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THE CONLEY INDEX AND SPECTRAL SEQUENCES

Piotr Bartłomiejczyk

Abstract. We define spectral sequences associated with Morse decompositions of a compact metric space. We prove the existence and uniqueness of such spectral sequences for continuous dynamical systems.

1. Introduction

Many ideas of the Conley index theory is motivated by Morse theory. One of them is Conley's idea of studying isolated invariant sets by decomposing them into invariant subsets (Morse sets) and connecting orbits between them. This structure is called a Morse decomposition of an isolated invariant set. To study its dynamics different tools are used, such as connection matrices and graphs. In [1], [2], [5], [7]–[9] the connection matrix theory is developed for both continuous and discrete dynamical systems. In [4] authors use connection graphs to investigate a variational problem.

In this paper we introduce another tool for studying the dynamics of the Morse decomposition, namely a spectral sequence of the Morse decomposition. This spectral sequence is related to a filtration of index pairs associated with the Morse decomposition and provides some information on the structure of the Morse decomposition. For example, its first differential gives an algebraic condition for the existence of connecting orbits between adjoining Morse sets.

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We prove the existence and uniqueness of such spectral sequences in the case of a flow on a compact metric space. The advantage of using spectral sequences lies in the fact that they are well-known algebraic objects with many nice properties. However, we must admit that we restrict our attention to total orderings on the Morse sets, because we want to apply the classical theory of spectral sequences.

The organization of the paper is as follows. Sections 2 and 3 contain a brief exposition of Zeeman's theory of spectral sequences [11]. In Sections 4 and 5 we recall some basic facts from the Conley index theory. In Section 6 we introduce the notion of the spectral sequence of the Morse decomposition. Finally, in Section 7 our main results are stated and proved.

The general references to the Conley index theory are [3], [6].

2. Filtered differential modules

If A is a module, then a sequence $\{A^i\}_0^n$ of submodeles of A such that the following inclusions hold

$$0 = A^0 \subset A^1 \subset \ldots \subset A^n = A$$

is called a *filtered module*. In fact, instead of modules we may consider abelian groups, vector spaces etc.

Definition 2.1. A *filtered differential module* is a filtered module A together with an endomorphism d such that $d^2 = 0$ and d preserves the filtration i.e. $dA^i \subset A^i$.

We have two natural filtrations associated with the filtered differential module. Namely,

$$\begin{aligned} 0 &= A^0 \subset A^1 \subset \ldots \subset A^n = A, \\ 0 &= dA^0 \subset dA^1 \subset \ldots \subset dA \subset d^{-1}0 \subset d^{-1}A^1 \subset \ldots \subset d^{-1}A^n = A. \end{aligned}$$

Moreover, we will use the following convenient notation:

$$\begin{aligned} A^k &= A \quad \text{for } k \geq n, \\ A^k &= 0 \quad \text{for } k \leq 0. \end{aligned}$$

A *homomorphism of filtered differential modules* is any homomorphism of modules $h: A \to \widehat{A}$ such that $h\widehat{d} = dh$ and h preserves the filtration i.e. $hA^i \subset \widehat{A}^i$. It is easy to see that filtered differential modules and their homomorphisms form a category, which will be denoted by $\mathcal{FDM}$.

3. Spectral sequences

Now we are ready to define spectral sequences of filtered differential modules. We introduce the following notation. Let

$$Z_p^r := A^p \cap d^{-1} A^{p-r}, \quad B_p^r := A^p \cap dA^{p+r}$$

for any $r \in \mathbb{Z}^+$ and $p \in \mathbb{Z}$. Since, as is easy to see, $Z_{p-1}^{r-1} \subset Z_p^r$ and $B_p^{r-1} \subset Z_p^r$, the quotient module

$$E_p^r := \frac{Z_p^r}{Z_{p-1}^{r-1} + B_p^{r-1}}$$

is well defined. Moreover, since the differential d induces homomorphisms

$$Z_p^r \to Z_{p-r}^r, \qquad Z_{p-1}^{r-1} + B_p^{r-1} \to Z_{p-r-1}^{r-1} + B_{p-r}^{r-1},$$

it also induces the homomorphism of quotient modules, which we will denote by d_p^r:

$$d_p^r \colon E_p^r \to E_{p-r}^r.$$

Observe that $d_p^r([z]) = [dz]$, where $[\,\cdot\,]$ denotes the respective equivalence class. From this we obtain $d_{p-r}^r d_p^r[z] = [ddz] = 0$ and so $d_{p-r}^r d_p^r = 0$. For a fixed r homomorphisms d_p^r induce the homomorphism

$$d^r \colon \bigoplus_p E_p^r \to \bigoplus_p E_p^r.$$

Hence d^r is a differential of a module $E^r = \bigoplus_p E_p^r$. A sequence of modules and differentials $\{E^r, d^r\}$, $r = 0, 1, \dots$, is called the *spectral sequence of the filtered differential module A*.

If $h \colon A \to \widehat{A}$ is a homomorphism of filtered differential modules and $\widehat{Z}_p^r$, $\widehat{B}_p^r$, $\widehat{E}_p^r$, denote the respective modules determined by $\widehat{A}$, then h induces homomorphisms $Z_p^r \to \widehat{Z}_p^r$, $B_p^r \to \widehat{B}_p^r$. Consequently, there exists an induced homomorphism

$$h_p^r \colon E_p^r \to \widehat{E}_p^r$$

given by $h_p^r[z] = [hz]$. Finally, homomorphisms h_p^r define a homomorphism of modules

$$h^r \colon E^r \to \widehat{E}^r$$

such that $h^r \widehat{d}^r = d^r h^r$.

It is easily seen that the above construction actually defines a functor from the category $\mathcal{FDM}$ to the category of modules $\mathcal{M}$ which maps a filtered differential module A to the module E^r and sends a homomorphism of filtered differential modules $h \colon A \longrightarrow \widehat{A}$ to the homomorphism of modules $h^r \colon E^r \to \widehat{E}^r$.

4. Index pairs and the Conley index

Now let's recall briefly some basic definitions from the Conley index theory. Assume that X is a compact metric space and φ is a flow on X. Given $N \subset X$ the *maximal invariant* set of N is defined by

$$\mathrm{Inv}(N) := \{x \in N \mid \varphi(t,x) \in N \text{ for all } t \in R\}.$$

A set S is an *isolated invariant set* if there exists a compact set N such that $S = \mathrm{Inv}(N) \subset \mathrm{int}(N)$. Let S be an isolated invariant set.

Definition 4.1. A pair of compact sets (N^1, N^0) is an *index pair* for S if:

(a) $N^0 \subset N^1$,
(b) $S = \mathrm{Inv}(\mathrm{cl}(N^1 \setminus N^0)) \subset \mathrm{int}(N^1 \setminus N^0)$,
(c) N^0 is positively invariant in N^1; that is if given $x \in N^0$ and $\varphi([0,t],x) \subset N^1$, then $\varphi([0,t],x) \subset N^0$,
(d) N^0 is an exit set for N^1; that is if given $x \in N^1$ and $t_1 > 0$ such that $\varphi(t_1,x) \notin N^1$, then there exists $t_2 \geq 0$ such that $\varphi([0,t_2],x) \subset N^1$ and $\varphi(t_2,x) \in N^0$.

Consequently, the Conley index of S is defined in terms of any index pair for S. For example, the *homological Conley index* of S is defined by

$$CH_*(S) := H_*(N^1, N^0),$$

where (N^1, N^0) is any index pair for S.

5. Morse decompositions and index filtration

Recall that given a point $x \in X$ the *positive omega limit set* of X is given by

$$\omega^+(x) := \bigcap_{t>0} \mathrm{cl}(\varphi([t,\infty),x))$$

and the *negative omega limit set* is

$$\omega^-(x) := \bigcap_{t<0} \mathrm{cl}(\varphi((-\infty,t],x)).$$

We give two more definitions from the Conley index theory. Let S be an isolated invariant set.

Definition 5.1. A collection $\{M_i\}_1^n$ of mutually disjoint compact invariant subsets of S is a *Morse decomposition* of S if for every $x \in S \setminus \bigcup_{i=1}^n M_i$ there are indices $i < j$ such that $\omega^+(x) \subset M_i$ and $\omega^-(x) \subset M_j$.

The sets M_i are called *Morse sets*. Moreover, we define generalized Morse sets for $i \le j$:

$$M_{ji} := \left\{ x \in S \;\middle|\; \omega^+(x) \cup \omega^-(x) \subset \bigcup_{k=i}^{j} M_k \right\}.$$

In particular, $M_{jj} = M_j$. It is easy to check that all M_{ji} are isolated invariant sets.

Definition 5.2. An *index filtration* for the Morse decomposition $\{M_i\}_1^n$ is a collection of compact sets $\{N^i\}_0^n$ such that

(a) $N^0 \subset \ldots \subset N^n$,
(b) for any $i \le j$, (N^j, N^{i-1}) is an index pair for M_{ji}.

Let us formulate natural

Theorem 5.3. *For any given Morse decomposition there exists an index filtration.*

This was proved by Salamon [10].

The simplest nontrivial case of a Morse decomposition of an isolated invariant set S is one consisting of two elements $\{M_1, M_2\}$. It is called an *attractor-repeller pair* in S. The *set of connecting orbits* from M_2 to M_1 in S is

$$C(M_2, M_1; S) := \{x \in S \mid \omega^+(x) \subset M_1,\ \omega^-(x) \subset M_2\}.$$

An index filtration for an attractor-repeller pair $\{M_1, M_2\}$ is reduced to an *index triple* $N^0 \subset N^1 \subset N^2$, where

$$\begin{array}{ll} (N^2, N^0) & \text{is an index pair for } S, \\ (N^2, N^1) & \text{is an index pair for } M_2, \\ (N^1, N^0) & \text{is an index pair for } M_1. \end{array}$$

Let ∂ denote the boundary map in a long exact homology sequence:

$$\cdots \to H_k(N^1, N^0) \to H_k(N^2, N^0) \to H_k(N^2, N^1) \xrightarrow{\partial} H_{k-1}(N^1, N^0) \to \cdots$$

The importance of the boundary map ∂ is given by the following

Theorem 5.4. *If $\partial \neq 0$, then $C(M_2, M_1; S) \neq 0$.*

Proof. If $C(M_2, M_1; S) = 0$, then $S = M_1 \cup M_2$, so

$$H_*(N^2, N^0) = CH_*(S) \simeq CH_*(M_1) \oplus CH_*(M_2) = H_*(N^1, N^0) \oplus H_*(N^2, N^1)$$

and, in consequence, $\partial = 0$. □

The next result will be needed until Section 7.

Lemma 5.5. *Let $N^0 \subset N^1 \subset N^2$ and $L^0 \subset L^1 \subset L^2$ be index triples for an attractor-repeller pair (A, R) in an isolated invariant set S. Then, there exist flow-defined homotopy equivalences φ_{ji} $(0 \le i < j \le 2)$ such that the following diagram commutes*

$$\begin{tikzcd} N^1/N^0 \arrow[r] \arrow[d, "\varphi_{10}"'] & N^2/N^0 \arrow[r] \arrow[d, "\varphi_{20}"] & N^2/N^1 \arrow[d, "\varphi_{21}"] \\ L^1/L^0 \arrow[r] & L^2/L^0 \arrow[r] & L^2/L^1 \end{tikzcd}$$

Proof. By Salamon [10, Lemma 4.7 and Theorem 4.10], for any $0 \le i < j \le 2$ there exists $T_{ji} \ge 0$ such that for $t \ge T_{ji}$ a flow-defined map

$$f^t_{ji}\colon \frac{N^j}{N^i} \to \frac{L^j}{L^i}$$

given by

$$f^t_{ji}([x]) = \begin{cases} [\varphi(3t, x)] & \text{if } \varphi([0, 2t], x) \subset N^j \setminus N^i \text{ and } \varphi([t, 3t], x) \subset L^j \setminus L^i, \\ [L^i] & \text{otherwise,} \end{cases}$$

is a homotopy equivalence. Taking $T = \max\{T_{ji} \mid 0 \le i < j \le 2\}$ and $\varphi_{ji} = f^T_{ji}$ we get the required maps. □

6. A spectral sequence for the Morse decomposition

Let $\emptyset = N^0 \subset N^1 \subset \ldots \subset N^n = N$ be a topological filtration. Let $C(N^k)$ be a module of singular chains in N^k and $i_k\colon C(N^k) \to C(N)$ be a homomorphism induced by the inclusion $N^k \subset N$.

Definition 6.1. A *spectral sequence for the topological filtration* $\{N^i\}_0^n$ is a spectral sequence of the filtered differential module $(\{i_k(C(N^k))\}_0^n, d)$, where d is a boundary map on singular chains.

Let X be a compact metric space.

Definition 6.2. A *spectral sequence for the Morse decomposition* $\{M_i\}_1^n$ of X is a spectral sequence for any index filtration for this Morse decomposition.

Some information about the Morse decomposition may be studied by its spectral sequence. For example:

Theorem 6.3. *If $d^1_p \ne 0$, then $C(M_p, M_{p-1}; X) \ne \emptyset$ for $p = 2, \ldots, n$.*

Proof. From the definition of the spectral sequence for the Morse decompositon $\{M_i\}_1^n$, we obtain the following commutative diagram

$$\begin{array}{ccc} E_p^1 & \xrightarrow{d_p^1} & E_{p-1}^1 \\ \downarrow & & \downarrow \\ H_*(N^p, N^{p-1}) & \xrightarrow[\partial]{} & H_*(N^{p-1}, N^{p-2}) \end{array}$$

in which vertical maps are cannonical isomorphisms. Observe that $N^{p-2} \subset N^{p-1} \subset N^p$ form an index triple for an attractor-repeller pair (M_{p-1}, M_p) in $M_{p,p-1}$ and $\partial \neq 0$, since $d_p^1 \neq 0$. By Theorem 5.4 the set $C(M_p, M_{p-1}; X)$ is nonempty. □

7. Existence and uniqueness

We can now formulate our main results.

Theorem 7.1 (Existence). *For any Morse decomposition of the compact metric space X there exists a spectral sequence.*

Proof. By Theorem 5.3, there is an index filtration such that $N^0 = \emptyset$ and $N^n = X$ for any Morse decomposition of X. Hence the construction of the spectral sequence, as presented in the previous section, poses no problem. □

Theorem 7.2 (Uniqueness). *Any two spectral sequences E, $\widehat{E}$ for the Morse decomposition $\{M_i\}_1^n$ are isomorphic in the following sense: if $N^0 \subset \ldots \subset N^n$ and $L^0 \subset \ldots \subset L^n$ are two index filtrations for the Morse decomposition $\{M_i\}_1^n$, then there is a flow-defined homomorphism of filtered differential modules $h: C(N^n) \to C(L^n)$ such that for all $r \geq 0$ induced maps $h^r: E^r \to \widehat{E}^r$ are isomorphisms.*

Proof. Repeated application of Lemma 5.5 enables us to write the following commutative diagram

$$\begin{array}{ccccccc} N^1/N^0 & \longrightarrow & N^2/N^0 & \longrightarrow & \cdots & \longrightarrow & N^n/N^0 \\ \downarrow{\scriptstyle\varphi_{10}} & & \downarrow{\scriptstyle\varphi_{20}} & & & & \downarrow{\scriptstyle\varphi_{n0}} \\ L^1/L^0 & \longrightarrow & L^2/L^0 & \longrightarrow & \cdots & \longrightarrow & L^n/L^0 \end{array}$$

where all φ_{ko} are flow-defined homotopy equivalences. But $N^0 = L^0 = \emptyset$, so $N^k/N^0 = N^k \cup \{*\}$ and $L^k/L^0 = L^k \cup \{*\}$ for any $1 \leq k \leq n$, where $*$ denotes the equivalence class consisting of the empty set. Consequently, we obtain the following commutative diagram

$$\begin{tikzcd}
(N^1 \cup \{*\}, \{*\}) \arrow[r] \arrow[d, "\varphi_{10}"'] & (N^2 \cup \{*\}, \{*\}) \arrow[r] \arrow[d, "\varphi_{20}"] & \cdots \arrow[r] & (N^n \cup \{*\}, \{*\}) \arrow[d, "\varphi_{n0}"] \\
(L^1 \cup \{*\}, \{*\}) \arrow[r] & (L^2 \cup \{*\}, \{*\}) \arrow[r] & \cdots \arrow[r] & (L^n \cup \{*\}, \{*\})
\end{tikzcd}$$

in which all φ_{k0} are homotopy equivalences of pairs. Observe that by the definition of relative singular chains, we get

$$C(P \cup \{*\}, \{*\}) = \frac{C(P \cup \{*\})}{C(\{*\})} \simeq \frac{C(P) + C(\{*\})}{C(\{*\})} \simeq \frac{C(P)}{C(P) \cap C(\{*\})} = C(P),$$

where both isomorphisms are cannonical. This means that the following diagram commutes

$$\begin{tikzcd}
C(N^1) \arrow[r] \arrow[d, "C(\varphi_{10})"'] & C(N^2) \arrow[r] \arrow[d, "C(\varphi_{20})"] & \cdots \arrow[r] & C(N^n) \arrow[d, "C(\varphi_{n0})"] \\
C(L^1) \arrow[r] & C(L^2) \arrow[r] & \cdots \arrow[r] & C(L^n)
\end{tikzcd}$$

According to the above diagram, $h = C(\varphi_{n0}) \colon C(N^n) \to C(L^n)$ preserves the filtration and, in consequence, is a homomorphism of filtered differential modules $\{i_k(C(N^k))\}_0^n$ and $\{i_k(C(L^k))\}_0^n$. Since φ_{n0} is a homotopy equivalence, it follows that all induced maps h_p^r are isomorphisms, which completes the proof. $\square$

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PIOTR BARTŁOMIEJCZYK
Institute of Mathematics
University of Gdańsk
Wita Stwosza 57
80-952 Gdańsk, POLAND

E-mail address: pb@math.univ.gda.pl