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TWIN POSITIVE PERIODIC SOLUTIONS OF SECOND ORDER SINGULAR DIFFERENTIAL SYSTEMS

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Abstract. In this paper, we study positive periodic solutions to singular second order differential systems. It is proved that such a problem has at least two positive periodic solutions. The proof relies on a nonlinear alternative of Leray–Schauder type and on Krasnosel'skiĭ fixed point theorem on compression and expansion of cones.

1. Introduction

In this paper, we consider the second order system

$$\begin{cases} x'' + a_1(t)x = f_1(x,y), \\ y'' + a_2(t)y = f_2(x,y). \end{cases} \tag{1.1}$$

The type of nonlinearity $f_i(x,y)$, $i = 1,2$ we are mainly interested in is when $f_i(x,y)$ has a singularity near $(x,y) = (0,0)$, although the main results of this paper apply also to a more general type of nonlinearity. We discuss the existence and multiplicity of positive periodic solutions of (1.1), i.e. positive solutions of (1.1) satisfying the periodic boundary condition

$$x(0) = x(1), \quad x'(0) = x'(1), \quad y(0) = y(1), \quad y'(0) = y'(1). \tag{1.2}$$

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Recently, the singular periodic problems have been studied extensively; see [1]–[5], [7]–[9], [11]–13] and the references therein. Motivated by [13], [14] we study (1.1) and establish the existence of two different positive periodic solutions to (1.1); see Theorems 3.1 and 3.3. The existence of the first solution is obtained using a nonlinear alternative of Leray–Schauder, and the second one is found using a fixed point theorem in cones.

2. Preliminaries and notation

Let us consider the linear periodic problem

$$\begin{cases} x'' + a(t)x = 0, \\ x(0) = x(1), \ x'(0) = x'(1). \end{cases} \tag{2.1}$$

In this section, we assume conditions under which the only solution of problem (2.1) is the trivial one. As a consequence of Fredholm's alternative, the nonhomogeneous problem

$$\begin{cases} x'' + a(t)x = h(t), \\ x(0) = x(1), \ x'(0) = x'(1), \end{cases}$$

admits a unique solution that can be written as

$$x(t) = \int_0^T G(t,s)h(s)\,ds,$$

where $G(t,s)$ is the Green's function of problem (2.1). The following two results follow from [13] directly (We write $a \succ 0$ if $a \geq 0$ almost everywhere on $[0,1]$ and is positive on a set of positive measure).

Lemma 2.1. *If $a(t) \prec 0$, then $G(t,s) < 0$ for all $(t,s) \in [0,1] \times [0,1]$.*

If on the contrary $a(t) \succ 0$, the following best Sobolev constants will be used

$$K(q) = \begin{cases} \dfrac{2\pi}{q}\left(\dfrac{2}{2+q}\right)^{1-2/q}\left(\dfrac{\Gamma(1/q)}{\Gamma(1/2+1/q)}\right)^2 & \text{if } 1 \leq q < \infty, \\ 4 & \text{if } q = \infty, \end{cases}$$

where Γ is the Gamma function. For a given p, let us define

$$p^* = \begin{cases} \dfrac{p}{p-1} & \text{if } 1 \leq q < \infty, \\ 1 & \text{if } q = \infty. \end{cases}$$

Lemma 2.2. *Assume that $a(t) \succ 0$ and $a \in L^p(0,1)$ for some $1 \leq p \leq \infty$. If*

$$\|a\|_p < K(2p^*), \tag{2.2}$$

then $G(t,s) > 0$ for all $(t,s) \in [0,1] \times [0,1]$.

Remark 2.3. If $p = \infty$ then hypothesis (2.2) is equivalent to $\|a\|_\infty < (\pi)^2$, which is a well-known criterion for the maximum principle used in the literature.

Let us define the sets of functions

$$\begin{aligned}\Lambda^- &= \{a \in L^1(0,1) : a \prec 0\},\\ \Lambda^+ &= \{a \in L^1(0,1) : a \succ 0,\ \|a\|_p < K(2p^*) \text{ for some } 1 \le p \le \infty\}.\end{aligned}$$

From the above, it is known that if $a \in \Lambda^+ \cup \Lambda^-$, then problem (2.1) has a Green's function $G(t,s)$ with a definite sign.

REMARK 2.4. As in [9], we can compute the maximum (M) and the minimum (m) of the Green's function when $a(t) = k^2 < (\pi)^2$, and we obtain

$$M = \frac{1}{2k\sin(\frac{k}{2})}, \quad m = \frac{1}{2k}\cot(\frac{k}{2}).$$

Throughout this paper, we assume that $G_i(t,s)$, $i = 1,2$, are the Green functions for the problems

$$\begin{aligned} x'' + a_1(t)x &= h_1(t), & x(0) &= x(1), & x'(0) &= x'(1),\\ y'' + a_2(t)y &= h_2(t), & y(0) &= y(1), & y'(0) &= y'(1), \end{aligned} \tag{2.3}$$

i.e.

$$\begin{aligned} x(t) &= (Lh_1)(t) = \int_0^1 G_1(t,s)h_1(s)\,ds,\\ y(t) &= (Lh_2)(t) = \int_0^1 G_2(t,s)h_2(s)\,ds. \end{aligned}$$

We also assume that

(A) $a_i \in \Lambda^+ \cup \Lambda^-$.

Under hypothesis (A), we always denote

$$A_i = \min_{0\le s,t\le 1} |G_i(t,s)|, \quad B_i = \max_{0\le s,t\le 1} |G_i(t,s)|, \quad \sigma_i = A_i/B_i, \quad i = 1,2. \tag{2.4}$$

Thus $B_i > A_i > 0$ and $0 < \sigma_i < 1$. We also use $w_i(t)$ to denote the unique periodic solution of (2.3) with $h_i(t) = 1$. In particular, $A_i \le \|w_i\|_\infty \le B_i$.

Here and henceforth, we denote the norm of $(x,y) \in R^2$ by $\|(x,y)\| = \max\{\|x\|, \|y\|\}$, and write $(x_1,y_1) > (x_2,y_2)$ ($(x_1,y_1) \ge (x_2,y_2)$), if $(x_1-x_2, y_1-y_2) \in \overline{R}_+^2$ ($(x_1-x_2,y_1-y_2) \in R_+^2$), $\overline{R}_+ = (0,\infty)$.

Further,we say that a vector (x,y) is positive (nonnegative) if $(x,y) > (0,0)$ ($(x,y) \ge (0,0)$).

In order to get the first periodic solution, we need the following nonlinear alternative of Laray–Schauder (see [11]).

THEOREM 2.5. *Assume Ω is a relatively open subset of a convex set K in a normed space X. Let $A\colon \overline{\Omega} \to K$ be a continuous and compact map with $0 \in \Omega$. Then either*

(A_1) *A has a fixed point in $\overline{\Omega}$, or*

(A_2) *there is a $x \in \partial\Omega$ and a $\lambda < 1$ such that $x = \lambda A(x)$.*

To obtain a second periodic solution of (1.1), we need the following well known fixed point theorem of compression and expansion of cones [10].

Theorem 2.6 ([10]). *Let X be a Banach space and K ($\subset X$) be a cone. Assume that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$T: K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a continuous and compact operator such that either

(a) $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$, *or*
(b) $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

In the applications below, we take $X_1 = C[0,1]$ with the supremum norm $\|\cdot\|$ and define

$$K_i = \{x \in X : x(t) \geq 0 \text{ for all } t \in [0,1] \text{ and } \min_{0 \leq t \leq 1} x(t) \geq \sigma_i \|x\|\}, \quad i = 1,2$$

where σ_i is as in (2.4). Let $X = X_1 \times X_1$, $K = K_1 \times K_2$, then $(X, \|\cdot\|)$ is a Banach space, and K is a cone in X.

Suppose now that $F_i: R \times R \to R$ is a continuous function and

$$G_i(t,s)F_i(x,y) \geq 0 \quad \text{for all } (t,s) \in [0,1] \times [0,1],\ (x,y) \in R^2.$$

Define an operator $T: X \to X$ by

$$T(x,y) = \bigg(\int_0^1 G_1(t,s)F_1(x(s),y(s))\,ds, \int_0^1 G_2(t,s)F_2(x(s),y(s))\,ds \bigg) \tag{2.5}$$

for $(x,y) \in X$.

Lemma 2.7. *T is well defined and maps X into K. Moreover, T is continuous and completely continuous.*

Proof. From [11], it is easy to see that T is continuous and completely continuous. Next, we show $T: X \to K$. Since

$$\begin{aligned} \int_0^1 G_1(t,s)F_1(x(s),y(s))\,ds &= \int_0^1 |G_1(t,s)F_1(x(s),y(s))|\,ds \\ &\geq A_1 \int_0^1 |F_1(x(s),y(s))|\,ds \end{aligned}$$

and

$$\int_0^1 |G_1(t,s)F_1(x(s),y(s))|\,ds \leq B_1 \int_0^1 |F_1(x(s),y(s))|\,ds,$$

we have

$$\bigg\| \int_0^1 |G_1(t,s)F_1(x(s),y(s))|\,ds \bigg\| \leq B_1 \int_0^1 |F_1(x(s),y(s))|\,ds,$$

and also

$$\int_0^1 |G_1(t,s)F_1(x(s),y(s))|\,ds \geq A_1 \int_0^1 |F_1(x(s),y(s))|\,ds$$
$$\geq \sigma_1 \bigg\| \int_0^1 |G_1(t,s)F_1(x(s),y(s))|\,ds \bigg\|,$$

i.e.

$$\int_0^1 G_1(t,s)F_1(x(s),y(s))\,ds \geq \sigma_1 \bigg\| \int_0^1 G_1(t,s)F_1(x(s),y(s))\,ds \bigg\|.$$

Similarly

$$\int_0^1 G_2(t,s)F_2(x(s),y(s))\,ds \geq \sigma_2 \bigg\| \int_0^1 G_2(t,s)F_2(x(s),y(s))\,ds \bigg\|,$$

so, $T(x,y) \in K_1 \times K_2$. □

Throughout this paper, we make the following hypotheses:

(H_1) $G_i(t,s)f_i(x,y) > 0$ for all $(t,s) \in [0,1] \times [0,1]$, $(x,y) \in [0,\infty)^2 \setminus (0,0)$.

(H_2) $|f_i(x,y)| \in C([0,\infty)^2 \setminus (0,0), (-\infty,\infty))$ and there exist continuous, positive functions $g_i(x,y)$ and $h_i(x,y)$ on $[0,\infty)^2 \setminus (0,0)$ such that

$$|f_i(x,y)| = g_i(x,y) + h_i(x,y) \quad \text{for all } (x,y) \in [0,\infty)^2 \setminus (0,0),\ i = 1,2$$

with $g_i > 0$ continuous and nonincreasing on $[0,\infty)^2 \setminus (0,0)$, $h_i \geq 0$ continuous on $[0,\infty)^2$ and h_i/g_i nondecreasing on $[0,\infty)^2 \setminus (0,0)$, for $i = 1,2$.

(H_3) There exists a positive r such that

$$\frac{r}{g_1(\sigma_1 r,\ 0)(1 + h_1(r,r)/g_1(r,r))} \geq \|\omega_1\|,$$
$$\frac{r}{g_2(0,\sigma_2 r)(1 + h_2(r,r)/g_2(r,r))} \geq \|\omega_2\|.$$

(H_4) There exists a positive $R > r$ such that

$$\frac{R}{\sigma_1 g_1(R,R)(1 + h_1(\sigma_1 R,0)/g_1(\sigma_1 R,0))} \leq \|\omega_1\|,$$
$$\frac{R}{\sigma_2 g_2(R,R)(1 + h_2(0,\sigma_2 R)/g_2(0,\sigma_2 R))} \leq \|\omega_2\|.$$

3. Main result and proof

THEOREM 3.1. *Suppose that a_i satisfies* (A) *and let* (H_1)–(H_3) *hold. Then the problem* (1.1) *has at least one positive periodic solution.*

PROOF. The existence is proved by using the Leray–Schauder alternative principle, together with a truncation technique.

Let $N_0 = \{n_0, n_0+1, \dots\}$, where $n_0 \in \{1, 2, \dots\}$ is chosen such that

$$\begin{aligned}\|\omega_1\| g_1(\sigma_1 r, 0)\bigg(1 + \frac{h_1(r,r)}{g_1(r,r)}\bigg) + \frac{1}{n_0} &< r,\\ \|\omega_2\| g_2(0, \sigma_2 r)\bigg(1 + \frac{h_2(r,r)}{g_2(r,r)}\bigg) + \frac{1}{n_0} &< r;\end{aligned}$$

see (H_3). Fix $n \in N_0$. Consider the systems

$$\begin{cases} x'' + a_1(t)x = \lambda f_1^n(x,y) + a_1(t)/n, \\ y'' + a_2(t)y = \lambda f_2^n(x,y) + a_2(t)/n, \end{cases} \tag{3.1}$$

where $\lambda \in [0,1]$ and $|f_i^n(x,y)| = g_i^*(x,y) + h_i(x,y)$. Here

$$g_1^*(x,y) = \begin{cases} g_1(x,y) & \text{for } x > 1/n, \\ g_1(1/n, y) & \text{for } x \le 1/n, \end{cases}$$

and

$$g_2^*(x,y) = \begin{cases} g_2(x,y) & \text{for } y > 1/n, \\ g_2(x, 1/n) & \text{for } y \le 1/n. \end{cases}$$

Problem (3.1)–(1.2) is equivalent to the following fixed point problem in $C[0,1] \times C[0,1]$

$$(x,y) = \lambda T_n(x,y) + \bigg(\frac{1}{n}, \frac{1}{n}\bigg), \tag{3.2}$$

where T_n denotes the operator defined by (2.5), with $F_i(x,y)$ replaced by $f_i^n(x,y)$.

We claim that any fixed point x of (3.2) for any $\lambda \in [0,1]$ must satisfy $\|(x,y)\| \ne r$. If not, assume that (x,y) is a solution of (3.2) for some $\lambda \in [0,1]$ such that $\|(x,y)\| = r$. Since

$$\|(x,y)\| = \max(\|x\|, \|y\|),$$

without loss of generality, we assume that $\|x\| = r$. Note that $f_i^n(x,y) \ge 0$. By Lemma 2.7, for all t,

$$x(t) \ge \frac{1}{n} \quad \text{and} \quad r \ge x(t) \ge \frac{1}{n} + \sigma_1 \bigg\|x - \frac{1}{n}\bigg\|.$$

By the choice of n_0, $1/n \le 1/n_0 < r$.

Hence, for all t, $x(t) \ge 1/n$, $y(t) \ge 1/n$ and

$$r \ge x(t) \ge \frac{1}{n} + \sigma_1 \bigg\|x - \frac{1}{n}\bigg\| \ge \frac{1}{n} + \sigma_1\bigg(r - \frac{1}{n}\bigg) > \sigma_1 r. \tag{3.3}$$

Note that

$$\int_0^1 |G_1(t,s)|\,ds = \bigg|\int_0^1 G_1(t,s)\,ds\bigg| = |\omega_1(t)|.$$

Using (3.3), we have from condition (H_2), for all t,

$$
\begin{aligned}
(3.4)\qquad x(t) &= \lambda \int_0^1 G_1(t,s) f_1^n(x(s),y(s))\,ds + \frac{1}{n} \\
&\le \int_0^1 |G_1(t,s)||f_1(x(s),y(s))|\,ds + \frac{1}{n} \\
&= \int_0^1 |G_1(t,s)| g_1(x(s),y(s)) \bigg(1 + \frac{h_1(x(s),y(s))}{g_1(x(s),y(s))}\bigg)\,ds + \frac{1}{n} \\
&\le g_1(\sigma_1 r, 0)\bigg(1 + \frac{h_1(r,r)}{g_1(r,r)}\bigg) \int_0^1 |G_1(t,s)|\,ds + \frac{1}{n_0} \\
&\le \|\omega_1\| g_1(\sigma_1 r, 0)\bigg(1 + \frac{h_1(r,r)}{g_1(r,r)}\bigg) + \frac{1}{n_0}.
\end{aligned}
$$

Therefore,

$$
r = \|x\| \le \|\omega_1\| g_1(\sigma_1 r, 0)\bigg(1 + \frac{h_1(r,r)}{g_1(r,r)}\bigg) + \frac{1}{n_0}.
$$

This is a contradiction to the choice of n_0 and the claim is proved.

From this claim, the nonlinear alternative of Leray–Schauder guarantees that (3.2) (with $\lambda = 1$) has a fixed point, denoted by (x_n, y_n), in $B_r = \{(x,y) : \|(x,y)\| < r\}$, i.e. (3.1) (with $\lambda = 1$) has a periodic solution (x_n, y_n) with $\|(x_n,y_n)\| < r$. Since (x_n,y_n) satisfies (3.2), $(x_n,y_n) \ge (1/n, 1/n)$ for all t. Thus (x_n,y_n) is a positive periodic solution of (3.1) (with $\lambda = 1$).

Next we claim that these solutions (x_n,y_n) have a uniform positive lower bound, i.e. there exists a constant vector $\delta = (\delta_1, \delta_2)$, $\delta > (0,0)$, independent of $n \in N_0$, such that

$$
(3.5)\qquad \min_t (x_n(t), y_n(t)) \ge \delta
$$

for all $n \in N_0$. To see this,we know from (H_1) that

$$
\begin{aligned}
x_n(t) &= \int_0^1 G_1(t,s) f_1^n(x_n(s), y_n(s))\,ds + \frac{1}{n} \\
&= \int_0^1 |G_1(t,s) f_1(x_n(s), y_n(s))|\,ds + \frac{1}{n} \\
&\ge \int_0^1 |G_1(t,s)| g_1(x_n(s), y_n(s))\,ds + \frac{1}{n} > A g_1(r,r) =: \delta_1.
\end{aligned}
$$

Similarly $y_n(t) > A_2 g_2(r,r) = \delta_2$, so we have $\min_t(x_n(t), y_n(t)) \ge \delta$.

To establish the existence to the original system(1.1), we need the following fact

$$
(3.6)\qquad \|(x_n', y_n')\| \le H
$$

for some constant $H > 0$ and for all $n \ge n_0$. First, we claim there is H_1, such that $\|x_n'\| \le H_1$. First from the boundary condition, $x_n'(t_0) = 0$ for some $t_0 \in [0,1]$.

Integrating the first equation of (3.1) (with $\lambda = 1$) from 0 to 1, we obtain

$$\int_0^1 a_1(t)\bigg(x_n(t) - \frac{1}{n}\bigg)\,dt = \int_0^1 f_1^n(x_n(s), y_n(s))\,ds.$$

Since $x_n(t) \geq 1/n$ and $a_1(t)f_1(x_n(s), y_n(s)) > 0$, then

$$\begin{aligned}
\|x_n'\| &= \max_{0\leq t\leq 1} |x_n'(t)| = \max_{0\leq t\leq 1} \bigg|\int_{t_0}^t x_n''(s)\,ds\bigg| \\
&= \max_{0\leq t\leq 1} \bigg|\int_{t_0}^t \bigg[f_1^n(x_n(s), y_n(s)) + a_1(s)\bigg(\frac{1}{n} - x_n(s)\bigg)\bigg]\,ds| \\
&\leq \int_0^1 |f_1^n(x_n(s), y_n(s))| + \bigg|a_1(s)\bigg(x_n(s) - \frac{1}{n}\bigg)\bigg|\,ds \\
&= 2\int_0^1 |a_1(s)x_n(s)|\,ds < 2r\|a_1\|_1 =: H_1.
\end{aligned}$$

Similarly, we have $\|y_n'\| \ \leq H_2$.

Let $H = \max\{H_1, H_2\}$, so $\|(x_n', y_n')\| \leq H$.

Now $\|(x_n, y_n)\| < r$ and (3.6) show that $\{(x_n, y_n)\}_{n\in N_0}$ is a bounded and equi-continuous family on $[0,1]$. The Arzela–Ascoli Theorem guarantees that $\{(x_n, y_n)\}_{n\in N_0}$ has a subsequence, $\{(x_{n_k}, y_{n_k})\}_{k\in N}$, converging uniformly on $[0,1]$ to a $(x, y) \in C[0,1] \times C[0,1]$. From $\|(x_n, y_n)\| < r$ and (3.5), (x, y) satisfies $\delta \leq (x(t), y(t)) \leq (r, r)$ for all t. Moreover, (x_{n_k}, y_{n_k}) satisfies the integral equation

$$\begin{cases}
x_{n_k}(t) = \displaystyle\int_0^1 G_1(t,s) f_1(x_{n_k}(s), y_{n_k}(s))\,ds + \frac{1}{n_k}, \\
y_{n_k}(t) = \displaystyle\int_0^1 G_2(t,s) f_2(x_{n_k}(s), y_{n_k}(s))\,ds + \frac{1}{n_k}.
\end{cases}$$

Letting $k \to \infty$, we arrive at

$$(x(t), y(t)) = \bigg(\int_0^1 G_1(t,s) f_1(x(s), y(s))\,ds, \int_0^1 G_2(t,s) f_2(x(s), y(s))\,ds\bigg)$$

where the uniform continuity of $f_i(x, y)$ on $[\delta_1, r] \times [\delta_2, r]$ is used. Therefore, (x, y) is a positive periodic solution of (1.1).

Finally it is easy to see that $\|(x, y)\| < r$, by noting that if $\|(x, y)\| = r$ an argument similar to the proof of the first claim will yield a contradiction. □

Example 3.2. Consider the singular problem

$$\text{(3.7)} \quad \begin{cases}
x''(t) + a_1(t)x(t) = \sqrt{(x^2 + y^2)^{-\alpha}} + \mu\sqrt{(x^2 + y^2)^{\beta}}, \\
y''(t) + a_2(t)y(t) = -\sqrt{(x^2 + y^2)^{-\alpha}} - \mu\sqrt{(x^2 + y^2)^{\beta}}, \qquad 0 < t < 1, \\
x(0) = x(1),\ x'(0) = x'(1),\ y(0) = y(1),\ y'(0) = y'(1), \quad \alpha > 0,\ \beta \geq 0,
\end{cases}$$

where $a_1 \in \Lambda^+$, $a_2 \in \lambda^-$. Then (3.7) has at least one positive periodic solution for each $0 < \mu < \mu_*$, where μ_* is some positive constant.

We will apply Theorem 3.1 with $g_i = \sqrt{(x^2+y^2)^{-\alpha}}$, $h_i = \mu\sqrt{(x^2+y^2)^{\beta}}$ $(i=1,2)$. Clearly, (H_1) and (H_2) hold. Now the condition (H_3) becomes

$$\mu < \frac{\sigma_1^\alpha \sqrt{2}^{-\alpha-\beta} r^{\alpha+1}/\|\omega_1\| - \sqrt{2}^{-\alpha-\beta}}{r^{\alpha+\beta}}$$

and

$$\mu < \frac{\sigma_2^\alpha \sqrt{2}^{-\alpha-\beta} r^{\alpha+1}/\|\omega_2\| - \sqrt{2}^{-\alpha-\beta}}{r^{\alpha+\beta}}$$

for some $r > 0$, so (3.7) has at least one positive period solution (x_1, y_1) for $0 < \mu < \mu^*$, if

$$\mu^* = \max\bigg\{\sup_{r>0} \frac{\sigma_1^\alpha \sqrt{2}^{-\beta} r^{\alpha+1}/\|\omega_1\| - \sqrt{2}^{-\alpha-\beta}}{r^{\alpha+\beta}}, \\ \sup_{r>0} \frac{\sigma_2^\alpha \sqrt{2}^{-\beta} r^{\alpha+1}|\omega_2\| - \sqrt{2}^{-\alpha-\beta}}{r^{\alpha+\beta}}\bigg\}.$$

We remark here that $\mu^* = \infty$ if $0 \le \beta < 1$, and $\mu^* < \infty$ if $\beta > 1$.

THEOREM 3.3. *Suppose that a_i satisfies* (A) *and let* (H_1)–(H_4) *hold. Then, besides the periodic solution x constructed in Theorem* 3.1*, the problem* (1.1) *has another positive periodic solution.*

PROOF. First we have $\|T(x,y)\| < \|(x,y)\|$ for $(x,y) \in K \cap \partial\Omega_1$, $\Omega_1 = B_r$. In fact, if $x \in K \cap \partial\Omega_1$, then $\|(x,y)\| = r$. Now the estimate $\|T(x,y)\| < r$ can be obtained following the ideas used to prove (3.4).

Next we show that $\|T(x,y)\| \ge \|(x,y)\|$ for $(x,y) \in K \cap \partial\Omega_2$, where $\Omega_2 = B_R = \{(x,y) \| (x,y)\| < R\}$, and R is as in (H_4). To see this, let $(x,y) \in K \cap \partial\Omega_2$. Then $\|(x,y)\| = R$ and without loss of generality we assume $\|x\| = R$, so $x(t) \ge \sigma_1 R$. As a result, it follows from (H_2) and (H_4) that, for $0 \le t \le 1$,

$$\begin{aligned}\int_0^1 G_1(t,s) f_1(x(s),y(s))\,ds &= \int_0^1 |G_1(t,s)| g_1(x(s),y(s)) \frac{1+h_1(x(s),y(s))}{g_1(x(s),y(s))}\,ds \\ &\ge \int_0^1 |G_1(t,s)| g_1(R,R) \frac{1+h_1(\sigma_1 R,0)}{g_1(\sigma_1 R,0)}\,ds \\ &= g_1(R,R)\frac{1+h_1(\sigma_1 R,0)}{g_1(\sigma_1 R,0)}\omega_1(t) \\ &\ge \sigma_1 \|\omega_1\| g_1(R,R) \frac{1+h_1(\sigma_1 R,0)}{g_1(\sigma_1 R,0)} \ge R.\end{aligned}$$

This implies $\|T(x,y)\| \ge \|(x,y)\|$.

Now Theorem 2.6 guarantees that T has a fixed point $\widetilde{(x,y)} \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$. Thus $r \le \|\widetilde{(x,y)}\| \le R$.

By the same argument as in Theorem 3.1 we see that there exist $(\delta_3, \delta_4) > (0,0)$ such that $\widetilde{(x,y)} > (\delta_3, \delta_4)$. $\square$

Let we consider (3.7) again with $\alpha > 0$, $\beta > 1$. Now the condition (H_4) becomes

$$\mu \geq \frac{\sigma_1^{-\alpha-\beta-1}\sqrt{2}^{\alpha} R^{\alpha+1}/\|\omega_1\| - \sigma_1^{-\alpha-\beta}\sqrt{2}^{-\beta}}{R^{\alpha+\beta}} \tag{3.8}$$

and

$$\mu \geq \frac{\sigma_2^{-\alpha-\beta-1}\sqrt{2}^{\alpha} R^{\alpha+1}/\|\omega_2\| - \sigma_2^{-\alpha-\beta}\sqrt{2}^{-\beta}}{R^{\alpha+\beta}}. \tag{3.9}$$

Since $\beta > 1$, the right-hand side goes to 0 as $R \to \infty$. Thus, for any given $0 < \mu < \mu^*$, it is always possible to find a $R > r$ such that (3.8) and (3.9) are satisfied. Thus, (3.7) has an additional periodic solution (x_2, y_2) such that $r < \|(x_2, y_2)\| \leq R$.

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