Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 26, 2005, 9–16

ON THE TOPOLOGY OF THE EIGENFIELDS

Vladimir I. Arnold

Dedicated to the memory of Olga A. Ladyzhenskaya

Abstract. Topological properties of the eigenfields dependence on the eigenvalue position is discussed for the cases, where the variety of the eigenfield vanishing does not divide the oscillating domain into pieces.

1. Introduction

R. Courant had proved that the zero level set of the n-th eigenfunction subdivides the oscillating domain in at most n connected components.

In the present note we consider the eigenfields case, where the zero level set's codimension is larger and where it does not subdivide the oscillating domain. For instance, if the field's components number is equal to the oscillating domain dimension the zeros places are generically isolated points. The theorem, proved below, provides some information on the topological complexity of this picture, depending on the eigenfields number (in the natural ordering $\lambda_1 \leq \lambda_2 \leq \ldots$ of the eigenvalues of the minus Laplace operator, where each eigenvalue is repeated according to its multiplicity, providing the number n of an eigenvalue λ to be the maximal m, for which $\lambda_m = \lambda$).

Consider a closed connected two-dimensional Riemannian manifold M and two eigefunctions u, v of the Laplace operator on it with the same eigenvalue Λ

2000 *Mathematics Subject Classification*. 3530, 51H20, 58C40.

Key words and phrases. Eigenvlue, Laplace operator, Betti numbers, variational principle, quadratic functionals, oscillating domains, nodes.

Partially supported by RFBR grant 02-01-00655.

of the minus Laplace operator:

$$\Delta u = -\Lambda u, \quad \Delta v = -\Lambda v.$$

DEFINITION. A connected component of the domain

$$G = \{z \in M : u(z)v(z) \neq 0\}$$

is called *positive*, if the integral of the scalar product of the gradients of the functions u^2 and v^2 along the domain G is not negative.

THEOREM. *The number N of the positive connected components of the domain G does not exceed the number n of the (independent) eigenfields, for which the number of the minus Laplace operator eigenvalue does not exceed the number 2Λ:*

$$N \leq n(2\Lambda).$$

The number N measures in some sense the topological complexity of the mapping $t\colon G \to P^1$ of the domain G into the projective line of the directions on the plane with coordinates u and v:

$$t(z) = [v(z)\colon u(z)].$$

The topological complexity of the mapping t might be evaluated by the degree of the natural mapping to P^1 of the one-dimensional complex, whose points are the connected components of the level lines $t^{-1}(T)$ for $T \in P^1$.

REMARK 1. I do not know, whether the total number $\widetilde{N}$ of all the connected components of the domain G is bounded in terms of Λ. One might imagine the eigenfields, having small number $n(\Lambda)$, defining domains G with arbitrary large numbers of the connected components (and of the zeros) for the convenient metrics on S^2 or on T^2.

REMARK 2. The conjectures on the boundedness of the topological complexity (in terms of the eigenvalue number) had been formulated by the author many times (see, e. g.,[2] and [3], containing also some ideas of such bounds, different from those, discussed in the present article).

In the general case of an m-dimensional field on a d-dimensional manifold the zeros set is generically of dimension $d-m$, and for $d > m$ one should bound rather the $d-m$-dimensional Betti number then the zeros number.

Of course, the fields are in the general case sections of a bundle with a d-dimensional base space and an m-dimensional fiber (describing, for instance, the oscillations of a d-dimensional manifold in its $d+m$-dimensional neighbouring).

The bound should be provided by the eigenfield's number in the space of the fields, rather than functions. In the case of an oscillating domain with boundary one should consider the eigenfields verifying some boundary conditions, providing the usual variational description of the eigenvalues, as being the critical values of some quadratic form (generalizing the Dirichlet form) along the constancy sphere of the second form (generalizing the function square's integral).

Below we shall only discuss the 2-dimensional case, $m = d = 2$, on a closed manifold with no boundaries (the periodic boundary conditions case being included).

The zeros of the field will be supposed to be nondegenerated, the set of the zeros will be therefore a smooth manifold of dimension $d - m$ (of dimension zero in our case). Even when $m = d$, the degeneration is able to produce the zeros set of dimension, greater than $d - m$ (or to produce singularities of this set).

Already in the infinitely-degenerated case (of dimension greater that $d - m$ of the zeros set) one may hope to find the upper bounds, say, for the number of the connected components of the zeros set (or for the $d - m$-dimensional Betti number of this manifold, whose dimension exceeds $d - m$), but these degenerations will not be studies in the present article.

A topological study of the degenerations of the zero level line of an eigenfunction of the Laplace operator on a two-dimensional manifold had been published by V. N. Karpushkin ([4], [5]). He proved, in particular, that the number of the (simplest) singular points does not exceed the eigenfunction number if the oscillating surface Euler characteristic vanishes.

However, already in dimension 3 there are no published bounds for the numbers (and for the multiplicities) of the zeros surface singularities of an eigenfunction.

It is interesting, however, that these singularities are the same, as those of the harmonic functions zeros sets. In the case of the oscillating surfaces Karpushkin associated to each singular point some *multiplicity*, similar to the Milnor number, but these multiplicities are still to be described in dimension 3 or higher.

In the case, when the number m of the components of the field exceeds the oscillating domain dimension, it is natural to expect some Sturm theory type bounds for the *linking numbers* of the field's graph with the zero section in terms of the eigenvalue, but these bounds had not been published yet (these generalized linking numbers are interesting homotopical invariants, rather than numbers, if $m > d + 1$).

2. The variational method of the estimation of the topological complexity of the eigenfield

The eigenfunctions (or eigenfields) of the minus Laplace operator with the eigenvalue Λ verify the relation between the Dirichlet integral and the Hilbert integral

$$\int (\nabla u)^2 \, ds = \Lambda \int u^2 \, ds,$$

being the eigenvectors of the Dirichlet quadratic form with respect to the Hilbert space metrics form in the space of the functions (of the fields).

Consider the integral

$$I_\Lambda[u] = \int (\nabla u)^2\, ds - \Lambda \int u^2\, ds. \tag{2.1}$$

In terms of this quadratic functional the eigenvalue Λ has the following (evident) variational definition: *if the integral I_Λ is nowhere positive on some N-dimensional real vector subspace of the space of the fields u then $N \leq n$* (a vector subspace of dimension n, where $I_\Lambda[a] \leq 0$, does exist: it is generated by the eigenfields, whose eigenvalues of the minus Laplace operator, do not exceed Λ).

Therefore, finding N different places, in which neighbourhoods one is able to construct the fields, at which the functional I_Λ is nonpositive, one deduces, that $N \leq n$: *the number N of such nonpositivity places is bounded from above by number n of the eigenvalue Λ* (*counting the eigenvalues taking the multiplicities into account*).

We shall now show, that the positive connected components of the complement to the union of the level lines ($\{u = 0\}$, $\{v = 0\}$) of two eigenfunctions with the same eigenvalue Λ are such nonpositivity places:

$$N \leq n(2\Lambda),$$

as is stated in the above theorem.

Remark. Simple topological reasoning below relates the total number $\widetilde{N}$ of the connected components of the manifold $\{uv \neq 0\}$ to the number d_0 of the points of the (transversal) intersection of the curves $\{u = 0\}$ and $\{v = 0\}$.

Topological Lemma. *If two closed smooth curves $\{u = 0\}$ and $\{v = 0\}$ on a smooth closed surface of Euler characteristic χ intersect transversally, the intersection points number being α_0, then holds the inequality*

$$\left(\alpha_0 = \left(\sum_{i=1}^{\widetilde{N}} \chi_i\right) - \chi\right) \leq (\widetilde{N} - \chi),$$

where $\widetilde{N}$ is the number of the connected components of the manifold $\{uv \neq 0\}$ and where the numbers χ_i are the Euler characteristics of these connected components.

The lemma is proved below, in Section 4.

We get from the lemma an upper bound for the number α_0 of the zeros of the eigenfield (u, v):

$$\alpha_0 \leq \widetilde{N} - \chi,$$

which would provide the upper bound for α_0 in terms of the eigenvalue Λ whenever the connected components number $\widetilde{N}$ would be bounded in terms of Λ.

3. The proof of the upper bound for the number of the positive connected components

Fixing a positive connected component, define the function

$$w = \begin{cases} uv & \text{in the fixed connected component,} \\ 0 & \text{otherwise.} \end{cases}$$

We shall see, that the value of the quadratic functional $I_{2\Lambda}$ at this function is nonpositive: $I_{2\Lambda}[w] \le 0$.

The variational method of Section 2 provides then the inequality

$$N \le n(2\Lambda),$$

bounding the number N of such positive components.

To calculate the value of the quadratic form $I_{2\Lambda}$ on the function w, note, that $w = 0$ on the boundary of the intergration domain. Therefore, the intergration by parts inside this domain provides the identity

$$\int (\nabla w)^2 \, ds = -\int w(\Delta w) \, ds. \tag{3.1}$$

We shall also use the identity

$$\Delta(uv) = u(\Delta v) + v(\Delta u) + 2(\nabla u, \nabla v), \tag{3.2}$$

following from the product's gradient calculation,

$$\nabla(uv) = u(\nabla v) + v(\nabla u),$$

taking into account the divergences of the summands:

$$\operatorname{div}(u \operatorname{grad} v) = u(\operatorname{div} \operatorname{grad} v) + (\operatorname{grad} u, \operatorname{grad} v) = u(\Delta v) + (\nabla u, \nabla v).$$

Now apply these formulas to calculate the quadratic form $I_{2\Lambda}$ value at the function w. Formulas (2.1), (3.1), (3.2) imply the expressions

$$\begin{aligned} I_{2\Lambda}[w] &= \int (\nabla w)^2 \, ds - 2\Lambda \int w^2 \, ds = -\int w(\Delta w) \, ds - 2\Lambda \int w^2 \, ds \\ &= -\int uv\big(u(\Delta v) + v(\Delta u)\big) \, ds - 2 \int uv(\nabla u, \nabla v) \, ds - 2\Lambda \int u^2 v^2 \, ds. \end{aligned}$$

In the case $\Delta u = -\Lambda u$, $\Delta v = -\Lambda v$ we deduce, that

$$\begin{aligned} I_{2\Lambda}[w] &= \int \Lambda uv \, uv \, ds + \int \Lambda vu \, uv \, ds - 2\Lambda \int v^2 u^2 \, ds - 2 \int vu(\nabla u, \nabla v) \, ds \\ &= -2 \int vu(\nabla u, \nabla v) \, ds. \end{aligned}$$

The last integral is nonnegative, since we integrate along a positive component. Thus, the quadratic form value $I_{2\Lambda}[w]$ is nonpositive, and therefore $N \le n(2\Lambda)$ according to the variational method of Section 2. The theorem is thus proved.

REMARK. The theorem remains valid for $m \le d$ eigenfunctions on a d-dimensional manifold having the same eigenvalue Λ of the minus Laplace operator: *the number of the positive connected components of the complement to the union of the m zero level hypersurfaces of the m eigenfunctions does not exceed the number $n(m\Lambda)$ of the (independent) eigenfields, whose eigenvalues do not exceed the quantity $m\Lambda$*: $N \le n(m\Lambda)$. The positivity of the connected component means here the nonnegativity of the integral along this component of the following function

$$Q = \sum_{i<j\le m} \left[\left(\prod_{k\ne i,j} (u_k^2)\right)(\nabla(u_i^2), \nabla(v_j^2))\right].$$

The proof differs from the one presented above (for $m = d = 2$) only in the use of the following formula for the Laplacian of the product:

$$\Delta\left(\prod_{i=1}^m u_i\right) = \sum_{i=1}^m \left(\left(\prod_{k\ne i} u_k\right)(\Delta u_i)\right) + 2\sum_{i<j}\left(\left(\prod_{k\ne i,j} u_k\right)(\nabla u_i, \nabla u_j)\right).$$

In the case of the eigenfunctions, for which

$$\Delta u_i = -\Lambda_i u_i,$$

the preceding identity provides for $w = \prod_{i=1}^m (u_i)$ the quadratic form value

$$-w(\Delta w) = \left(\sum_{i=1}^m \Lambda_i\right)\left(\prod_{j=1}^m u_j^2\right) - 2Q.$$

For m equal eigenvalues $\Lambda_i \equiv \Lambda$ we get for the coefficients of the term w^2, written above, the value $\sum_{i=1}^m \Lambda_i = m\Lambda$. It provides the upper bound $n(m\Lambda)$ for the connected components number, since the integral of $-2Q$ along our positive domain is non positive.

The method, described above, provides also the upper bounds for the numbers of the connected components of the subsets of the m-fold product of the oscillating domain with itself, the subsets being defined either by the condition $\big(u_1(x_1)u_2(x_2)\dots u_m(x_m)\big) \ne 0$, or by the condition $\det\big(u_i(x_j)\big) \ne 0$.

The second condition is related to the Fermi–Dirac antisymmetric eigenfunctions of an m-particles system and with the upper bound of the topological complexity of the linear combinations of the eigenfunctions, corresponding to different eigenvalues.

These upper bounds include rather the products, than the sums, of the numbers $n(m\Lambda)$ since one have to consider rather the tensor products of the eigenfunctions spaces, than their direct products.

The upper bound, published in [1] of the number of the connected components of the domain, where a linear combination of the eigenfunctions is different from zero (by the highest number of the eigenfunction, involved in the combination) is wrong for $d > 1$, as O. Viro had shown already for the spherical functions on S^2, when I had explained him the real algebraic geometry corollaries of the bounds of [1].

4. Topological bounds

To prove the topological Lemma, formulated in Section 2, denote by α_1 the number of the segments, into which the intersection points subdivide the union of the two curves $\{u = 0\}$ and $\{v = 0\}$ (intersecting transversally).

The Euler characteristics additivity implies the (Euler) identity

$$\alpha_0 - \alpha_1 + \sum \chi_i = \chi. \tag{4.1}$$

The α_1 segments have $2\alpha_1$ ends. The ends number is also equal to $4\alpha_0$, each transversal intersection point being the end point of four connected parts of the intersecting curves.

Thus, $2\alpha_1 = 4\alpha_0$, $\alpha_1 = 2\alpha_0$, and therefore the identity (4.1) takes the form $\alpha_0 - 2\alpha_0 + \sum \chi_i = \chi$, that is the form $\alpha_0 = (\sum \chi_i) - \chi$, proving the Topological Lemma of Section 2, since the Euler characteristic χ_i of each of the $\widetilde{N}$ connected components of the complement to the union of the curves does not exceed one, and thus $\sum \chi_i \leq \widetilde{N}$.

REMARK. In the case of 2 eigenfunctions (u, v) on a three-dimensional oriented closed connected manifold (where $m = 2$, $d = 3$ in the notation of Section 2) our reasoning also provides some upper bound for the number of the smooth intersection curves of the eigenfunctions vanishing surfaces, $\{z : u(z) = v(z) = 0\}$.

The number of the curves of zero level of both eigenfunctions does not exceed the sum $\widetilde{N} + (b_1)/2$, where b_1 denotes the one-dimensional Betti number of the zeros surface of one of the eigenfunctions, $\widetilde{N}$ being the number of the connected components of the domain $\{z : u(z)v(z) \neq 0\}$.

To prove this bound, note, that if the eigenfunction v is sufficiently close to u, then the different connected components of the surface $\{z : u(z) = 0,\ v(z) \neq 0\}$ generate different small semineighbourhood components, forming *different* connected components of the three-dimensional manifold $\{z : u(z)v(z) \neq 0\}$.

We obtain this way the upper bound for the number of the connected closed curves $\{u = v = 0\}$, being the sum of the number $\widetilde{N}$ of the three-dimensional components and of the dimension of the image of the mapping from the one-dimensional homology space of the curves union into the one-dimensional homology space of the surface $\{u = 0\}$.

The dimension of this image does not exceed the number $(b_1)/2$, since a surface of genus $g = (b_1)/2$ carries no sets of $g + 1$ mutually disjoint closed curves (which property had been the Riemann initial definition of the genus).

Unfortunately, I have no universal bound for the Betti number b_1 (of the zeros surface of an eigenfunction) in terms of the eigenvalue. I do not know, whether this number b_1 might be arbitrary large for a suitable choice of the Riemann metrics (say, on S^3 or on T^3) for the eigenfunction, whose eigenvalue number is small (even for the first nonconstant eigenfunction).

The author is greatefull to the reviewer, who insisted that the positivity condition should be carefully discussed.

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Manuscript received October 1, 2004

VLADIMIR I. ARNOLD
Steklov Mathematical Institute
8, Gubkina Street
119991 Moscow, GSP-1, RUSSIA