overline{u} \in \mathcal{V}$ and $w, \overline{w} \in \mathcal{W}$,

$$
\begin{aligned}
\frac{\partial J}{\partial w}(u, w)[\overline{w}] &= DJ(u, w)[0, \overline{w}] \\
&= -\int_{\mathbb{R}} f'(|u + \nabla w|^2)(u + \nabla w | \nabla \overline{w})\, dx,
\end{aligned}
\tag{2.17}
$$

$$
\begin{aligned}
\frac{\partial J}{\partial u}(u, w)[\overline{u}] &= DJ(u, w)[\overline{u}, 0] \\
&= \int_{\mathbb{R}} (\nabla u | \nabla \overline{u})\, dx - \int_{\mathbb{R}} f'(|u + \nabla w|^2)(u + \nabla w | \overline{u})\, dx.
\end{aligned}
\tag{2.18}
$$

Using (2.15) and (2.16), we can show the variational nature of the problem (1.4)

Theorem 2.4. *If the couple* $(u, w) \in \mathcal{V} \times \mathcal{W}$ *solves the system*

$$
\frac{\partial J}{\partial w}(u, w) = 0, \tag{2.19}
$$

$$
\frac{\partial J}{\partial v}(u, w) = 0 \tag{2.20}
$$

then $\mathbf{A} = u + \nabla w \in \mathcal{F}$ *is a finite energy, weak solution of* (1.4).

Proof. Let $(u,w) \in \mathcal{V} \times \mathcal{W}$ be a solution of (2.19) and (2.20). Then, by (2.17) and (2.18), for any $\overline{u} \in \mathcal{V}$ and $\overline{w} \in \mathcal{W}$

$$\int_{\mathbb{R}} f'(|u+\nabla w|^2)(u+\nabla w|\nabla\overline{w})\,dx = 0, \tag{2.21}$$

$$\int_{\mathbb{R}} (\nabla u|\nabla\overline{u})\,dx - \int_{\mathbb{R}} f'(|u+\nabla w|^2)(u+\nabla w|\overline{u})\,dx = 0. \tag{2.22}$$

We want to show that $\mathbf{A} = u + \nabla w$ is a weak solution of (1.4), namely for all $\varphi \in \Lambda^1(\mathbb{R}^n)$

$$DJ(\mathbf{A})[\varphi] = \int_{\mathbb{R}} \langle d\mathbf{A}, d\varphi\rangle\,dx - \int_{\mathbb{R}} f'(|\mathbf{A}|^2)(\mathbf{A}|\varphi)\,dx = 0. \tag{2.23}$$

Actually, it is enough to prove (2.23) just for every $\varphi \in \mathcal{D}_r$, since $\mathcal{D}_r$ is a natural constraint for J. In fact observe that, if we denote by $\mathcal{T}$ the group of isometric transformations on $\mathcal{D}(\mathbb{R}^n)$ defined as follows

$$G \in \mathcal{T} \text{ if and only if there exists } g \in T \text{ such that } G(\mathbf{A})(x) = g^{-1}\mathbf{A}(gx)$$
$$\text{for all } \mathbf{A} \in \mathcal{D} \text{ and for almost every } x \in \mathbb{R}^n.$$

then $\mathcal{D}_r$ is the subspace of the fix points of $\mathcal{D}(\mathbb{R}^n)$ under the action of $\mathcal{T}$ and

$$J(G(\mathbf{A})) = J(\mathbf{A}) \quad \text{for all } G \in \mathcal{T} \text{ and all } \mathbf{A} \in \mathcal{D}(\mathbb{R}^n).$$

Then, by the Palais' Principle of symmetric criticality (see [16]), $\mathcal{D}_r$ is a natural constraint. Let $\varphi \in \mathcal{D}_r$. As in (2.2), we can split the function φ and obtain

$$\varphi = v + dh = v + \nabla h \tag{2.24}$$

where $v \in \mathcal{V}$ and $h \in \mathcal{W}$. Writing (2.21) and (2.22) with respectively $\overline{u} = v$ and $\overline{w} = h$, we get

$$\int_{\mathbb{R}} f'(|u+\nabla w|^2)(u+\nabla w|\nabla h)\,dx = 0, \tag{2.25}$$

$$\int_{\mathbb{R}} (\nabla u|\nabla v)\,dx - \int_{\mathbb{R}} f'(|u+\nabla w|^2)(u+\nabla w|v)\,dx = 0, \tag{2.26}$$

so, subtracting (2.25) from (2.26), by (2.24) we have

$$\int_{\mathbb{R}} (\nabla u|\nabla v)\,dx - \int_{\mathbb{R}} f'(|u+\nabla w|^2)(u+\nabla w|\varphi)\,dx = 0. \tag{2.27}$$

Since $\delta v = 0$, then

$$\delta d\varphi = \delta d(v + dh) = \delta dv = -\Delta v, \tag{2.28}$$

where $-\Delta := d\delta + \delta d$ is the Laplace–Beltrami operator. From (2.27) and (2.28), we deduce that

$$\begin{aligned}\int_{\mathbb{R}} \langle du, d\varphi\rangle\,dx &- \int_{\mathbb{R}} f'(|u+\nabla w|^2)(u+\nabla w|\varphi)\,dx\\ &= \int_{\mathbb{R}} \langle u, \delta d\varphi\rangle\,dx - \int_{\mathbb{R}} f'(|u+\nabla w|^2)(u+\nabla w|\varphi)\,dx\\ &= -\int_{\mathbb{R}} (u|\Delta v)\,dx - \int_{\mathbb{R}} f'(|u+\nabla w|^2)(u+\nabla w|\varphi)\,dx\\ &= \int_{\mathbb{R}} (\nabla u|\nabla v)\,dx - \int_{\mathbb{R}} f'(|u+\nabla w|^2)(u+\nabla w|\varphi)\,dx = 0.\end{aligned}$$

Since $u \in \mathcal{D}(\mathbb{R}^n)$ and $w \in \mathcal{D}^{p,q}(\mathbb{R}^n)$, by (2.13) the energy of $\mathbf{A}$ is finite.

Finally, since $u, \nabla w \in \mathcal{F}$, also $\mathbf{A} \in \mathcal{F}$. □

3. Proof of the main theorem

Set

$$\mathcal{C}_1 := \left\{(u,w) \in \mathcal{V}\times\mathcal{W} : \frac{\partial J}{\partial w}(u,w) = 0\right\}, \tag{3.1}$$

$$\mathcal{C}_2 := \left\{(u,w) \in \mathcal{V}\times\mathcal{W} : \frac{\partial J}{\partial u}(u,w) = 0\right\}. \tag{3.2}$$

By Theorem 2.4, we are interested in finding the couples $(u,w) \in \mathcal{C}_1 \cap \mathcal{C}_2$. Rendering (3.2) explicit we have that

(3.3) $(u,w) \in \mathcal{C}_2$ for all $\overline{u} \in \mathcal{V}$

$$\int_{\mathbb{R}} (\nabla u|\nabla \overline{u})\,dx - \int_{\mathbb{R}} f'(|u+\nabla w|^2)(u+\nabla w|\overline{u})\,dx = 0.$$

The following theorem characterizes the set $\mathcal{C}_1$

Theorem 3.1. *There exists a compact map $\Phi\colon \mathcal{V} \to \mathcal{W}$ such that*

$$\mathcal{C}_1 = \{(u,\Phi(u)) : u \in \mathcal{V}\}. \tag{3.4}$$

Moreover the map Φ is characterized by the following property:

(3.5) *for every $u \in \mathcal{V}$, $\Phi(u)$ is the unique function in $\mathcal{W}$ such that*

$$F_u(\Phi(u)) = \min_{w\in\mathcal{W}} F_u(w).$$

Before we prove the Theorem 3.1, we need the following

Lemma 3.2. *If*

$$\zeta_n \rightharpoonup \zeta \quad \text{in } L^p + L^q \quad \text{and} \quad F(\zeta_n) \to F(\zeta), \tag{3.6}$$

then

$$\zeta_n \to \zeta \quad \text{in } L^p + L^q. \tag{3.7}$$

Proof. Let $(\zeta_n)_n$ be a sequence in $L^p + L^q$ and $\zeta \in L^p + L^q$ such that (3.6) hold. Using (f_2) for $(\zeta_n)_x$ and $(\zeta)_x$ for all $x \in \mathbb{R}^n$ and $n \geq 1$, we have that the following inequality holds pointwise:

$$f(|\zeta_n|^2) - f(|\zeta|^2) - 2f'(|\zeta|^2)(\zeta|\zeta_n - \zeta) \geq c_1 \min(|\zeta_n - \zeta|^p, |\zeta_n - \zeta|^q). \tag{3.8}$$

Set $\Omega_n := \{x \in \mathbb{R}^n : |\zeta_n - \zeta| > 1\}$. Integrating in inequality (3.8), by Theorem 2.2 we get

$$\begin{aligned} F(|\zeta_n|^2) - F(|\zeta|^2) &- DF(\zeta)(\zeta_n - \zeta) \\ &\geq c_1 \int_{\Omega_n} |\zeta_n - \zeta|^p \, dx + c_1 \int_{\mathbb{R}^n - \Omega_n} |\zeta_n - \zeta|^q \, dx \\ &= c_1 \left(\|\zeta_n - \zeta\|^p_{L^p(\Omega_n)} + \|\zeta_n - \zeta\|^q_{L^q(\mathbb{R}^n - \Omega_n)}\right), \end{aligned} \tag{3.9}$$

By (3.6) and (3.9) we have that

$$\|\zeta_n - \zeta\|^p_{L^p(\Omega_n)} + \|\zeta_n - \zeta\|^q_{L^q(\mathbb{R}^n - \Omega_n)} \to 0,$$

and then we get (3.7) by (2.7). □

Proof of Theorem 3.1. Let $u \in \mathcal{V}$ and consider F_u defined as in (2.14). By Remarks 2.3 and 1.3, F_u is continuous and strictly convex. Then F_u is weakly lower semicontinuous.

Moreover, F_u is also coercive. In fact, if $w \in \mathcal{W}$ and we set

$$\Omega := \{x \in \mathbb{R}^n : |u(x) + \nabla w(x)| > 1\},$$

then, by (2.12), we have

$$\begin{aligned} F_u(w) &= \int_{\mathbb{R}} f(|u + \nabla w|^2) \, dx \\ &= \int_{\mathbb{R}^n - \Omega} f(|u + \nabla w|^2) \, dx + \int_{\Omega} f(|u + \nabla w|^2) \, dx \\ &\geq c_1 \int_{\mathbb{R}^n - \Omega} |u + \nabla w|^q \, dx + c_1 \int_{\Omega} |u + \nabla w|^p \, dx. \end{aligned} \tag{3.10}$$

By (3.10) and (2.7) we deduce that F_u is coercive and then, by Weierstrass theorem, F_u possesses a minimizer in $\mathcal{W}$. So, let Φ be the map defined as follows

$$\Phi \colon \mathcal{V} \to \mathcal{W} \quad \text{such that, for all } u \in \mathcal{V}, \Phi(u) \text{ minimizes } F_u. \tag{3.11}$$

Since F_u is strictly convex, for all $u \in \mathcal{V}$ the minimizer of the functional F_u is unique, and then the map Φ is well defined and satisfies (3.5).

Now, before we prove the compactness of $\Phi \colon \mathcal{V} \to \mathcal{W}$, first we show that the functional

$$u \in \mathcal{V} \mapsto \int_{\mathbb{R}^n} f(|u + \nabla\Phi(u)|^2) \, dx \tag{3.12}$$

is weakly continuous. Let

$$u_n \rightharpoonup u \quad \text{in } \mathcal{V}, \tag{3.13}$$

then, by Theorem 2.1(e),

$$u_n \to u \quad \text{in } L^p + L^q. \tag{3.14}$$

Since

$$0 \le F(u_n + \nabla\Phi(u_n)) = F_{u_n}(\Phi(u_n)) \le F_{u_n}(0) = F(u_n),$$

by (3.14) and the continuity of F, the sequence $\{F(u_n + \nabla\Phi(u_n))\}$ is bounded. Since F is coercive, then

$$u_n + \nabla\Phi(u_n) \quad \text{is bounded in } L^p + L^q,$$

so, by (3.14),

$$\nabla\Phi(u_n) \quad \text{is bounded in } L^p + L^q. \tag{3.15}$$

Set

$$\begin{aligned}
\alpha_n &:= \int_{\mathbb{R}} f(|u_n + \nabla\Phi(u_n)|^2)\,dx - \int_{\mathbb{R}} f(|u + \nabla\Phi(u_n)|^2)\,dx \\
\beta_n &:= \int_{\mathbb{R}} f(|u_n + \nabla\Phi(u_n)|^2)\,dx - \int_{\mathbb{R}} f(|u + \nabla\Phi(u)|^2)\,dx \\
\gamma_n &:= \int_{\mathbb{R}} f(|u_n + \nabla\Phi(u)|^2)\,dx - \int_{\mathbb{R}} f(|u + \nabla\Phi(u)|^2)\,dx
\end{aligned}$$

By (3.11), certainly we have

$$\alpha_n \le \beta_n \le \gamma_n. \tag{3.16}$$

Moreover, by Lagrange theorem,

$$\begin{aligned}
\alpha_n &= \int_{\mathbb{R}} (f(|u_n + \nabla\Phi(u_n)|^2) - f(|u + \nabla\Phi(u_n)|^2))\,dx \\
&= 2\int_{\mathbb{R}} f'(|\theta_n|^2)(\theta_n|u_n - u)\,dx
\end{aligned} \tag{3.17}$$

where θ_n is a suitable convex combination of $u_n+\nabla\Phi(u_n)$ and $u+\nabla\Phi(u_n)$. Since $\{u_n\}$ and $\{\nabla\Phi(u_n)\}$ are bounded in $L^p + L^q$, certainly also $\{\theta_n\}$ is bounded in $L^p + L^q$. Then, by Theorem 2.2 and (3.14), from (3.17) we deduce that

$$\alpha_n \to 0. \tag{3.18}$$

Analogously we also have that

$$\gamma_n \to 0, \tag{3.19}$$

so, by (3.16), (3.18) and (3.19), we get $\beta_n \to 0$ and then (3.12) is weakly continuous.

Now, we prove the compactness of Φ. Consider again $(u_n)_{n\geq 1}$ in $\mathcal{V}$ such that (3.13) holds. By (3.15), there exists $w \in \mathcal{W}$ such that (up to a subsequence)

$$\nabla\Phi(u_n) \rightharpoonup \nabla w \quad \text{in } L^p + L^q. \tag{3.20}$$

From (3.14) and (3.20) we deduce that

$$u_n + \nabla\Phi(u_n) \rightharpoonup u + \nabla w \quad \text{in } L^p + L^q \tag{3.21}$$

so, using the weak continuity of (3.12) and the weak lower semicontinuity of F we have

$$\begin{aligned} F_u(\Phi(u)) &= F(u + \nabla\Phi(u)) \\ &= \lim_n F(u_n + \nabla\Phi(u_n)) \geq F(u + \nabla w) = F_u(w). \end{aligned} \tag{3.22}$$

By the uniqueness of the minimizer of F_u, from (3.22) we deduce that $w = \Phi(u)$, so, by (3.21), we have

$$u_n + \nabla\Phi(u_n) \rightharpoonup u + \nabla\Phi(u) \quad \text{in } L^p + L^q. \tag{3.23}$$

But using the weak continuity of (3.12), by (3.13) we also have

$$\int_{\mathbb{R}} f(|u_n + \nabla\Phi(u_n)|^2)\,dx \to \int_{\mathbb{R}} f(|u + \nabla\Phi(u)|^2)\,dx \tag{3.24}$$

so, by Lemma 3.2, from (3.23) and (3.24) we deduce that

$$u_n + \nabla\Phi(u_n) \longrightarrow u + \nabla\Phi(u) \quad \text{in } L^p + L^q. \tag{3.25}$$

Now, comparing (3.25) with (3.14), we deduce that $\Phi(u_n) \longrightarrow \Phi(u)$ in $\mathcal{W}$ and then Φ is compact.

Finally, we prove (3.4). Observe that, since $(\partial J/\partial w)(u,w) = DF_u(w)$, then

$$(u,w) \in \mathcal{C}_1 \text{ if and only if } DF_u(w) = 0. \tag{3.26}$$

But since F_u is convex, its critical points are minimizers, and then

$$DF_u(w) = 0 \text{ if and only if } w = \Phi(u), \tag{3.27}$$

so we have (3.4) by (3.26) and (3.27). $\square$

Consider the functional $\widehat{J}\colon \mathcal{V} \to \mathbb{R}$

$$\widehat{J}(u) := J(u, \Phi(u)) = \frac{1}{2}\int_{\mathbb{R}} |\nabla u|^2\,dx - \frac{1}{2}F(u + \nabla\Phi(u)). \tag{3.28}$$

The following regularity result holds:

Theorem 3.3. *The functional $\widehat{J}$ is continuously differentiable and its Frechet differential $D\widehat{J}\colon \mathcal{V} \to \mathcal{V}'$ has this expression*

$$D\widehat{J}(u)[\overline{u}] = \int_{\mathbb{R}} (\nabla u|\nabla \overline{u})\, dx - \int_{\mathbb{R}} f'(|u + \nabla\Phi(u)|^2)(u + \nabla\Phi(u)|\overline{u})\, dx. \tag{3.29}$$

Proof. Set $\widehat{F}\colon u \in \mathcal{V} \mapsto F(u + \nabla\Phi(u))$. We will prove that $\widehat{F} \in C^1$ so that, clearly, also $\widehat{J} \in C^1$.

Let $u \in \mathcal{V}$. We claim that for all $\overline{u} \in \mathcal{V} - \{0\}$ the functional $\widehat{F}$ is derivable at u in the direction $\overline{u}$, and the directional derivative (i.e. the Gâteaux derivative $D_G\widehat{F}$) is

$$D_G\widehat{F}(u)[\overline{u}] = 2\int_{\mathbb{R}} f'(|u + \nabla\Phi(u)|^2)(u + \nabla\Phi(u)|\overline{u})\, dx. \tag{3.30}$$

In fact, let $t \in \mathbb{R} - \{0\}$ and set

$$\begin{aligned}
\alpha(t) &:= F(u + t\overline{u} + \nabla\Phi(u + t\overline{u})) - F(u + \nabla\Phi(u + t\overline{u})),\\
\beta(t) &:= F(u + t\overline{u} + \nabla\Phi(u + t\overline{u})) - F(u + \nabla\Phi(u)),\\
\gamma(t) &:= F(u + t\overline{u} + \nabla\Phi(u)) - F(u + \nabla\Phi(u)).
\end{aligned}$$

By (3.5) we know that

$$\begin{aligned}
F(u + t\overline{u} + \nabla\Phi(u + t\overline{u})) &\le F(u + t\overline{u} + \nabla\Phi(u)),\\
F(u + \nabla\Phi(u)) &\le F(u + \nabla\Phi(u + t\overline{u})),
\end{aligned}$$

and then, certainly, for every $t \in \mathbb{R} - \{0\}$

$$\alpha(t) \le \beta(t) \le \gamma(t). \tag{3.31}$$

Now, for every $t \in \mathbb{R} - \{0\}$, set

$$\widetilde{\alpha}(t) = \frac{\alpha(t)}{t}, \qquad \widetilde{\beta}(t) = \frac{\beta(t)}{t}, \qquad \widetilde{\gamma}(t) = \frac{\gamma(t)}{t}$$

and observe that (3.30) means that

$$\lim_{t\to 0} \widetilde{\beta}(t) = 2\int_{\mathbb{R}} f'(|u + \nabla\Phi(u)|^2)(u + \nabla\Phi(u)|\overline{u})\, dx. \tag{3.32}$$

From (3.31) we deduce that

$$\begin{aligned}
\widetilde{\alpha}(t) &\le \widetilde{\beta}(t) \le \widetilde{\gamma}(t) \quad \text{if } t > 0,\\
\widetilde{\gamma}(t) &\le \widetilde{\beta}(t) \le \widetilde{\alpha}(t) \quad \text{if } t < 0,
\end{aligned}$$

and then

$$\min(\widetilde{\alpha}(t), \widetilde{\gamma}(t)) \le \widetilde{\beta}(t) \le \max(\widetilde{\alpha}(t), \widetilde{\gamma}(t)). \tag{3.33}$$

Now, by Lagrange theorem, we have that

$$\begin{aligned}\widetilde{\alpha}(t) &= \frac{1}{t}\int_{\mathbb{R}} (f(|u+t\overline{u}+\nabla\Phi(u+t\overline{u})|^2) - f(|u+\nabla\Phi(u+t\overline{u})|^2))\,dx \\ &= \frac{2}{t}\int_{\mathbb{R}} f'(|\theta_t|^2)(\theta_t|t\overline{u})\,dx = 2\int_{\mathbb{R}} f'(|\theta_t|^2)(\theta_t|\overline{u})\,dx = DF(\theta_t)[\overline{u}]\end{aligned} \tag{3.34}$$

where θ_t is a suitable convex combination of $u+t\overline{u}+\nabla\Phi(u+t\overline{u})$ and $u+\nabla\Phi(u+t\overline{u})$.

Since Φ is continuous, we have that

$$\begin{aligned}\lim_{t\to 0} u+t\overline{u}+\nabla\Phi(u+t\overline{u}) &= u+\nabla\Phi(u) \quad \text{in } L^p+L^q, \\ \lim_{t\to 0} u+\nabla\Phi(u+t\overline{u}) &= u+\nabla\Phi(u) \quad \text{in } L^p+L^q,\end{aligned}$$

and then

$$\lim_{t\to 0}\theta_t = u+\nabla\Phi(u) \quad \text{in } L^p+L^q. \tag{3.35}$$

By continuity, from (3.34) and (3.35) we deduce that

$$\begin{aligned}\lim_{t\to 0}\widetilde{\alpha}(t) &= DF(u+\nabla\Phi(u))[\overline{u}] \\ &= 2\int_{\mathbb{R}} f'(|u+\nabla\Phi(u)|^2)(u+\nabla\Phi(u)|\overline{u})\,dx.\end{aligned} \tag{3.36}$$

By the same arguments, we can see that

$$\lim_{t\to 0}\widetilde{\gamma}(t) = 2\int_{\mathbb{R}} f'(|u+\nabla\Phi(u)|^2)(u+\nabla\Phi(u)|\overline{u})\,dx, \tag{3.37}$$

so, by (3.33), (3.36) and (3.37) we get (3.32), i.e. and the existence of the directional derivative.

Now observe that from (3.30) we have

$$D_G\widehat{F}(u) \in \mathcal{V}', \quad \text{for all } u \in \mathcal{V}$$

and the map

$$D\widehat{F}_G\colon u \in \mathcal{V} \mapsto 2\int_{\mathbb{R}} f'(|u+\nabla\Phi(u)|^2)(u+\nabla\Phi(u)|\,\cdot\,)\,dx \in \mathcal{V}' \tag{3.38}$$

is continuous by Theorem 2.2 and the continuity of Φ. Then $\widehat{F}$ is Frechet differentiable, and, for all $u,\overline{u} \in \mathcal{V}$

$$D\widehat{F}(u)[\overline{u}] = 2\int_{\mathbb{R}} f'(|u+\nabla\Phi(u)|^2)(u+\nabla\Phi(u)|\overline{u})\,dx. \tag{3.39}$$

From (3.39) we have (3.29). $\square$

Theorem 3.4. *If $u \in \mathcal{V}$ is a nontrivial critical point of $\widehat{J}$, then $\mathbf{A} = u + \nabla\Phi(u) \in \mathcal{F}$ is a finite energy, nontrivial weak solution of* (1.4).

Proof. Let $u \in \mathcal{V}$ be a critical point of $\widehat{J}$. By (3.29) we have that

$$\int_{\mathbb{R}} (\nabla u | \nabla \overline{u})\, dx - \int_{\mathbb{R}} f'(|u + \nabla\Phi(u)|^2)(u + \nabla\Phi(u) | \overline{u})\, dx = 0$$

so, by (3.3), the couple $(u, \Phi(u)) \in \mathcal{C}_2$. Since by Theorem 3.1 we also have that $(u, \Phi(u)) \in \mathcal{C}_1$, then, by Theorem 2.4, $\mathbf{A} = u + \nabla\Phi(u)$ is a finite energy, weak solution. Moreover, if $u \neq 0$, then

$$u + \nabla\Phi(u) \neq 0. \tag{3.40}$$

In fact, if

$$u = -\nabla\Phi(u), \tag{3.41}$$

then

$$-\Delta\Phi(u) = \nabla \cdot u = 0,$$

and this should imply

$$\int_{\mathbb{R}} |\nabla\Phi(u)|^2\, dx = 0,$$

that is

$$\nabla\Phi(u) = 0. \tag{3.42}$$

But (3.41) and (3.42) contradict the fact that $u \neq 0$, so (3.40) holds. □

By Theorem 3.4 we are reduced to find the critical points of $\widehat{J}$, so we are going to study the geometry and the compactness properties of the functional in order to apply the symmetrical mountain pass theorem (see [1], [5]).

Theorem 3.5. *$\widehat{J}$ satisfies the following Palais–Smale condition:*

(PS) *if $\{u_n\} \in \mathcal{V}$ is a sequence such that for $M \geq 0$*

$$\widehat{J}(u_n) \leq M, \quad \text{for all } n \geq 1 \tag{3.43}$$

and

$$D\widehat{J}(u_n) \to 0, \tag{3.44}$$

then $\{u_n\} \in \mathcal{V}$ is precompact.

Proof. Let $\{u_n\} \in \mathcal{V}$ be a sequence such that (3.43) and (3.44) hold. Since Φ is compact and the embedding $\mathcal{V} \hookrightarrow L^p + L^q$ is compact, we have that the map (3.38) is compact, so, by standard arguments we are reduced to prove that $\{u_n\}$ is bounded.

Rendering (3.43) explicit, we have

$$\frac{1}{2}\int_{\mathbb{R}}|\nabla u_n|^2\,dx-\frac{1}{2}\int_{\mathbb{R}}f(|u_n+\nabla\Phi(u_n)|^2)\,dx\le M. \tag{3.45}$$

Moreover, from (3.44) we deduce that $D\widehat{J}(u_n)[u_n/\|u_n\|_{\mathcal{D}}]\longrightarrow 0$, that is, there exists $\varepsilon_n\to 0$ such that

$$\begin{aligned}\int_{\mathbb{R}}|\nabla u_n|^2\,dx-\int_{\mathbb{R}}f'(|u_n+\nabla\Phi(u_n)|^2)(u_n+\nabla\Phi(u_n)\big|u_n)\,dx&\\ =\varepsilon_n\|u_n\|_{\mathcal{D}}.&\end{aligned} \tag{3.46}$$

Now, by (3.5), certainly we have that, for every $w\in\mathcal{W}$,

$$0=DF_{u_n}(\Phi(u_n))[w]=\int_{\mathbb{R}}f'(|u_n+\nabla\Phi(u_n)|^2)(u_n+\nabla\Phi(u_n)|\nabla w)\,dx,$$

so (3.46) can be written as follows

$$\int_{\mathbb{R}}|\nabla u_n|^2\,dx-\int_{\mathbb{R}}f'(|v_n|^2)|v_n|^2\,dx=\varepsilon_n\|u_n\|_{\mathcal{D}}, \tag{3.47}$$

where we have set $v_n=u_n+\nabla\Phi(u_n)$. Now, multiplying (3.45) by α and subtracting (3.47) we get

$$\begin{aligned}\Big(\frac{\alpha}{2}-1\Big)\int_{\mathbb{R}}|\nabla u_n|^2\,dx+\int_{\mathbb{R}}\Big[f'(|v_n|^2)|v_n|^2-\frac{\alpha}{2}f(|v_n|^2)\Big]\,dx&\\ \le M-\varepsilon_n\|u_n\|_{\mathcal{D}}.&\end{aligned} \tag{3.48}$$

Using (f_4), from (3.48) we deduce that $\{u_n\}$ is bounded. □

Theorem 3.7. *There exist $\rho>0$ and $C>0$ such that*

$$\widehat{J}(u)>C,\quad \text{for all } u\in\mathcal{V}\cap S_\rho,$$

where $S_\rho:=\{u\in\mathcal{D}:\|u\|_{\mathcal{D}}=\rho\}$.

Proof. Let $u\in\mathcal{V}$ and consider Ω_u defined as in (2.6). Since $p<2^*<q$ we have that

$$|u(x)|^p\le|u(x)|^{2^*}\quad\text{if } x\in\Omega_u, \tag{3.49}$$

$$|u(x)|^q\le|u(x)|^{2^*}\quad\text{if } x\in\mathbb{R}^n-\Omega_u, \tag{3.50}$$

so, by (3.49) and (3.50), using (2.12), (3.5) and the continuous embedding

$$(\mathcal{D},\|\cdot\|_{\mathcal{D}})\hookrightarrow(L^{2^*},|\cdot|_{2^*}),$$

for a suitable $k > 0$ we have

$$\begin{aligned}\widehat{J}(u) &= \frac{1}{2}\int_{\mathbb{R}} |\nabla u|^2\,dx - \frac{1}{2}\int_{\mathbb{R}} f(|u+\nabla\Phi(u)|^2)\,dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}} |\nabla u|^2\,dx - \frac{1}{2}\int_{\mathbb{R}} f(|u|^2)\,dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}} |\nabla u|^2\,dx - \frac{c_2}{2}\int_{\Omega_u} |u|^p\,dx - \frac{c_2}{2}\int_{\mathbb{R}^n-\Omega_u} |u|^q\,dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}} |\nabla u|^2\,dx - \frac{c_2}{2}\int_{\Omega_u} |u|^{2^*}\,dx - \frac{c_2}{2}\int_{\mathbb{R}^n-\Omega_u} |u|^{2^*}\,dx \\ &= \frac{1}{2}\|u\|^2_{\mathcal{D}} - \frac{c_2}{2}|u|^{2^*}_{2^*} \geq \frac{1}{2}\|u\|^2_{\mathcal{D}} - k\|u\|^{2^*}_{\mathcal{D}}.\end{aligned}$$

Then $\widehat{J}(u) > C$ for $u \in S_\rho$ with ρ small enough. □

Now, before we prove that also the second geometrical assumption of the symmetrical mountain pass theorem is satisfied, we need a preliminary result. For every $\gamma > 1$ and $u \in \mathcal{V}$ set

$$\widetilde{F}_u : w \in \mathcal{W} \mapsto \|u + \nabla w\|^\gamma_{L^p+L^q}.$$

We have the following:

Lemma 3.8. *For every $u \in \mathcal{V}$ there exists a unique $\Phi_\gamma(u) \in \mathcal{W}$ such that*

$$\widetilde{F}_u(\Phi_\gamma(u)) = \min_{w\in\mathcal{W}} \widetilde{F}_u(w).$$

Moreover, for every $V \subset \mathcal{V}$ such that $\dim V < \infty$ we have

$$\text{there exists } \widetilde{C}_\gamma(V) > 0 \text{ such that } \|u + \nabla\Phi_\gamma(u)\|^\gamma_{L^p+L^q} \geq \widetilde{C}_\gamma \|u\|^\gamma_{\mathcal{D}} \tag{3.51}$$

uniformly for $u \in V$.

Proof. Since $\widetilde{F}_u$ is strictly convex, continuous and coercive on $\mathcal{W}$, by Weierstrass theorem there exists a unique minimizer $\Phi_\gamma(u)$.

Actually the minimizing map $\Phi_\gamma : u \to \Phi_\gamma(u)$ is compact from $\mathcal{V}$ into $\mathcal{W}$.

In fact, consider $u_n \rightharpoonup u$ in $\mathcal{V}$. Since $\mathcal{V} \hookrightarrow L^p + L^q$ compactly, certainly

$$u_n \to u \text{ in } L^p + L^q. \tag{3.52}$$

Moreover, by the definition of Φ_γ,

$$0 \leq \|u_n + \nabla\Phi_\gamma(u_n)\|^\gamma_{L^p+L^q} \leq \|u_n\|^\gamma_{L^p+L^q}$$

so,

$$u_n + \nabla\Phi_\gamma(u_n) \quad \text{is bounded in } L^p + L^q. \tag{3.53}$$

From (3.52) and (3.53) we deduce that $\{\Phi_\gamma(u_n)\}$ is bounded in $\mathcal{W}$, so there exists $\overline{w} \in \mathcal{W}$ such that (up to a subsequence)

$$\nabla \Phi_\gamma(u_n) \rightharpoonup \nabla \overline{w} \quad \text{in } L^p + L^q. \tag{3.54}$$

Now we prove that

(1) $\lim_n \|u_n + \nabla \Phi_\gamma(u_n)\|_{L^p+L^q} = \|u + \nabla \Phi_\gamma(u)\|_{L^p+L^q}$;
(2) $\nabla \Phi_\gamma(u_n) \rightharpoonup \nabla \Phi_\gamma(u)$ in $L^p + L^q$.

Observe that, by the definition of Φ_γ and the triangular inequality,

$$\begin{aligned}\|u + \nabla \Phi_\gamma(u)\|^\gamma_{L^p+L^q} &\leq \|u + \nabla \Phi_\gamma(u_n)\|^\gamma_{L^p+L^q} \\ &\leq (\|u - u_n\|_{L^p+L^q} + \|u_n + \nabla \Phi_\gamma(u_n)\|_{L^p+L^q})^\gamma\end{aligned}$$

and then, by (3.52)

$$\|u + \nabla \Phi_\gamma(u)\|^\gamma_{L^p+L^q} \leq \liminf_n \|u_n + \nabla \Phi_\gamma(u_n)\|^\gamma_{L^p+L^q}. \tag{3.55}$$

On the other hand, by definition of Φ_γ

$$\|u_n + \nabla \Phi_\gamma(u_n)\|^\gamma_{L^p+L^q} \leq \|u_n + \nabla \Phi_\gamma(u)\|^\gamma_{L^p+L^q}$$

and then, by (3.52)

$$\limsup_n \|u_n + \nabla \Phi_\gamma(u_n)\|^\gamma_{L^p+L^q} \leq \|u + \nabla \Phi_\gamma(u)\|^\gamma_{L^p+L^q}. \tag{3.56}$$

The claim (1) follows from (3.55) and (3.56).

Since $\|\cdot\|^\gamma_{L^p+L^q}$ is weakly lower semicontinuous, from (3.52), (3.54) and the claim (1) we deduce

$$\|u + \nabla \overline{w}\|^\gamma_{L^p+L^q} \leq \liminf_n \|u_n + \nabla \Phi_\gamma(u_n)\|^\gamma_{L^p+L^q} = \|u + \nabla \Phi_\gamma(u)\|^\gamma_{L^p+L^q}. \tag{3.57}$$

By the uniqueness of the minimizer of $\widetilde{F}_u$, the inequality (3.57) implies that $\overline{w} = \Phi_\gamma(u)$ and then the claim (2) is a consequence of (3.54).

By a well known theorem, the claims (1) and (2) and (3.52) imply that

$$\nabla \Phi_\gamma(u_n) \to \nabla \Phi_\gamma(u) \text{ in } L^p + L^q$$

and then Φ_γ is compact.

Now, let $V \subset \mathcal{V}$ such that $\dim V < \infty$. By Weierstrass theorem there exists

$$\widetilde{C}_\gamma := \min_{\substack{\|u\|_{\mathcal{D}}=1 \\ u \in V}} \|u + \nabla \Phi_\gamma(u)\|^\gamma_{L^p+L^q} \geq 0.$$

Actually, $\widetilde{C}_\gamma > 0$. In fact, if $\widetilde{C}_\gamma = 0$, then there should exist $\overline{u} \in V$ such that $\|\overline{u}\|_{\mathcal{D}} = 1$ and $\overline{u} + \nabla \Phi_\gamma(\overline{u}) = 0$, but it is not possible as we have already seen in the proof of Theorem 3.4. Now, if we consider $u \in V - \{0\}$ and set $\widetilde{u} = u/\|u\|_{\mathcal{D}}$, since $\|\widetilde{u}\|_{\mathcal{D}} = 1$, we have that

$$
\begin{aligned}
(3.58)\qquad \frac{\|u + \nabla\Phi_\gamma(u)\|^\gamma_{L^p+L^q}}{\|u\|^\gamma_{\mathcal{D}}} &= \bigg\|\widetilde{u} + \nabla\bigg(\frac{\Phi_\gamma(u)}{\|u\|_{\mathcal{D}}}\bigg)\bigg\|^\gamma_{L^p+L^q} \\
&\geq \|\widetilde{u} + \nabla\Phi_\gamma(\widetilde{u})\|^\gamma_{L^p+L^q} \geq \widetilde{C}_\gamma.
\end{aligned}
$$

So (3.51) follows from (3.58). □

Theorem 3.9. *For all $V \subset \mathcal{V}$ such that $\dim V < \infty$ we have*

$$\sup_{u\in V} \widehat{J}(u) < \infty.$$

Proof. Let $V \subset \mathcal{V}$ such that $\dim V < \infty$. Consider $u \in V$ and set

$$\Omega := \{x \in \mathbb{R}^n : |(u + \nabla\Phi(u))(x)| > 1\}.$$

Since inequality (2.7) implies that

$$\|u + \nabla\Phi(u)\|^p_{L^p+L^q} \leq |u + \nabla\Phi(u)|^p_{L^p(\Omega)}$$

or

$$\|u + \nabla\Phi(u)\|^q_{L^p+L^q} \leq |u + \nabla\Phi(u)|^q_{L^q(\mathbb{R}^n-\Omega)},$$

certainly

$$
\begin{aligned}
(3.59)\qquad &\min(\|u + \nabla\Phi(u)\|^p_{L^p+L^q}, \|u + \nabla\Phi(u)\|^q_{L^p+L^q}) \\
&\qquad \leq \max(|u + \nabla\Phi(u)|^p_{L^p(\Omega)}, |u + \nabla\Phi(u)|^q_{L^q(\mathbb{R}^n-\Omega)}).
\end{aligned}
$$

By (3.59) and Lemma 3.8

$$
\begin{aligned}
&\int_{\mathbb{R}} f(|u+ \nabla\Phi(u)|^2)\,dx \\
&\quad \geq c_1 \int_\Omega |u + \nabla\Phi(u)|^p\,dx + c_1 \int_{\mathbb{R}^n-\Omega} |u + \nabla\Phi(u)|^q\,dx \\
&\quad = c_1|u + \nabla\Phi(u)|^p_{L^p(\Omega)} + c_1|u + \nabla\Phi(u)|^q_{L^q(\mathbb{R}^n-\Omega)} \\
&\quad \geq c_1 \max(|u + \nabla\Phi(u)|^p_{L^p(\Omega)}, |u + \nabla\Phi(u)|^q_{L^q(\mathbb{R}^n-\Omega)}) \\
&\quad \geq c_1 \min(\|u + \nabla\Phi(u)\|^p_{L^p+L^q}, \|u + \nabla\Phi(u)\|^q_{L^p+L^q}) \\
&\quad \geq c_1 \min(\|u + \nabla\Phi_p(u)\|^p_{L^p+L^q}, \|u + \nabla\Phi_q(u)\|^q_{L^p+L^q}) \\
&\quad \geq c_1 \min(\widetilde{C}_p\|u\|^p_{\mathcal{D}}, \widetilde{C}_q\|u\|^q_{\mathcal{D}}) \\
&\quad \geq c_1 \min(\widetilde{C}_p, \widetilde{C}_q) \min(\|u\|^p_{\mathcal{D}}, \|u\|^q_{\mathcal{D}}),
\end{aligned}
$$

and then

$$
\begin{aligned}
(3.60)\qquad \widehat{J}(u) &= \frac{1}{2}\|u\|^2_{\mathcal{D}} - \frac{1}{2}\int_{\mathbb{R}} f(|u + \nabla\Phi(u)|^2)\,dx \\
&\leq \frac{1}{2}\|u\|^2_{\mathcal{D}} - c_1 \min(\widetilde{C}_p, \widetilde{C}_q) \min(\|u\|^p_{\mathcal{D}}, \|u\|^q_{\mathcal{D}}).
\end{aligned}
$$

Since $2 < p < q$, we get our conclusion from (3.60). $\square$

PROOF OF THEOREM 1.1. Since $\widehat{J}$ is C^1 and even, by Theorems 3.5, 3.7, 3.9 and the symmetrical version of the mountain pass theorem (see [1], [5]) certainly it possesses infinitely many critical points. Then the conclusion is a consequence of Theorem 3.4. $\square$

4. Appendix

As we have seen, in order to have infinitely many solutions for the problem (1.4) we need some assumptions on the growth and on the convexity of the nonlinearity. Here we want to show an example of function satisfying those assumptions.

Consider the function $f\colon [0,\infty[\to \mathbb{R}$ such that

$$f(x) = \begin{cases} ax^p + b & \text{if } x > 1, \\ cx^q & \text{if } x \le 1. \end{cases} \tag{4.1}$$

where $2 < p < 2^* < q$ and the set of three numbers $(a,b,c) \in \mathbb{R}^2 \times]0,\infty[$ is any solution of the system

$$\begin{cases} a + b = c, \\ ap = cq. \end{cases} \tag{4.2}$$

LEMMA 4.1. *There exist $\delta > 0$ and $K_1 > 0$ such that*

$$f(x) - f(y) - (f'(y)|x-y) \ge K_1|x-y|^q \tag{4.3}$$

for all $(x,y) \in]1, 1+\delta \times]1-\delta, 1[$.

PROOF. Consider the function $h\colon]1,\infty[\times]0,1]$ such that

$$h(x,y) = \frac{f(x) - f(y) - (f'(y)|x-y)}{|x-y|^q}. \tag{4.4}$$

that is

$$h(x,y) = \frac{ax^p + b + (q-1)cy^q - qcxy^{q-1}}{|x-y|^q}.$$

Dividing numerator and denominator by y^q and setting $z = x/y$, we get the new function

$$\widetilde{h}(z,y) = \frac{az^p y^{p-q} + by^{-q} + (q-1)c - qcz}{|z-1|^q}$$

defined in the domain $\{(z,y) \in]1,\infty[\times]0,1] : y > 1/z\}$.

We claim that

$$\widetilde{h}(z,1) = \min_{y>1/z} \widetilde{h}(z,\,\cdot\,) \quad \text{for all } z > 1.$$

We compute

$$\frac{\partial \widetilde{h}}{\partial y}(z,y) = \frac{a(p-q)z^p y^{p-q-1} - bqy^{-q-1}}{|z-1|^q} = \frac{a(p-q)z^p y^p - bq}{y^{q+1}|z-1|^q} = \frac{g(zy)}{y^{q+1}|z-1|^q} \tag{4.5}$$

where $g(t) = a(p-q)t^p - bq$. By (4.2) we deduce that

$$\begin{aligned} g(1) &= 0, \\ g'(t) &< 0 \quad \text{if } t > 1, \end{aligned}$$

so $g(zy) < 0$ because $zy > 1$. By (4.5) we can conclude that the function $\widetilde{h}(z,\,\cdot\,)$ is decreasing in $]1/z, 1]$ and then

$$\widetilde{h}(z,y) \geq \widetilde{h}(z,1) \quad \text{for all } z > 1. \tag{4.6}$$

Now, by (4.6) and using twice De l'Hôpital's rule, we compute

$$\begin{aligned} \lim_{(x,y)\to(1^+,1^-)} h(x,y) &= \lim_{(z,y)\to(1^+,1^-)} \widetilde{h}(z,y) \geq \lim_{(z,y)\to(1^+,1^-)} \widetilde{h}(z,1) \\ &= \lim_{z\to 1^+} \frac{az^p + b + (q-1)c - qcz}{(z-1)^q} \\ &= \lim_{z\to 1^+} \frac{ap(p-1)z^{p-2}}{q(q-1)(z-1)^{q-2}} = \infty. \end{aligned}$$

The inequality (4.3) is a consequence of the previous limit. □

Theorem 4.2. *There exists $K_2 > 0$ such that for every nonnegative numbers x, y*

$$f(x) - f(y) - f'(y)(x-y) \geq K_2 \min(|x-y|^p, |x-y|^q). \tag{4.7}$$

Proof. We distinguish the following three cases:

(1) $0 \leq y \leq 1 < x$ or $0 \leq x \leq 1 < y$;
(2) $1 < x, y$;
(3) $0 \leq x, y \leq 1$.

(1) If $0 \leq y \leq 1 < x$, then we consider these three possibilities

- $(x,y) \in \,]1, 1+\delta[\, \times \,]1-\delta, 1[$,
- $(x,y) \in \,]1, 1+\delta] \times [0, 1-\delta]$,
- $(x,y) \in [1+\delta, \infty[\times[0,1]$,

where δ is the same as in Lemma 4.1.

By Lemma 4.1, certainly (4.7) holds in $]1, 1+\delta[\, \times \,]1-\delta, 1[$.

Since the function h defined in (4.4) is continuous in $[1, 1+\delta] \times [0, 1-\delta]$, by Weierstrass' theorem there exists $\min\{h(x,y) \mid (x,y) \in [1, 1+\delta] \times [0, 1-\delta]\}$ and then the inequality (4.7) holds also in $]1, 1+\delta] \times [0, 1-\delta]$.

Finally, suppose $(x,y) \in [1+\delta,\infty[\times [0,1]$. Since, for every $x \in [1+\delta,\infty[$,

$$\min_{y\in[0,1]} y^{q-1}(c(q-1)y - cqx) = c(q-1) - cqx,$$

then, by (4.2),

$$\begin{aligned} f(x) - f(y) - f'(y)(x-y) &= ax^p + b + y^{q-1}(c(q-1)y - cqx) \\ &\geq ax^p + b + c(q-1) - cqx = ax^p - ap(x-1) - a. \end{aligned} \tag{4.8}$$

But

$$C_1 := \inf_{x\geq 1+\delta} \frac{ax^p - ap(x-1) - a}{x^p} > 0 \tag{4.9}$$

so, by (4.8) and (4.9),

$$f(x) - f(y) - f'(y)(x-y) \geq C_1 x^p \geq C_1(x-y)^p$$

and then the inequality (4.7) holds also in $[1+\delta,\infty[\times [0,1]$.

We can use similar arguments for the case $0 \leq x \leq 1 < y$.

(2) Suppose $1 < x, y$. We have that

$$f(x) - f(y) - f'(y)(x-y) = a(x^p - y^p - py^{p-1}(x-y)). \tag{4.10}$$

In [15] (see the proof of Theorem 4, Chapter VIII) the following inequality has been proved: for all $r > 2$ there exists a positive constant $C_2(r)$ such that for any $u \in \mathbb{R}$

$$|u+1|^r \geq 1 + ru + C_2(r)|u|^r. \tag{4.11}$$

If we set $r = p$ and replace u by $(x-y)/y$, then by some calculus we get

$$x^p \geq y^p + py^{p-1}(x-y) + C_2(p)|x-y|^p. \tag{4.12}$$

Inequality (4.7) follows from (4.10) and (4.12).

(3) Suppose $0 \leq x, y \leq 1$. Then

$$f(x) - f(y) - f'(y)(x-y) = c(x^q - y^q - qy^{q-1}(x-y))$$

so we get again (4.7) using (4.11) as before. □

Theorem 4.3. *Let $\widehat{f}$ be the even extension of f, i.e.*

$$\widehat{f}(x) = \begin{cases} f(x) & \text{if } x \geq 0, \\ f(-x) & \text{if } x < 0, \end{cases}$$

Then there exists $K_3 > 0$ such that for all $(x,y) \in \mathbb{R}^2$

$$\widehat{f}(x) - \widehat{f}(y) - \widehat{f}'(y)(x-y) \geq K_3 \min(|x-y|^p, |x-y|^q). \tag{4.13}$$

Proof. We distinguish some cases.

(1) $x, y \leq 0$. Since $\widehat{f}$ is even, certainly $\widehat{f}'$ is odd and then, by (4.7),

$$\begin{aligned}\widehat{f}(x) - \widehat{f}(y) \; - \widehat{f}'(y)(x-y) & \\ &= \widehat{f}(-x) - \widehat{f}(-y) - \widehat{f}'(-y)(-x-(-y)) \\ &= f(-x) - f(-y) - f'(-y)(-x-(-y)) \\ &\geq K_2 \min(|-x-(-y)|^p, |-x-(-y)|^q) \\ &= K_2 \min(|x-y|^p, |x-y|^q).\end{aligned}$$

(2) $x \leq 0$ and $y \geq 0$. We have that

$$\widehat{f}(x) - \widehat{f}(y) - \widehat{f}'(y)(x-y) = \widehat{f}(x) - \widehat{f}'(y)x - \widehat{f}(y) + \widehat{f}'(y)y. \tag{4.14}$$

Since $\widehat{f}'(y) \geq 0$, the property (f_3) (that can be easily proved) implies that

$$\widehat{f}(x) - \widehat{f}'(y)x \geq \widehat{f}(x) \geq c_1 \min(|x|^p, |x|^q), \tag{4.15}$$

and, on the other hand, by (4.7)

$$widehat f(y) + \widehat{f}'(y)y = f(0) - f(y) - f'(y)(0-y) \geq K_2 \min(|y|^p, |y|^q). \tag{4.16}$$

Comparing (4.14), (4.15) and (4.16) we get

$$\begin{aligned}\widehat{f}(x) - \widehat{f}(y) \; - \widehat{f}'(y)(x-y) & \\ &\geq C_3(\min(|x|^p, |x|^q) + \min(|y|^p, |y|^q)) \\ &\geq C_4 \min(|x|^p + |y|^p, |x|^q + |y|^q) \\ &\geq C_5 \min((|x| + |y|)^p, (|x| + |y|)^q) \\ &= C_5 \min(|x-y|^p, |x-y|^q)\end{aligned}$$

where C_3, C_4 and C_5 are positive constants.

(3) $x \geq 0$ and $y \leq 0$. The inequality (4,13) can be proved by similar arguments as before.

(4) $x, y \geq 0$. The inequality (4,13) follows directly from (4.7). □

Finally, define $\overline{f}\colon \mathbb{R}^n \to \mathbb{R}$ as the radial extension of f, namely

$$\overline{f}(x) = f(|x|), \quad \text{for all } x \in \mathbb{R}^n. \tag{4.17}$$

Theorem 4.4. *The function $\overline{f}$ defined by* (4.17) *and* (4.1) *satisfies the inequality*

$$\overline{f}(x) - \overline{f}(y) - (\overline{f}\,'(y)|x-y) = c_1 \min(|x-y|^p, |x-y|^q) \tag{4.18}$$

for some positive constant c_1 which doesn't depend on $x, y \in \mathbb{R}^n$.

Proof. It is very easy to verify that $\overline{f}$ satisfies the inequality

$$f(x) \geq c_1 \min(|x|^p, |x|^q) \quad \text{for all } x \in \mathbb{R}^n. \tag{4.19}$$

If $y = 0$, then (4.18) follows trivially from (4.19).

If $y \neq 0$, observe that for all $x \in \mathbb{R}^n$

$$\overline{f}(x) - \overline{f}(y) - (\overline{f}'(y)|x-y) = f(|x|) - f(|y|) - \frac{f'(|y|)}{|y|}(x|y) + f'(|y|)|y|. \tag{4.20}$$

Now consider the following three cases

(1) $x = ty$, $t \geq 0$;
(2) $x = ty$, $t < 0$;
(3) $x \neq ty$, $t \in \mathbb{R}$.

If $x = ty$ for $t \geq 0$, then $(x|y) = |x||y|$ and $|x-y| = ||x|-|y||$ so, by (4.20) and (4.7),

$$\begin{aligned}\overline{f}(x) - \overline{f}(y) - (\overline{f}'(y)|x-y) &= f(|x|) - f(|y|) - f'(|y|)(|x|-|y|)\\ &\geq K_2 \min(||x|-|y||^p, ||x|-|y||^q) = K_2 \min(|x-y|^p, |x-y|^q).\end{aligned}$$

If $x = ty$ for $t < 0$, then $(x|y) = -|x||y|$ and $|x-y| = |x|+|y|$ so, by (4.20) and (4,13),

$$\begin{aligned}\overline{f}(x) - \overline{f}(y) - (\overline{f}\,'(y)|x-y) &= \widehat{f}(|x|) - \widehat{f}(-|y|) - \widehat{f}'(-|y|)(|x|-(-|y|))\\ &\geq K_3 \min(||x|+|y||^p, ||x|+|y||^q) = K_3 \min(|x-y|^p, |x-y|^q).\end{aligned}$$

Finally, if $x \notin \{ty : t \in \mathbb{R}\}$, then $x = x_1 + x_2$ where $x_1 \in \{ty : t \in \mathbb{R}\}$ and $(x_2|y) = 0$. Since $x_1 \parallel y$, from the previous cases we have

$$\overline{f}(x_1) - \overline{f}(y) - (\overline{f}\,'(y)\big|x_1 - y) \geq C_6 \min(|x_1-y|^p, |x_1-y|^q) \tag{4.21}$$

where $C_6 = \min(K_2, K_3)$. Moreover, observe that for all $a, b \geq 0$ the following inequality holds

$$f(\sqrt{a+b}) \geq f(\sqrt{a}) + f(\sqrt{b}), \tag{4.22}$$

so, by (4.22), (4.21) and property (f_3), we have

$$\begin{aligned}\overline{f}(x) - \overline{f}(y) \ &- (\overline{f}'(y)|x-y)\\ &= f(\sqrt{|x_1|^2+|x_2|^2}) - \overline{f}(y) - \overline{f}'(y)|x_1-y)\\ &\geq f(|x_1|) + f(|x_2|) - \overline{f}(y) - (\overline{f}'(y)|x_1-y)\\ &= \overline{f}(x_1) - \overline{f}(y) - (\overline{f}'(y)|x_1-y) + \overline{f}(x_2)\\ &\geq C_6 \min(|x_1-y|^p, |x_1-y|^q) + c_1 \min(|x_2|^p, |x_2|^q)\\ &\geq C_7 \min((|x_1-y|^2)^{p/2} + (|x_2|^2)^{p/2}, (|x_1-y|^2)^{q/2} + (|x_2|^2)^{q/2})\\ &\geq C_8 \min((|x_1-y|^2+|x_2|^2)^{p/2}, (|x_1-y|^2+|x_2|^2)^{q/2})\\ &= C_8 \min(|x-y|^p, |x-y|^q)\end{aligned}$$

where C_7 and C_8 are suitable positive constants. □

Now, consider the function $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)=\begin{cases} a|x|^{p/2}+b & \text{if } |x|>1,\\ c|x|^{q/2} & \text{if } |x|\le 1,\end{cases}$$

where $2<p<2^*<q$ and $(a,b,c)\in\mathbb{R}^2\times\,]0,\infty[$ is any solution of the system

$$\begin{cases} a+b=c,\\ ap=cq.\end{cases}$$

It is easy to verify that f satisfies (f_1), (f_3) and (f_4). Moreover, applying Theorem 4.1 to the function

$$g:\xi\in\mathbb{R}^n\mapsto f((\xi|\xi))\in\mathbb{R},$$

we verify that f satisfies also (f_2).

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