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EXISTENCE OF SOLUTIONS TO SOME ELLIPTIC SYSTEM IN SOBOLEV SPACES WITH THE WEIGHT AS A POWER OF THE DISTANCE FROM SOME AXIS

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Abstract. We examine some overdetermined elliptic system in a domain in $\mathbb{R}^3$ which contains an axis. Assuming that data functions belong to Sobolev spaces with weights equal to a power of the distance from the axis we prove existence of solutions in the corresponding kind of weighted Sobolev spaces.

1. Introduction

In this paper we prove the existence and some regularity properties of solutions to the following overdetermined elliptic system (see also [6])

$$
\begin{aligned}
\operatorname{rot} v &= \omega && \text{in } \Omega,\\
\operatorname{div} v &= 0 && \text{in } \Omega,\\
v\cdot\overline{n} &= 0 && \text{on } \partial\Omega,
\end{aligned}
\tag{1.1}
$$

where Ω is a bounded domain in $\mathbb{R}^3$, $v=(v_1(x),v_2(x),v_3(x))$, $x=(x_1,x_2,x_3)$ and $\overline{n}$ is a unit outward normal vector to the boundary $\partial\Omega$.

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The given vector $\omega = (\omega_1(x), \omega_2(x), \omega_3(x)) \in \mathbb{R}^3$ must satisfy the compatibility condition

$$\operatorname{div} \omega = 0 \quad \text{in } \Omega. \tag{1.2}$$

Assume that $\omega \in H^k_\mu(\Omega; L)$ (or $W^k_{2,\mu}(\Omega; L)$), $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mu \in \mathbb{R}_+$, which is a Sobolev space with the weight equal to the μ-power of the distance from an axis L passing through Ω (for the notation see Section 2). Then we show the existence of such solutions to (1.1) that $v \in H^{k+1}_\mu(\Omega; L)$ (or $W^{k+1}_{2,\mu}(\Omega; L)$) and the corresponding estimate holds.

Unfortunately we do not know how to solve problem (1.1) directly. Therefore we introduce potentials for v. By [1, Lemma 1] there exists a vector u such that

$$v = \operatorname{rot} u, \quad \operatorname{div} u = 0, \quad u \cdot \overline{\tau}_\alpha|_{\partial\Omega} = 0, \tag{1.3}$$

where $\overline{\tau}_\alpha$, $\alpha = 1, 2$, are tangent vectors to $\partial\Omega$.

The vector u is defined as

$$u = u^1 + u^2, \tag{1.4}$$

where

$$u^1(x) = \frac{1}{4\pi} \operatorname{rot} \int_\Omega \frac{v(y)}{|x-y|}\, dy, \quad u^2(x) = \nabla\psi(x),$$

where ψ is a solution to the Dirichlet problem

$$\Delta\psi = 0, \quad \psi|_{\partial\Omega} = -\psi_0, \tag{1.5}$$

with

$$u^1 \cdot \overline{\tau}_\alpha|_{\partial\Omega} = \psi_{0,\tau_\alpha}, \quad \alpha = 1, 2, \tag{1.6}$$

and τ_α is the curvilinear coordinate along the curve tangent to vector $\overline{\tau}_\alpha$, $\alpha = 1, 2$.

Using the potential u we write (1.1) in the form

$$\begin{gathered} -\Delta u = \omega, \\ u \cdot \overline{\tau}_\alpha|_{\partial\Omega} = 0, \quad \alpha = 1, 2, \\ \operatorname{div} u|_{\partial\Omega} = 0, \end{gathered} \tag{1.7}$$

where we have taken into account that

$$\Delta \operatorname{div} u = 0, \quad \operatorname{div} u|_{\partial\Omega} = 0 \tag{1.8}$$

implies

$$\operatorname{div} u = 0. \tag{1.9}$$

In a curvilinear system of coordinates (τ_1, τ_2, n) and in a neighbourhood of $\partial\Omega$ the vector u can be written in the form

$$u = \sum_{\mu=1}^{2} u_{\tau_\mu} \overline{\tau}_\mu + u_n \overline{n},$$

where $u_{\tau_\mu} = u \cdot \overline{\tau}_\mu$, $u_n = u \cdot \overline{n}$ and n is the coordinate along the curve tangent to the normal vector $\overline{n}$. Then (1.7) can be replaced by

$$\begin{aligned} -\Delta u &= \omega && \text{in } \Omega, \\ u_{\tau_\alpha} &= 0, \quad \alpha = 1,2, && \text{on } \partial\Omega, \\ \overline{n} \cdot \nabla u_n + u_n \operatorname{div} \overline{n} &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.10}$$

From now on we shall restrict our considerations to problem (1.10). To prove existence and regularity of solutions to (1.10) we prove the existence of weak solutions and next using a partition of unity we increase regularity locally. From [6] we have

Definition 1.1. A weak solution to problem (1.10) is defined to be a function $u \in H^1(\Omega)$, $u_\tau|_{\partial\Omega} = 0$, such that

$$\int_\Omega \nabla u \cdot \nabla \eta \, dx + \int_{\partial\Omega} u_n \eta_n \operatorname{div} \overline{n} \, ds = \int_\Omega \omega \cdot \eta \, dx, \tag{1.11}$$

holds for all $\eta \in H^1(\Omega)$, $\eta_\tau|_{\partial\Omega} = 0$, where $\operatorname{div} \omega = 0$. Moreover, see [6].

Lemma 1.2. *Let $\omega \in L_2(\Omega)$. Then there exists a weak solution to* (1.10) *such that $u \in H^1(\Omega)$, $u_\tau|_{\partial\Omega} = 0$ and*

$$\|u\|_{H^1(\Omega} \le c \|\omega\|_{L_2(\Omega)}. \tag{1.12}$$

Since any increasing of regularity of the weak solutions in nontrivial in a neighbourhood of the axis only we shall restrict examining of (1.10) to such neighbourhoods. Taking the idea from [8] we apply the Kondratiev technique [2], [5] to the following artificial problem

$$\begin{aligned} -\Delta u &= \omega, \\ u|_{\Gamma_0} &= u|_{\Gamma 2\pi}, \\ \overline{n} \cdot \nabla u|_{\Gamma_0} &= -\overline{n} \cdot \nabla u|_{\Gamma 2\pi}, \end{aligned} \tag{1.13}$$

where ω has a compact support, $\Gamma_0 = \Gamma_{2\pi} = \{x \in \mathbb{R}^3 : \ x_2 = 0\}$, $\overline{n}|_{\Gamma_0} = (0, -1, 0)$, $\overline{n}|_{\Gamma 2\pi} = (0, 1, 0)$.

2. Notation and auxiliary results

We introduce a system of local coordinates (x_1, x_2, x_3) such that the axis x_3 is the distinguished axis L in Ω. We denote $x' = (x_1, x_2)$ and $|x'| = \sqrt{x_1^2 + x_2^2}$. Following Kondratiev [2] there are introduced spaces $H^k_\mu(\mathbb{R}^2; 0)$ and $H^k_\mu(\mathbb{R}^3; L)$, $k \in \mathbb{N}_0$, $\mu \in \mathbb{R}_+$ with the norms

$$\|u\|_{H^k_\mu(\mathbb{R}^2;0)} = \Big(\sum_{|\alpha'|\le k}\int_{\mathbb{R}^2} |D^{\alpha'}_{x'}u|^2 |x'|^{2(\mu-(k-|\alpha'|))}\,dx'\Big)^{1/2},$$

and

$$\|u\|_{H^k_\mu(\mathbb{R}^3;L)} = \Big(\sum_{|\alpha|\le k}\int_{\mathbb{R}^3} |D^{\alpha}_{x}u|^2 |x'|^{2(\mu-(k-|\alpha|))}\,dx\Big)^{1/2},$$

where $\alpha' = (\alpha_1, \alpha_2)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ are multiindices, $D^{\alpha'}_{\alpha'} = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}$, $D^\alpha_x = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}$, $|\alpha'| = \alpha_1+\alpha_2$, $|\alpha| = \alpha_1+\alpha_2+\alpha_3$. Moreover, $H^0_\mu(\mathbb{R}^2; 0) = L_{2,\mu}(\mathbb{R}^2; 0)$, $H^0_\mu(\mathbb{R}^3; L) = L_{2,\mu}(\mathbb{R}^3; L)$.

Let $\zeta(t) \in C^\infty_0(\mathbb{R}_+)$ be a monotone function such that $\zeta(t) = 1$ for $t \le 1/2$ and $\zeta(t) = 0$ for $t \ge 1$. Moreover,

$$\|u\|_{H^k_\mu(\Omega;L)} = \Big(\sum_{|\alpha|\le k}\int_{\Omega} |D^{\alpha}_{x}u(x)|^2 (\varrho(x,L))^{2(\mu-(k-|\alpha|))}\,dx\Big)^{1/2},$$

where $\varrho(x, L) = \mathrm{dist}(x, L)$, and

$$\|u\|_{W^k_{2,\mu}(\Omega;L)} = \Big(\sum_{|\alpha|\le k}\int_{\Omega} |D^{\alpha}_{x}u(x)|^2 (\varrho(x,L))^{2\mu}\,dx\Big)^{1/2}.$$

We introduce also

$$\|u\|_{\widetilde{H}^k_\mu(\mathbb{R}^3;0)} = \Big(\sum_{|\alpha|\le k}\int_{\mathbb{R}^3} |D^{\alpha}_{x}u|^2 |x|^{2(\mu-(k-|\alpha|))}\,dx\Big)^{1/2},$$

where $|x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

Finally we introduce spaces $L^k_{2,\mu}(\mathbb{R}^2; 0)$ and $L^k_{2,\mu}(\mathbb{R}^3; L)$ with the norms

$$\|u\|_{L^k_{2,\mu}(\mathbb{R}^2;0)} = \Big(\sum_{|\alpha'|= k}\int_{\mathbb{R}^2} |D^{\alpha'}_{x'}u|^2 |x'|^{2\mu}\,dx'\Big)^{1/2}$$

and

$$\|u\|_{L^k_{2,\mu}(\mathbb{R}^3;L)} = \Big(\sum_{|\alpha|= k}\int_{\mathbb{R}^3} |D^{\alpha}_{x}u|^2 |x'|^{2\mu}\,dx\Big)^{1/2}.$$

Next we recall the following Hardy inequality. Let

$$u^{(j)} = \sum_{|\alpha'|=\alpha_1+\alpha_2\le j} D^{\alpha'}_{x'}u(x)|_{x'=0}\frac{x_1^{\alpha_1}}{\alpha_1!}\frac{x_2^{\alpha_2}}{\alpha_2!}.$$

Then

$$\|u-u^{(j)}\|_{L_{2,\mu-k}(\mathbb{R}^3;L)} \le c\|u\|_{L^k_{2,\mu}(\mathbb{R}^3;L)}, \tag{2.1}$$

where $k-\mu-2<j<k-\mu-1$, $\mu\in(0,1)$. We need also the Hardy inequality in the form

$$\int_0^\infty |u|^2 r^{2\mu}\,dr \le c\int_0^\infty |u_{,r}|^2 r^{2\mu+2}\,dr, \quad \mu\in\mathbb{R}, \tag{2.1$'$}$$

$\mu\neq -1/2$, where the both sides of (2.1′) exist and $u_{,r}=\partial_r u$.

Now we recall some results from [2]. Let us consider the problem

$$\begin{aligned} -\Delta' u &= f \quad \text{in } \mathbb{R}^2,\\ u|_{\gamma_0} &= u|_{\gamma_{2\pi}},\\ u_{,\varphi}|_{\gamma_0} &= u_{,\varphi}|_{\gamma_{2\pi}}, \end{aligned} \tag{2.2}$$

where $\Delta'=\partial^2_{x_1}+\partial^2_{x_2}$, $\gamma_0=\gamma_{2\pi}=\{x\in\mathbb{R}^2: x_2=0\}$, r,φ are the polar coordinates. First we find solutions of the homogenous problem (2.2). Expressing homogeneous problem (2.2) in the polar coordinates we have

$$\begin{gathered} \frac{1}{r}\partial_r(ru_r)+\frac{1}{r^2}u_{,\varphi\varphi}=0,\\ u|_{\varphi=0}=u|_{\varphi=2\pi},\\ u_\varphi|_{,\varphi=0}=u_\varphi|_{,\varphi=2\pi}. \end{gathered} \tag{2.3}$$

A general solution of $(2.3)_1$ has the form

$$u=r^\alpha(a_1\sin\alpha\varphi+a_2\cos\alpha\varphi), \tag{2.4}$$

where a_1,a_2 are arbitrary parameters. Using (2.4) in $(2.3)_{2,3}$ yields

$$\sin 2\pi\alpha=0, \quad \cos 2\pi\alpha=1, \tag{2.5}$$

so α is an arbitrary integer number. Therefore solutions of (2.3) have the form

$$u_{1k}=r^k\sin k\varphi, \quad u_{2k}=r^k\cos k\varphi, \tag{2.6}$$

where $k\in\mathbb{Z}$.

Introducing new variable $\tau=-\ln r$ and the new quantity $v(\tau,\varphi)=u(e^{-\tau},\varphi)$ we write (2.2) in the form

$$\begin{aligned} v_{,\tau\tau}+v_{,\varphi\varphi} &= fe^{-2\tau}\equiv F,\\ v|_{\varphi=0} &= v|_{\varphi=2\pi},\\ v_{,\varphi}|_{\varphi=0} &= v_{,\varphi}|_{\varphi=2\pi}. \end{aligned} \tag{2.7}$$

Applying the Fourier transform

$$v(\tau,\varphi)=\int_{-\infty}^{\infty}e^{i\lambda\tau}\widetilde{v}(\lambda,\varphi)\,d\lambda \tag{2.8}$$

to (2.7) and putting $\sigma=-i\lambda$ yield

$$\begin{gathered}\sigma^2\widetilde{v}+\widetilde{v}_{,\varphi\varphi}=\widetilde{F},\\ \widetilde{v}|_{\varphi=0}=\widetilde{v}|_{\varphi=2\pi},\\ \widetilde{v}_{,\varphi}|_{\varphi=0}=\widetilde{v}_{,\varphi}|_{\varphi=2\pi}.\end{gathered} \tag{2.9}$$

In view of the above considerations we see that solutions of homogeneous problem (2.9) have the form (2.6) when $\sigma\in\mathbb{Z}$.

Assume that $f\in H^k_\mu(\mathbb{R}^2,0)$. Then

$$\sum_{s=0}^{k}\int_{-\infty+ih}^{\infty+ih}|\lambda|^{2s}\|\widetilde{F}\|^2_{H^{k-s}(0,2\pi)}\,d\lambda\le\|f\|^2_{H^k_\mu(\mathbb{R}^2,0)}, \tag{2.10}$$

where $h=1+k-\mu$. Then from [2] we have

Theorem 2.1. *Assume that $f\in H^k_\mu(\mathbb{R}^2;0)$, $\mu\in(0,1)$, $k\in\mathbb{N}_0$, $h\ne0$. Then there exists a unique solution $u\in H^{k+2}_\mu(\mathbb{R}^2;0)$ of* (2.2) *such that*

$$\|u\|_{H^{k+2}_\mu(\mathbb{R}^2;0)}\le c\|f\|_{H^k_\mu(\mathbb{R}^2;0)}. \tag{2.11}$$

Moreover,

Theorem 2.2. *Assume that $f\in H^k_\mu(\mathbb{R}^2;0)\cap H^{k'}_{\mu'}(\mathbb{R}^2;0)$, $\mu,\mu'\in(0,1)$, $k,k'\in\mathbb{N}_0$ and*

$$h(k',\mu')=h'=1+k'-\mu'>1+k-\mu=h=h(k,\mu),$$

where $h,h'\notin\mathbb{Z}$. Assume that there exist integer numbers such that $l_1,\dots,l_\mu\in(h,h')$. Then there exists two solutions of problem (2.2) *$u\in H^{k+2}_\mu(\mathbb{R}^2;0)$ and $u'\in H^{k'+2}_\mu(\mathbb{R}^2;0)$ such that*

$$\|u\|_{H^{k+2}_\mu(\mathbb{R}^2;0)}\le c\|f\|_{H^k_\mu(\mathbb{R}^2;0)},\quad \|u'\|_{H^{k'+2}_{\mu'}(\mathbb{R}^2;0)}\le c\|f\|_{H^{k'}_{\mu'}(\mathbb{R}^2;0)}, \tag{2.12}$$

and

$$u=\sum_{\sigma=l_1}^{l_\mu}(a_\sigma r^\sigma\sin\sigma\varphi+b_\sigma r^\sigma\cos\sigma\varphi)+u'. \tag{2.13}$$

Let $\mathcal{B}^l_2(\mathbb{R}^n)$ be a closure of smooth functions with compact supports in the seminorm

$$\langle u\rangle^{(l)}_{2,\mathbb{R}^n}=\bigg(\sum_{|\alpha|=[l]}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\bigg|D^\alpha u(x)-2D^\alpha u\Big(\frac{x+y}{2}\Big)+D^\alpha u(y)\bigg|^2\frac{dx\,dy}{|x-y|^{n+2(l-[l])}}\bigg)^{1/2},$$

where $[l]$ is the integer part of l (the seminorm in the main part of the norm of the Besov space $B_2^l(\mathbb{R}^n)$).

Theorem 2.3. *Let* $u \in L^k_{2,\mu}(\mathbb{R}^3, L)$, $\mu > -1$ *and* $|\alpha| < k-\mu-1$. *Then* $D^\alpha u|_L \in \mathcal{B}_2^{k-\mu-|\alpha|-1}(L)$ *and*

$$\langle D^\alpha u\rangle_{2,L}^{(k-\mu-|\alpha|-1)} \le c\|u\|_{L^k_{2,\mu}(\mathbb{R}^3;L)}.$$

Let $\varphi_\alpha \in \mathcal{B}_2^{k-\mu-|\alpha|-1}(L)$, where $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| < k-\mu-1$, be given functions with compact supports on L.

Then there exists a function $u \in L^k_{2,\mu}(\mathbb{R}^3; L)$ with a compact support such that $D^\alpha_{x'} u|_L = \varphi_\alpha$ and

$$\|u\|_{L^k_{2,\mu}(\mathbb{R}^3;L)} \le c\sum_\alpha \langle\varphi_\alpha\rangle_{2,L}^{(k-\mu-|\alpha|-1)}.$$

3. Regularity problem for (1.13)

In view of the form of (1.13) we can treat u and ω as scalar valued functions, however they are vector valued.

Lemma 3.1. *Assume that* $\omega \in L_{2,\mu}(\mathbb{R}^3; L)$, $\mu \in (0,1)$, *has a compact support. Then there exists a solution to problem* (1.13) *such that*

$$\|u - u(0)\|_{H^{*2}_\mu(\mathbb{R}^3;L)} \le c\|\omega\|_{L_{2,\mu}(\mathbb{R}^3;L)}, \tag{3.1}$$

where $u(0) = u|_{r=0}$ *and* $H^{*2}_\mu(\mathbb{R}^2; L)$ *is defined by* (3.23).

Proof. By a weak solution to problem (1.13) we mean a function $u \in H^1(\mathbb{R}^3)$ satisfying the integral identity

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla\varphi\, dx = \int_{\mathbb{R}^3} \omega\cdot\varphi\, dx, \tag{3.2}$$

which holds for any $\varphi \in H^1(\mathbb{R}^3)$. Inserting $\varphi = u$ in (3.2), passing to the spherical coordinates, using the Hardy inequality (2.1′) with $\mu = 1$ and compactness of the support of ω we obtain

$$\int_{\mathbb{R}^3} |\nabla u|^2\, dx \le c\int_{\mathbb{R}^3} |\omega|^2\, dx. \tag{3.3}$$

Using the Hardy inequality (2.1') once again we have existence of weak solutions (3.2) in $\widetilde{H}^1_0(\mathbb{R}^3; 0)$ and the estimate

$$\|u\|_{\widetilde{H}^1_0(\mathbb{R}^3;0)} \le c\|\omega\|_{L_2(\mathbb{R}^3)}. \tag{3.4}$$

Applying the Fourier transform

$$u(x) = \int_{-\infty}^{\infty} e^{ix_3\xi} \widetilde{u}(x', \xi)\, d\xi \tag{3.5}$$

to (1.13) implies

$$\begin{aligned} -\Delta' \widetilde{u} + \xi^2 \widetilde{u} &= \widetilde{\omega}, \\ \widetilde{u}|_{\gamma_0} &= \widetilde{u}|_{\gamma_{2\pi}}, \\ \frac{\partial \widetilde{u}}{\partial x_2}\bigg|_{\gamma_0} &= \frac{\partial \widetilde{u}}{\partial x_2}\bigg|_{\gamma_{2\pi}}. \end{aligned} \tag{3.6}$$

In view of the Parseval identity the identity (3.2) takes the form

$$\int_{\mathbb{R}^2} (\nabla' \widetilde{u} \cdot \nabla' \varphi + \xi^2 \widetilde{u} \cdot \varphi)\, dx' = \int_{\mathbb{R}^2} \widetilde{\omega} \cdot \varphi\, dx', \tag{3.7}$$

which holds for any $\varphi \in H^1(\mathbb{R}^2)$, where $\widetilde{\varphi}$ was replaced by φ for simplicity. To increase regularity of weak solutions determined by (3.7) we need some estimates involving parameter ξ to apply Kondratiev results. From [5] we have the estimate

$$\xi^2 \int_{\mathbb{R}^2} (|\nabla' \widetilde{u}|^2 + \xi^2 |\widetilde{u}|^2) |x'|^{2\mu}\, dx' \le c \int_{\mathbb{R}^2} |\widetilde{\omega}|^2 |x'|^{2\mu}\, dx', \tag{3.8}$$

where $\mu \in (0, 1)$.

In view of (3.3) and the Parseval identity we obtain the estimate

$$\int_{\mathbb{R}^2} (|\nabla' \widetilde{u}|^2 + \xi^2 |\widetilde{u}|^2)\, dx' \le c \int_{\mathbb{R}^2} |\widetilde{\omega}|^2\, dx'. \tag{3.9}$$

From (3.8) after applying the Hardy inequality we get

$$\xi^2 \|\widetilde{u}\|_{H^1_\mu(\mathbb{R}^2;0)} + \xi^4 \|\widetilde{u}\|^2_{L_{2,\mu}(\mathbb{R}^2;0)} \le c \|\widetilde{\omega}\|^2_{L_{2,\mu}(\mathbb{R}^2;0)}. \tag{3.10}$$

Let us introduce the function

$$\widetilde{u}_R = \widetilde{u} \zeta \left(\frac{|\xi|\, |x'|}{R} \right), \tag{3.11}$$

where R will be chosen large enough. Then $\widetilde{u}_R$ is a solution to the problem

$$\begin{aligned} -\Delta' \widetilde{u}_R + \xi^2 \widetilde{u}_R &= \widetilde{\omega} \zeta - 2\nabla' \widetilde{u} \nabla' \zeta - \widetilde{u} \Delta' \zeta \equiv h_R, \\ \widetilde{u}_R|_{\gamma_0} &= \widetilde{u}_R|_{\gamma_{2\pi}}, \\ \widetilde{u}_{R,\varphi}|_{\gamma_0} &= \widetilde{u}_{R,\varphi}|_{\gamma_{2\pi}}, \end{aligned} \tag{3.12}$$

where u in the r.h.s. is the weak solution.

Since $|\nabla'\zeta| \le c|\dot\zeta|/|x'|$, $|\nabla'^2\zeta| \le \frac{c}{|x'|^2}(|\dot\zeta| + |\ddot\zeta|)$, where c does not depend on R and dot denotes the derivative with respect to the argument,we obtain that

$$\begin{aligned}(3.13)\quad &\|2\nabla'\widetilde u\nabla'\zeta + \widetilde u\nabla'^2\zeta\|^2_{L_{2,\mu}(\mathbb{R}^2;0)}\\ &\le c\frac{|\xi|^2}{R^2}\int\limits_{\mathbb{R}^2}(|\nabla'\widetilde u|^2|x'|^{2\mu} + |\widetilde u|^2|x'|^{2\mu-2})\,dx' \le \frac{c}{R^2}\int\limits_{\mathbb{R}^2}|\widetilde\omega|^2|x'|^{2\mu}\,dx',\end{aligned}$$

where (3.8) was used. Hence $h_R \in L_{2,\mu}(\mathbb{R}^2;0)$ and

$$\|h_R\|_{L_{2,\mu}(\mathbb{R}^2;0)} \le c\left(1 + \frac{1}{R^2}\right)\|\widetilde\omega\|_{L_{2,\mu}(\mathbb{R}^2;0)}. \tag{3.14}$$

Therefore for $\widetilde u_R$ we obtain the estimates (3.8), (3.9) and (3.10).

Multiplying $(3.12)_1$ by $\overline{\widetilde u}_R|x'|^{2\mu}$ and integrating the result over $\mathbb{R}^2$ yield

$$\begin{aligned}(3.15)\quad &\int\limits_{\mathbb{R}^2}(|\nabla'\widetilde u_R|^2 + \xi^2|\widetilde u_R|^2)|x'|^{2\mu}\,dx' \le c\int\limits_{\mathbb{R}^2}|\widetilde u_R|^2|x'|^{2\mu-2}\,dx'\\ &+\int\limits_{\mathbb{R}^2}|\widetilde\omega|\,|\widetilde u_R|\,|x'|^{2\mu}\zeta\,dx' + c\int\limits_{\mathbb{R}^2}(|\nabla'\widetilde u|\,|x'|^{-1}|\dot\zeta| + |\widetilde u|\,|x'|^{-2}(|\dot\zeta| + |\ddot\zeta|)|\widetilde u_R|\,|x'|^{2\mu}\,dx',\end{aligned}$$

where the constants c do not depend on $\mathbb{R}$. Since $\widetilde u_R$ vanishes for $|\xi|\,|x'| \ge R$, the first integral on the r.h.s. of (3.15) can be estimated by

$$\frac{c}{R^2}\int\limits_{\mathbb{R}^2}\xi^2|\widetilde u_R|^2|x'|^{2\mu}\,dx'.$$

Using compactness of the support of $\widetilde\omega$ the second integral on the r.h.s. of (3.15) we estimate by

$$\begin{aligned}c\int\limits_{\mathbb{R}^2}|\widetilde\omega|\,|\widetilde u_R|\,|x'|^{2\mu-1}\,dx' &\le \varepsilon\int\limits_{\mathbb{R}^2}|\widetilde u_R|^2|x'|^{2\mu-2}\,dx' + c(\varepsilon)\int\limits_{\mathbb{R}^2}|\widetilde\omega|^2|x'|^{2\mu}\,dx'\\ &\le \varepsilon c\int\limits_{\mathbb{R}^2}|\nabla'\widetilde u_R|^2|x'|^{2\mu}\,dx' + c(\varepsilon)\int\limits_{\mathbb{R}^2}|\widetilde\omega|^2|x'|^{2\mu}\,dx'.\end{aligned}$$

Finally the last term on the r.h.s. of (3.15) we estimate by

$$\begin{aligned}c(\varepsilon)\int\limits_{\mathbb{R}^2}|\widetilde u_R|^2|x'|^{2\mu-2}(|\dot\zeta|^2 + |\ddot\zeta|^2)\,dx' &+ \varepsilon\int\limits_{\mathbb{R}^2}(|\nabla'\widetilde u|^2|x'|^{2\mu} + |\widetilde u|^2|x'|^{2\mu-2})\,dx'\\ &\le \frac{c(\varepsilon)}{R^2}\int\limits_{\mathbb{R}^2}|\xi|^2|\widetilde u_R|^2|x'|^{2\mu}\,dx' + \varepsilon c\int\limits_{\mathbb{R}^2}|\nabla'\widetilde u|^2|x'|^{2\mu}\,dx',\end{aligned}$$

where in the first integral we used $|x'|^{-2} \le c|\xi|^2/R^2$ and in the second the Hardy inequality.

Assuming that R is large enough we obtain from the above considerations the inequality

$$\int_{\mathbb{R}^2}(|\nabla'\widetilde{u}_R|^2+\xi^2|\widetilde{u}_R|^2)|x'|^{2\mu}\,dx' \le c\int_{\mathbb{R}^2}|\widetilde{\omega}|^2|x'|^{2\mu}\,dx'+\varepsilon c\int_{\mathbb{R}^2}|\nabla'\widetilde{u}|^2|x'|^{2\mu}\,dx'. \tag{3.16}$$

Passing with $R\to\infty$ in (3.16) yields

$$\int_{\mathbb{R}^2}(|\nabla'\widetilde{u}|^2+\xi^2|\widetilde{u}|^2)|x'|^{2\mu}\,dx'\le c\int_{\mathbb{R}^2}|\widetilde{\omega}|^2|x'|^{2\mu}\,dx'. \tag{3.17}$$

Finally, from (3.16) and (3.17), we have

$$\int_{\mathbb{R}^2}(|\nabla'\widetilde{u}_R|^2+\xi^2|\widetilde{u}_R|^2)|x'|^{2\mu}\,dx'\le c\int_{\mathbb{R}^2}|\widetilde{\omega}|^2|x'|^{2\mu}\,dx'. \tag{3.18}$$

Let us consider the problem

$$\begin{aligned} -\Delta'\widetilde{u}_R&=\widetilde{\omega}\zeta-2\nabla'\widetilde{u}\nabla'\zeta-\widetilde{u}\nabla'^2\zeta-\xi^2\widetilde{u}_R\equiv\widetilde{g}_R,\\ \widetilde{u}_R|_{\gamma_0}&=\widetilde{u}_R|_{\gamma_{2\pi}},\\ \widetilde{u}_{R,\varphi}|_{\gamma_0}&=\widetilde{u}_{R,\varphi}|_{\gamma_{2\pi}}. \end{aligned} \tag{3.19}$$

In view of the above considerations we have that $\widetilde{g}_R\in L_{2,\mu}(\mathbb{R}^2;0)$ and

$$\|\widetilde{g}_R\|_{L_{2,\mu}(\mathbb{R}^2;0)}\le c\|\widetilde{\omega}\|_{L_{2,\mu}(\mathbb{R}^2;0)}. \tag{3.20}$$

Since $h_R\in L_{2,\mu}(\mathbb{R}^2;0)$ and (3.14) solutions of (3.12) satisfy (3.8). From the estimate we obtain (3.20).

Using compactness of the support of $\widetilde{\omega}$ and (3.17) we have

$$\begin{aligned} \|\widetilde{g}_R\|_{L_{2,\mu+1}(\mathbb{R}^2;0)}&\le c(\|\widetilde{\omega}\|_{L_{2,\mu+1}(\mathbb{R}^2;0)}+\|\nabla'\widetilde{u}\|_{L_{2,\mu}(\mathbb{R}^2;0)}\\ &\quad+\|\widetilde{u}\|_{L_{2,\mu-1}(\mathbb{R}^2;0)}+\|\xi^2\widetilde{u}_R\|_{L_{2,\mu+1}(\mathbb{R}^2;0)})\\ &\le c(\|\widetilde{\omega}\|_{L_{2,\mu}(\mathbb{R}^2;0)}+\|\xi^2\widetilde{u}_R\|_{L_{2,\mu+1}(\mathbb{R}^2;0)}). \end{aligned} \tag{3.21}$$

Finally the last norm we estimate in the way

$$\int_{\mathbb{R}^2}\xi^4|\widetilde{u}_R|^2|x'|^{2\mu+2}\,dx'\le c\int_{\mathbb{R}^2}\xi^2|\widetilde{u}_R|^2|x'|^{2\mu}\,dx'\le c\int_{\mathbb{R}^2}|\widetilde{\omega}|^2|x'|^{2\mu}\,dx',$$

where we used that $\widetilde{u}_R$ vanishes for $|\xi|\,|x'|\ge R$ and (3.18)

Considering problem (3.19) we see that $\widetilde{g}_R\in L_{2,\mu}(\mathbb{R}^2;0)\cap L_{2,\mu+1}(\mathbb{R}^2;0)$. Therefore, by Theorem 2.2, we have two solutions $\widetilde{u}_R^1\in H^2_\mu(\mathbb{R}^2;0)$ and $\widetilde{u}_R^2\in H^2_{\mu+1}(\mathbb{R}^2;0)$.

Moreover, $1-\mu>0>1-(1+\mu)=-\mu$, so Theorem 2.2 implies that

$$\widetilde{u}_R^2=\widetilde{u}_R^1+c_0,$$

where we assume that $c_0 = \widetilde{u}|_{r=0} = \widetilde{u}(0)$ because $\widetilde{u}_R^1|_{r=0} = 0$ and $\widetilde{u}_R^2|_{r=0} \neq 0$ and corresponds to the weak solution.

Therefore the weak solution is such that $\widetilde{u} - \widetilde{u}(0) \in H_\mu^2(\mathbb{R}^2;0)$ and the estimate holds

$$\|\widetilde{u} - \widetilde{u}(0)\|_{H_\mu^2(\mathbb{R}^2;0)} \leq c\|\widetilde{\omega}\|_{L_{2,\mu}(\mathbb{R}^2;0)}. \tag{3.22}$$

Finally, by the Parseval identity, we have that

$$\begin{aligned}\|u - u(0)\|_{H_\mu^{*2}(\mathbb{R}^3;L)} \equiv \int_{\mathbb{R}^1} (|\xi|^4\|\widetilde{u}\|^2_{L_{2,\mu}(\mathbb{R}^2;0)} + |\xi|^2\|\widetilde{u}\|^2_{H_\mu^1(\mathbb{R}^2;0)} \\ + \|\widetilde{u} - \widetilde{u}(0)\|^2_{H_\mu^2(\mathbb{R}^2;0)})\,d\xi \leq c\|\omega\|^2_{L_{2,\mu}(\mathbb{R}^3;L)}.\end{aligned} \tag{3.23}$$

This concludes the proof. □

Next we have

LEMMA 3.2. *Assume that $\omega \in H_\mu^1(\mathbb{R}^3;L)$, $\mu \in (0,1)$ and has a compact support. Then there exists a solution to problem* (1.13) *such that $u - u(0) - u_{x_1}(0)x_1 - u_{x_2}(0)x_2 \in H_\mu^{*3}(\mathbb{R}^3;L)$ and*

$$\|u - u(0) - u_{x_1}(0)x_1 - u_{x_2}(0)x_2\|_{H_\mu^{*3}(\mathbb{R}^3;L)} \leq c\|\omega\|_{H_\mu^1(\mathbb{R}^3;L)}, \tag{3.24}$$

where

$$\begin{aligned}\|u - u(0) &- u_{x_1}(0)x_1 - u_{x_2}(0)x_2\|^2_{H_\mu^{*3}(\mathbb{R}^3;L)} \\ &= \int_{\mathbb{R}^1} (|\xi|^6\|\widetilde{u}\|^2_{L_{2,\mu}(\mathbb{R}^2;0)} + |\xi|^4\|\widetilde{u}\|^2_{H_\mu^1(\mathbb{R}^2;0)} + |\xi|^2\|\widetilde{u} - \widetilde{u}(0)\|^2_{H_\mu^2(\mathbb{R}^2;0)} \\ &\quad + \|\widetilde{u} - \widetilde{u}(0) - \widetilde{u}_{x_1}(0)x_1 - \widetilde{u}_{x_2}(0)x_2\|^2_{H_\mu^3(\mathbb{R}^2;0)})\,d\xi.\end{aligned}$$

PROOF. From (3.8) we obtain the estimate

$$\xi^4 \int_{\mathbb{R}^2} (|\nabla'\widetilde{u}|^2 + \xi^2|\widetilde{u}|^2)|x'|^{2\mu}\,dx' \leq c\xi^2 \int_{\mathbb{R}^2} |\widetilde{\omega}|^2|x'|^{2\mu}\,dx'. \tag{3.25}$$

Now we consider problem (3.19) where the third term on the r.h.s. takes the form $(\widetilde{u} - \widetilde{u}(0))\nabla'^2\zeta$. It means that in the r.h.s. of (3.19) is the function $\widetilde{u} - \widetilde{u}(0) \in H_\mu^2(\mathbb{R}^2;0)$. First we show that $\widetilde{g}_R \in H_\mu^1(\mathbb{R}^2;0)$. For this purpose we examine

$$\begin{aligned}\|\nabla'\widetilde{u}\nabla'\zeta\|^2_{H_\mu^1(\mathbb{R}^2,0)} &\leq c(\|\nabla'^2\widetilde{u}\dot{\zeta}\|^2_{L_{2,\mu-1}(\mathbb{R}^2;0)} + \|\nabla'\widetilde{u}(|\dot{\zeta}| + |\ddot{\zeta}|)\|^2_{L_{2,\mu-2}(\mathbb{R}^2;0)}) \\ &\leq c\xi^2 \int_{\mathbb{R}^2} |\nabla'^2\widetilde{u}|^2|x'|^{2\mu}\,dx' + c\xi^2 \int_{\mathbb{R}^2} |\nabla'\widetilde{u}|^2|x'|^{2\mu-2}\,dx' \\ &\leq c\xi^2 \int_{\mathbb{R}^2} |\widetilde{\omega}|^2|x'|^{2\mu}\,dx',\end{aligned}$$

where we used that $|\xi|^{-1}|x'|^{-1} \le c$ and the estimates from the proof of Lemma 3.1. Next

$$
\begin{aligned}
\|\widetilde{u}\nabla'^2\zeta\|^2_{H^1_\mu(\mathbb{R}^2;0)} &\le c\|\nabla'\widetilde{u}(|\ddot{\zeta}|+|\dot{\zeta}|)\|_{L_{2,\mu-2}(\mathbb{R}^2;0)} \\
&\quad + c\|(\widetilde{u}-\widetilde{u}(0))(|\dot{\zeta}|+|\ddot{\zeta}|+|\dddot{\zeta}|)\|^2_{L_{2,\mu-3}(\mathbb{R}^2;0)} \\
&\le c\xi^2 \int\limits_{\mathbb{R}^2} |\nabla'\widetilde{u}|^2|x'|^{2\mu-2}\,dx' + c\xi^2 \int\limits_{\mathbb{R}^2} |\widetilde{u}-\widetilde{u}(0)|^2|x'|^{2\mu-4}\,dx' \\
&\le c\xi^2 \int\limits_{\mathbb{R}^2} |\widetilde{\omega}|^2|x'|^{2\mu}\,dx'.
\end{aligned}
$$

Finally, from the proof of Lemma 3.1, it follows that $\xi^2\widetilde{u}_R \in H^1_\mu(\mathbb{R}^2;0)$. Hence $\widetilde{g}_R \in H^1_\mu(\mathbb{R}^2;0)$ is shown.

Now we prove that $\widetilde{g}_R \in H^1_{1+\mu}(\mathbb{R}^2;0)$. For this purpose we examine

$$
\begin{aligned}
\|\nabla'\widetilde{u}\nabla'\zeta\|_{H^1_{1+\mu}(\mathbb{R}^2;0)} &\le c\|\nabla'^2\widetilde{u}\dot{\zeta}\|^2_{L_{2,\mu}(\mathbb{R}^2;0)} + c\|\nabla'\widetilde{u}(|\dot{\zeta}|+|\ddot{\zeta}|)\|^2_{L_{2,\mu-1}(\mathbb{R}^2;0)} \\
&\le c\|\widetilde{\omega}\|^2_{L_{2,\mu}(\mathbb{R}^2;0)}.
\end{aligned}
$$

Continuing

$$
\begin{aligned}
\|(\widetilde{u}-\widetilde{u}(0))\nabla'^2\zeta\|^2_{H_{1+\mu}(\mathbb{R}^2;0)} &\le c\|\nabla'\widetilde{u}(|\ddot{\zeta}|+|\dot{\zeta}|)\|^2_{L_{2,\mu-1}(\mathbb{R}^2;0)} \\
&\quad + c\|(\widetilde{u}-\widetilde{u}(0))(|\dot{\zeta}|+|\ddot{\zeta}|+|\dddot{\zeta}|)\|^2_{L_{2,\mu-2}(\mathbb{R}^2;0)} \\
&\le c\|\widetilde{\omega}\|^2_{L_{2,\mu}(\mathbb{R}^2;0)}.
\end{aligned}
$$

Finally, in view of (3.18) and that $\widetilde{u}_R$ vanishes for $|\xi|\,|x'| \ge R$, we obtain

$$
\begin{aligned}
\|\xi^2\widetilde{u}_R\|^2_{H_{1+\mu}(\mathbb{R}^2;0)} &\le c\int\limits_{\mathbb{R}^2} \xi^4|\nabla'\widetilde{u}_R|^2|x'|^{2(1+\mu)}\,dx' + c\int\limits_{\mathbb{R}^2} \xi^4|\widetilde{u}_R|^2|x'|^{2\mu}\,dx' \\
&\le c\xi^2\Big(\int\limits_{\mathbb{R}^2} |\nabla'\widetilde{u}_R|^2|x'|^{2\mu}\,dx' + \int\limits_{\mathbb{R}^2} |\widetilde{u}_R|^2|x'|^{2\mu-2}\,dx'\Big) \\
&\le c\xi^2\int\limits_{\mathbb{R}^2} |\widetilde{\omega}|^2|x'|^{2\mu}\,dx'.
\end{aligned}
$$

Hence $\widetilde{g}_R \in H^1_{1+\mu}(\mathbb{R}^2;0)$ is proved. Therefore from (3.19) and Theorem 2.2 we have existence of two solutions $\widetilde{u}^1_R \in H^3_\mu(\mathbb{R}^2;0)$ and $\widetilde{u}^2_R \in H^3_{1+\mu}(\mathbb{R}^2;0)$ and because $2-\mu > 1 > 2-(1+\mu) = 1-\mu$ we have that

$$
\widetilde{u}^2_R = \widetilde{u}^1_R + c_{11}x_1 + c_{12}x_2, \|\widetilde{u}_R - c_0 - c_{11}x_1 - c_{12}x_2\|_{H^3_\mu(\mathbb{R}^2;0)} \le c\|\widetilde{\omega}\|_{H^1_\mu(\mathbb{R}^2;0)},
$$

where $c_0 = u|_{r=0}$, $c_{11} = u_{,x_1}|_{r=0}$, $c_{12} = u_{,x_2}|_{r=0}$. Applying the Parseval identity and adding necessary norms we obtain (3.24). This concludes the proof. □

Next we consider the case $\omega \in W^k_{2,\mu}(\mathbb{R}^3;L)$, where $k > 1$. For this purpose we need

Lemma 3.3. *Assume that* $\omega \in L^k_{2,\mu}(\mathbb{R}^3;L)$, $k \geq 0$. *Then there exists a function* $v \in L^{k+2}_{2,\mu}(\mathbb{R}^3;L)$ *such that* $f = \Delta v + \omega \in H^k_\mu(\mathbb{R}^3;L)$ *and the estimate holds*

$$\|v\|_{L^{k+2}_{2,\mu}(\mathbb{R}^3;L)} + \|f\|_{H^k_\mu(\mathbb{R}^3;L)} \leq c\|\omega\|_{L^k_{2,\mu}(\mathbb{R}^3;L)}. \tag{3.26}$$

Proof. Let

$$\omega_s = \sum_{|\alpha|=s} D^\alpha_{x'}\omega|_{x'=0}\frac{x_1^{\alpha_1}}{\alpha_1!}\frac{x_2^{\alpha_2}}{\alpha_2!},$$

where $\alpha = (\alpha_1,\alpha_2)$ is the multiindex and $D^\alpha_{x'} = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}$. We introduce homogenous polynomials R_s defined by the relations (see [4], [5])

$$\begin{aligned} -\nabla'^2 R_s &= \nabla''^2 R_{s-2} + \omega_{s-2}, \\ R_s|_{\gamma_0} &= R_s|_{\gamma_{2\pi}}, \\ R_{s,\varphi}|_{\gamma_0} &= R_{s,\varphi}|_{\gamma_{2\pi}}, \end{aligned} \tag{3.27}$$

where $\nabla'' = \partial_{x_3}$. We assume that $R_0 = R_1 = 0$. In view of Theorem 2.3 we have

$$\sum_{|\alpha|=[k+1-\mu]} \langle D^\alpha_{x'}R_{|\alpha|}\rangle^{(k+1-\mu-|\alpha|)}_{2,L} \leq c\|\omega\|_{L^k_{2,\mu}(\mathbb{R}^3;L)}. \tag{3.28}$$

We choose a function v in such a way that

$$v|_{x'=0} = 0, \quad D^\alpha_{x'}v|_{x'=0} = D^\alpha_{x'}R_{|\alpha|}, \tag{3.29}$$

where $0 < |\alpha| \leq [k+1-\mu]$ and

$$\|v\|_{L^{k+2}_{2,\mu}(\mathbb{R}^3;L)} \leq c\sum_{|\alpha|\leq[k+1-\mu]} \langle D^\alpha_{x'}R_{|\alpha|}\rangle^{(k+1-\mu-|\alpha|)}_{2,L} \leq c\|\omega\|_{L^k_{2,\mu}(\mathbb{R}^3;L)}. \tag{3.30}$$

Moreover, for $|\beta| < k-1-\mu$, we have

$$D^\beta_{x'}(\Delta v + \omega) = \nabla''^2 D^\beta_{x'}(v - R_{|\beta|}) + \nabla'^2 D^\beta_{x'}(v - R_{|\beta|+2}) + D^\beta_{x'}(\omega - \omega_{|\beta|}).$$

Hence $D^\beta_{x'}(\Delta v+\omega)|_{x'=0} = 0$ so $\Delta v + \omega \in H^k_\mu(\mathbb{R}^3;L)$ and

$$\|\Delta v + \omega\|_{H^k_\mu(\mathbb{R}^3;L)} \leq c\|\Delta v + \omega\|_{L^k_{2,\mu}(\mathbb{R}^3;L)}.$$

This concludes the proof. □

In view of Lemma 3.3 we can consider problem (1.13) with $\omega \in H^k_\mu(\mathbb{R}^3;L)$.

Theorem 3.4. *Assume that* $\omega \in H^k_\mu(\mathbb{R}^3;L)$, $\mu \in (0,1)$. *Then the problem* (1.13) *has a solution* $u \in L^{k+2}_{2,\mu}(\mathbb{R}^3;L)$ *and*

$$\|u\|_{L^{k+2}_{2,\mu}(\mathbb{R}^3;L)} \leq c\|\omega\|_{H^k_\mu(\mathbb{R}^3;L)}, \tag{3.31}$$

holds. Moreover, there exists a polynomial $P_{(k)}(u)$ *of degree* k *such that*

$$\|u\|_{H^{*k+2}_\mu(\mathbb{R}^3;L)} \leq c\|\omega\|_{H^k_\mu(\mathbb{R}^3;L)}, \tag{3.32}$$

where $H_\mu^{*k+2}(\mathbb{R}^3;L)$ *is defined by* (3.45) *and* (3.46). *The polynomial* $P_{(k)}(u)$ *depends on derivatives of* u *up to the order* k *calculated on the axis* L.

Proof. We examine problem (1.13) in the form (3.6). We show the lemma step by step starting from the regularity of weak solutions. Let $\omega\in L_{2,\mu}(\mathbb{R}^3;L)$. Then Lemma 3.1 implies that $u-P_0\in H_\mu^{*2}(\mathbb{R}^3;L)$, $P_0=u|_{r=0}$ and

$$\|u-P_0\|_{H_\mu^{*2}(\mathbb{R}^3;L)}\le c\|\omega\|_{L_{2,\mu}(\mathbb{R}^3;L)}. \tag{3.33}$$

Let $\omega\in H_\mu^1(\mathbb{R}^3;L)$. Then Lemma 3.2 gives that $u-P_0-P_1\in H_\mu^{*3}(\mathbb{R}^3;L)$ and

$$\|u-P_0-P_1\|_{H_\mu^{*3}(\mathbb{R}^3;L)}\le c\|\omega\|_{H_\mu^1(\mathbb{R}^3;L)}, \tag{3.34}$$

where $P_1=u_{x_1}(0)x_1+u_{x_2}(0)x_2$. Assume that $\omega\in H_\mu^k(\mathbb{R}^3;L)$. Then $\widetilde{\omega}\in\mathcal{E}_\mu^k(\mathbb{R}^2;0)$, where

$$\|\widetilde{u}\|^2_{\mathcal{E}_\mu^k(\mathbb{R}^2;0)}=\sum_{j\le k}|\xi|^{2j}\|\widetilde{\omega}\|^2_{H_\mu^{k-j}(\mathbb{R}^2;0)}. \tag{3.35}$$

The meaning of the space $\mathcal{E}_\mu^k(\mathbb{R}^2;0)$ is such that

$$\|u\|^2_{H_\mu^k(\mathbb{R}^3;\Gamma)}=\int_{-\infty}^{\infty}d\xi\|\widetilde{u}\|^2_{\mathcal{E}_\mu^k(\mathbb{R}^2;0)},$$

by the Parseval identity. To simplify considerations we introduce

$$u_0=u-P_0,\quad u_1=u-P_0-P_1,\quad\text{so}\quad u_0\in H_\mu^{*2}(\mathbb{R}^3;L),\quad u_1\in H_\mu^{*3}(\mathbb{R}^3;L).$$

To increase regularity we are looking for solutions of the problems

$$\begin{aligned}-\Delta'\widetilde{u}_j&=-\xi^2\widetilde{u}_{j-2}+\widetilde{\omega},\\ \widetilde{u}_j|_{\gamma_0}&=\widetilde{u}_j|_{\gamma_{2\pi}},\\ \widetilde{u}_{j,\varphi}|_{\gamma_0}&=\widetilde{u}_{j,\varphi}|_{\gamma_{2\pi}},\end{aligned}$$

where $j\ge2$, and homogeneous polynomials $\widetilde{P}_{s,s+2j}$ of degree $s+2j$, $s=0,1$, which are solutions to the problems

$$\begin{aligned}-\Delta'\widetilde{P}_{s,s+2j}&=-\xi^2\widetilde{P}_{s,s+2(j-1)},\\ \widetilde{P}_{s,s+2j}|_{\gamma_0}&=\widetilde{P}_{s,s+2j}|_{\gamma_{2\pi}},\\ \widetilde{P}_{s,s+2j,\varphi}|_{\gamma_0}&=\widetilde{P}_{s,s+2j,\varphi}|_{\gamma_{2\pi}},\end{aligned}$$

where $\widetilde{P}_{0,0}=\widetilde{P}_0$, $\widetilde{P}_{1,1}=\widetilde{P}_1$. Looking for solutions in H_μ^4 we introduce

$$\begin{aligned}v'&=\widetilde{u}_1-\zeta(|x'|)\widetilde{P}_{0,2},\\ v''&=\widetilde{u}_2+(1-\zeta(|x'|))\widetilde{P}_{0,2}.\end{aligned}$$

They are solutions to the same problem

$$
\begin{aligned}
-\Delta' v &= h, \\
v|_{\gamma_0} &= v|_{\gamma_{2\pi}}, \qquad (3.36)\\
v_{,\varphi}|_{\gamma_0} &= v_{,\varphi}|_{\gamma_{2\pi}},
\end{aligned}
$$

because

$$
\begin{aligned}
h' &= -\xi^2\widetilde{u} + \widetilde{\omega} + \zeta\xi^2\widetilde{P}_{0,2} + 2\nabla'\zeta\nabla'\widetilde{P}_{0,2} + \nabla'^2\zeta\widetilde{P}_{0,2}, \\
h'' &= -\xi^2\widetilde{u}_0 + \widetilde{\omega} - \xi^2\widetilde{P}_0 + \zeta\xi^2\widetilde{P}_{0,2} + \nabla'^2\zeta\widetilde{P}_{0,2},
\end{aligned}
$$

are equal. Let us introduce the polynomials

$$
Q_k = a_k r^k \sin k\varphi + b_k r^k \cos k\varphi. \qquad (3.37)
$$

Since $h = h' = h'' \in H^1_\mu(\mathbb{R}^2;0) \cap H^2_\mu(\mathbb{R}^2;0)$ we have that $h(1) = 2-\mu < 2 < 3-\mu = h(2)$. Applying Theorem 2.2 we obtain that $v' = v'' + \widetilde{Q}_2$, so

$$
\widetilde{u}_1 = \widetilde{u}_2 + \widetilde{P}_{0,2} + \widetilde{Q}_2,
$$

or

$$
\widetilde{u} = \widetilde{u}_2 + \widetilde{P}_0 + \widetilde{P}_1 + \widetilde{P}_{0,2} + \widetilde{Q}_2. \qquad (3.38)
$$

Moreover, we have

$$
\|\widetilde{u}_2\|_{H^4_\mu(\mathbb{R}^2;0)} \le c\|\widetilde{\omega}\|_{\mathcal{E}^2_\mu(\mathbb{R}^2;0)}. \qquad (3.39)
$$

To show regularity in $H^5_\mu(\mathbb{R}^2;0)$ we introduce

$$
v' = \widetilde{u}_2 - \zeta\widetilde{P}_{1,2}, \quad v'' = \widetilde{u}_3 + (1-\zeta)\widetilde{P}_{1,2},
$$

which are solutions of the same problem (3.36) because $\widetilde{u}_1 + \widetilde{P}_1 = \widetilde{u}_0$ implies that $h' = h''$.

Since $h \in H^2_\mu(\mathbb{R}^2;0) \cap H^3_\mu(\mathbb{R}^2;0)$ and $h(2) = 3-\mu < 3 < 4-\mu = h(3)$ we have, by Theorem 2.2, that $v' = v'' + \widetilde{Q}_3$, so

$$
\widetilde{u}_2 = \widetilde{u}_3 + \widetilde{P}_{1,2} + \widetilde{Q}_3.
$$

Hence

$$
\widetilde{u} = \widetilde{u}_3 + \widetilde{P}_0 + \widetilde{P}_1 + \widetilde{P}_{0,2} + \widetilde{P}_{1,2} + \widetilde{Q}_2 + \widetilde{Q}_3 \qquad (3.40)
$$

and

$$
\|\widetilde{u}_3\|_{H^5_\mu(\mathbb{R}^2;0)} \le c\|\widetilde{\omega}\|_{\mathcal{E}^3_\mu(\mathbb{R}^2;0)}. \qquad (3.41)
$$

To increase further regularity we introduce

$$
\begin{aligned}
v' &= \widetilde{u}_3 - \zeta(\widetilde{P}_{0,4} + \widetilde{Q}_{2,4}), \\
v'' &= \widetilde{u}_4 + (1-\zeta)(\widetilde{P}_{0,4} + \widetilde{Q}_{2,4}),
\end{aligned}
$$

where $\widetilde{Q}_{2,4}$ is the fourth degree polynomial calculated from the problem

$$\begin{aligned}-\nabla'^2\widetilde{Q}_{s,s+2j} &= -\xi^2\widetilde{Q}_{s,s+2(j-1)},\\ \widetilde{Q}_{s,s+2j}|_{\gamma_0} &= \widetilde{Q}_{s,s+2j}|_{\gamma_{2\pi}},\\ \widetilde{Q}_{s,s+2j,\varphi}|_{\gamma_0} &= Q_{s,s+2j,\varphi}|_{\gamma_{2\pi}},\end{aligned}$$

where $s=2$ and $j=1$ and $\widetilde{Q}_{s,s}=\widetilde{Q}_s$. To apply Theorem 2.2 we see that $h(3)=4-\mu<4<5-\mu=h(4)$, so

$$\widetilde{u}_3=\widetilde{u}_4+\widetilde{P}_{0,4}+\widetilde{Q}_{2,4}+\widetilde{Q}_4. \tag{3.42}$$

Continuing the above considerations we have

$$\widetilde{u}=\widetilde{u}_k+\sum_{2j\le k}\widetilde{P}_{0,2j}+\sum_{1+2j\le k}\widetilde{P}_{1,1+2j}+\sum_{\substack{s\le k\\ s+2j\le k}}\widetilde{Q}_{s,s+2j}, \tag{3.43}$$

and

$$\|\widetilde{u}_k\|_{H_\mu^{k+2}(\mathbb{R}^2;0)}\le c\|\widetilde{\omega}\|_{\mathcal{E}_\mu^k(\mathbb{R}^2;0)}. \tag{3.44}$$

Moreover, in view of (3.38)–(3.44), we obtain

$$\begin{aligned}\|\widetilde{u}\|^2_{H_\mu^{*k+2}(\mathbb{R}^2;0)} &\equiv \xi^{2(k+1)}\|\widetilde{u}\|^2_{L_{2,\mu}(\mathbb{R}^2;0)}+\xi^{2k}\|\widetilde{u}\|^2_{H_\mu^1(\mathbb{R}^2;0)}\\ &\quad+|\xi|^{2k-2}\|\widetilde{u}_0\|^2_{H_\mu^2(\mathbb{R}^2;0)}+\cdots+\|\widetilde{u}_k\|^2_{H_\mu^{k+2}(\mathbb{R}^2;0)}\\ &\le c\|\widetilde{\omega}\|^2_{\mathcal{E}_\mu^k(\mathbb{R}^2;0)}.\end{aligned} \tag{3.45}$$

Hence in view of the Parseval equality, we obtain (3.32) where $H_\mu^{*k+2}(\mathbb{R}^3;L)$ is defined by

$$\|u\|^2_{H_\mu^{*k+2}(\mathbb{R}^3;L)}=\int\limits_{\mathbb{R}^1}d\xi\|\widetilde{u}\|^2_{H_\mu^{*k+2}(\mathbb{R}^2;0)}. \tag{3.46}$$

This concludes the proof. □

From Lemma 3.3 and Theorem 3.4 we have

THEOREM 3.5. *Assume that* $\omega\in L_{2,\mu}^k(\mathbb{R}^3;L)$, $\mu\in(0,1)$. *Then there exists a solution to* (1.13) *such that* $u\in L_{2,\mu}^{k+2}(\mathbb{R}^3;L)$ *and*

$$\|u\|_{L_{2,\mu}^{k+2}(\mathbb{R}^3;L)}\le c\|\omega\|_{L_{2,\mu}^k(\mathbb{R}^3;L)}. \tag{3.47}$$

Moreover, there exists a function v *determined by Lemma* 3.3 *such that*

$$\|u\|_{H_\mu^{*k+2}(\mathbb{R}^3;L)}\le c\|\omega-v\|_{H_\mu^k(\mathbb{R}^3;L)}. \tag{3.48}$$

We have to underline that the boundary conditions in the problem (1.13) are artificial and are used to apply the technique of weighted Sobolev spaces introduced by Kondratiev only (see [2]). It can be shown similarly as in [8] that the solution remains regular passing through the plane $\Gamma_0=\Gamma_{2\pi}$.

4. Existence of solutions to (1.10)

To prove the existence of solutions to problem (1.10) and to find an appropriate extimate we use the existence of weak solutions (see Lemma 1.2) and the estimate (1.12). To show higher regularity we apply local considerations. Therefore we distinguish four different kinds of neighourhoods:

(1) near internal points of L,
(2) near the points where L meets the boundary,
(3) near internal points of Ω but in a positive distance from L,
(4) near boundary points but in a positive distance from the points where L meets $\partial\Omega$.

In the cases (3) and (4) the weighted spaces are not necessary. We shall restrict to cases (1) and (2) which can be treated similarly. Let ζ be a smooth function with the support near an internal point of L. Let $\widetilde{u} = u\zeta$, $\widetilde{\omega} = \omega\zeta$. Then problem (1.10) takes the form

$$-\Delta\widetilde{u} = \widetilde{\omega} - 2\nabla\zeta\nabla u - \Delta\zeta u \equiv \widetilde{\omega}_1, \tag{4.1}$$

where we can add additionally the boundary conditions $(1.13)_{2,3}$. In view of the weak solution and for $\omega \in L_2(\Omega)$ we have that $\widetilde{\omega}_1 \in L_2(\Omega)$ too. Therefore we can repeat the consideratons from Section 3.

Theorem 4.1. *Assume that* $\omega \in W^k_{2,\mu}(\Omega; L)$, $\mu \in (0,1)$. *Then there exists a solution to problem* (1.13) such that $u \in W^{k+2}_{2,\mu}(\Omega; L)$ and the estimate holds

$$\|u\|_{W^{k+2}_{2,\mu}(\Omega;L)} \le c\|\omega\|_{W^k_{2,\mu}(\Omega;L)}. \tag{4.2}$$

Let $p \in L$. Then there exists a neighbourhood $\Omega(p)$ of p sufficiently small and a function $v = v(p)$ such that

$$\|u\|_{H^{*k+2}_\mu(\Omega(p);L)} \le c\|\omega - v(p)\|_{H^k_\mu(\Omega(p);L)}. \tag{4.3}$$

5. Existence of solutions to problem (1.1)

In view of results of Sections 3 and 4 we have

Theorem 5.1. *Assume that* $\omega \in W^k_{2,\mu}(\Omega; L)$, $k \in \mathbb{N}_0$, $\mu \in (0,1)$. *Then there exists a solution* $v \in W^{k+1}_{2,\mu}(\Omega; L)$ *such that*

$$\|v\|_{W^{k+1}_{2,\mu}(\Omega;L)} \le c\|\omega\|_{W^k_{2,\mu}(\Omega;L)}. \tag{5.1}$$

Moreover, in any sufficiently small neighbourhood $\Omega(p)$ *of a point* $p \in L$, *there exist* $v_*(\omega, p)$ *such that*

$$\|v\|_{H^{*k+1}_\mu(\Omega(p);L)} \le c\|\omega - v_*(\omega,p)\|_{H^k_\mu(\Omega(p);L)} + \|w\|_{L_{2,\mu}(\Omega;L)}. \tag{5.2}$$

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