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HIGHER-ORDER NECESSARY OPTIMALITY CONDITIONS FOR EXTREMUM PROBLEMS IN TOPOLOGICAL VECTOR SPACES

Leon Mikołajczyk — Marcin Studniarski

Dedicated to the memory of Juliusz P. Schauder

Abstract. We present a higher-order extension of the well-known theorem of Ben–Tal and Zowe on second-order necessary optimality conditions in topological vector spaces. We also examine the connection between this extension and the results of Furukawa and Yoshinaga which are stated in terms of higher-order variational sets and Neustadt derivatives.

1. Introduction

Ben–Tal and Zowe [1, Theorem 2.1] have proved a general theorem on second-order necessary optimality conditions for the following abstract optimization problem in topological vector spaces:

$$\min\{f(x) \mid g(x) \in -K,\ h(x) = 0\}, \tag{1.1}$$

where $f : X \to U$, $g : X \to V$ and $h : X \to W$ are continuous maps, X, U, V and W are real topological vector spaces, K is a convex cone in V with nonempty topological interior ($\operatorname{int} K \neq \emptyset$), and U is ordered by a proper cone C with $\operatorname{int} C \neq \emptyset$. This result is a second-order extension of the classical Dubovitskiĭ–Milyutin theorem (see [3]).

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In this paper, the theorem of Ben–Tal and Zowe is generalized so as to include necessary conditions of arbitrary order n. Although the generalization itself is rather simple, one important question arises. Since the n-order necessary conditions for x_0 to be a local solution of (1.1) are formulated for a given sequence of points $x_1, \dots, x_{n-1}$, one should specify for which sequences the n-order information can be nontrivial. We discuss this question in Section 2.

In Section 3, we study the connection between the results of Section 2 and the n-order necessary optimality conditions of Furukawa and Yoshinaga [2, Theorem 5.2]. We prove that both types of optimality conditions are essentially equivalent if they are applied to problem (1.1) (note that in [2], the optimization problem is formulated in a Banach space, but the formulation includes an arbitrary subset B with nonempty interior instead of a convex cone K). This equivalence enables one to extend the conditions of Furukawa–Yoshinaga type to problems of the form (1.1) in an arbitrary real topological vector space.

We end this section by setting some notation which will be used throughout the paper. The zero vectors in the topological vector spaces X, U, V and W will be denoted, respectively, by 0_X, 0_U, 0_V and 0_W. For a given vector x, we denote by $\mathcal{N}(x)$ the collection of all neighbourhoods of x. For the order relations in U, we use the following notation: we write $u_1 \succeq u_2$ (or $u_2 \preceq u_1$) if $u_1 - u_2 \in C$ and $u_1 \succ u_2$ ($u_2 \prec u_1$) if $u_1 - u_2 \in \operatorname{int} C$.

In addition to the assumptions of [1] (which are quoted above), we will assume, throughout the paper, that the cone C is convex.

Finally, we associate with a subset S of X its so-called *support functional* $\delta^*(\,\cdot\,|S)$ defined on the topological dual X^* of X with values in the extended real line $\mathbb{R} \cup \{\pm\infty\}$:

$$\delta^*(x^*|S) := \sup\{x^*(x) \mid x \in S\} \quad \text{for } x^* \in X^*.$$

(If $S = \emptyset$, then by convention, $\delta^*(\,\cdot\,|S) = -\infty$.) The *effective domain* of $\delta^*(\,\cdot\,|S)$ is denoted by $\Lambda(S)$, i.e., $\Lambda(S) := \{x^* \in X^* \mid \delta^*(x^*|S) < \infty\}$. (See [1] for more information about these notions.)

2. A generalization of the theorem of Ben–Tal and Zowe

We start with some definitions needed for the formulation of the main result of this section. Let us denote by F the *feasible set* for problem (1.1), i.e.,

$$F := \{x \in X \mid g(x) \in -K,\ h(x) = 0_W\}.$$

Definition 1. Let $x_0 \in F$. We say that x_0 is a *weak local minimum point* for problem (1.1) if there exists $N \in \mathcal{N}(x_0)$ such that

$$f(x) \notin f(x_0) - \operatorname{int} C \quad \text{for all } x \in N \cap F. \tag{2.1}$$

In particular, if $U = \mathbb{R}$ and $C = \{\lambda \in \mathbb{R} \mid \lambda \geq 0\}$, then (1.1) is a usual minimization problem, and (2.1) amounts to $f(x) \geq f(x_0)$ for all $x \in N \cap F$, which means that x_0 is a usual local minimum point for (1.1).

Definition 2.

(a) A vector $x_1 \in X$ is called a *direction of quasidecrease* at x_0 of the objective function $f : X \to U$ if, for every $u \succ 0$ in U, there exists a real $t_0 > 0$ such that

$$f(x_0 + tx_1) \preceq f(x_0) + t\,u \quad \text{for all } t \in (0, t_0].$$

(b) A vector $x_1 \in X$ is called a *quasifeasible direction* at x_0 for the constraint $g(x) \in -K$ if, for every $v \in \operatorname{int} K$, there exists a real $t_0 > 0$ such that

$$g(x_0 + tx_1) \in -K + tv \quad \text{for all } t \in (0, t_0].$$

The cone of all directions of quasidecrease of f (respectively, quasifeasible directions for $g(x) \in -K$) at x_0 is denoted by $D_f(x_0)$ (respectively, $D_g(x_0)$).

Extending the second-order approximations from [1], we now define some n-order approximating sets associated with f, g and h. In the Banach space context, these approximating sets have been defined by Ledzewicz and Schättler in [5], [6] (where approximating sets in the space X have been used to construct approximating cones in the extended space $X \times \mathbb{R}$).

Definition 3. Let $n \geq 1$ be an integer, and let $x_0, \ldots, x_{n-1} \in X$.

(a) A vector $x_n \in X$ is called an *n-order direction of decrease* of f at x_0 with respect to $(x_1, \ldots, x_{n-1})$ if there exist $u \succ 0$, $N \in \mathcal{N}(x_n)$ and $t_0 > 0$ such that

$$f\Big(\sum_{i=0}^{n-1} t^i x_i + t^n y\Big) \preceq f(x_0) - t^n u \quad \text{for all } y \in N \text{ and } t \in (0, t_0]. \tag{2.2}$$

(b) A vector $x_n \in X$ is called an *n-order feasible direction* for the constraint $g(x) \in -K$ at x_0 with respect to $(x_1, \ldots, x_{n-1})$ if there exist $v \in \operatorname{int} K$, $N \in \mathcal{N}(x_n)$ and $t_0 > 0$ such that

$$g\Big(\sum_{i=0}^{n-1} t^i x_i + t^n y\Big) \in -K - t^n v \quad \text{for all } y \in N \text{ and } t \in (0, t_0]. \tag{2.3}$$

(c) A vector $x_n \in X$ is called an *n-order tangent direction* to the constraint $h(x) = 0$ at x_0 with respect to $(x_1, \ldots, x_{n-1})$ if there exist $t_0 > 0$ and a function $r : (0, t_0] \to X$ such that

$$h\Big(\sum_{i=0}^{n} t^i x_i + r(t)\Big) = 0 \quad \text{for all } t \in (0, t_0] \tag{2.4}$$

and, for every $N \in \mathcal{N}(0_X)$, there exists $t_1 \in (0, t_0]$ such that

$$t^{-n} r(t) \in N \quad \text{for all } t \in (0, t_1]. \tag{2.5}$$

The set of all n-order directions of decrease of f (respectively, feasible directions for $g(x) \in -K$ and tangent directions to $h(x) = 0$) at x_0 with respect to $(x_1, \ldots, x_{n-1})$ is denoted by $Q_f(x_0, \ldots, x_{n-1})$ (respectively, $Q_g(x_0, \ldots, x_{n-1})$ and $V_h(x_0, \ldots, x_{n-1})$). In particular, the first order sets $Q_f(x_0)$, $Q_g(x_0)$ and $V_h(x_0)$ are defined for $n = 1$, with the sequences $(x_1, \ldots, x_{n-1})$ absent in the formulations of (a), (b) and (c), respectively.

Definition 4. We say that:

(a) f is $(x_1, \ldots, x_{n-1})$-*regular* at x_0 if $Q_f(x_0, \ldots, x_{n-1})$ is nonempty and convex,
(b) g is $(x_1, \ldots, x_{n-1})$-*regular* at x_0 if $Q_g(x_0, \ldots, x_{n-1})$ is nonempty and convex,
(c) h is $(x_1, \ldots, x_{n-1})$-*regular* at x_0 if $V_h(x_0, \ldots, x_{n-1})$ is nonempty and convex.

(If the respective condition holds for $n = 1$, we simply say that f, g or h is *regular* at x_0.)

We can now formulate the n-order version of [1, Theorem 2.1].

Theorem 5. *Let x_0 be a weak local minimum point for problem* (1.1). *Then, for every sequence $(x_1, \ldots, x_{n-1})$ of elements of X, the following conditions hold:*

(a) $Q_f(x_0, \ldots, x_{n-1}) \cap Q_g(x_0, \ldots, x_{n-1}) \cap V_h(x_0, \ldots, x_{n-1}) = \emptyset$,
(b) *if f, g and h are $(x_1, \ldots, x_{n-1})$-regular, then there exist continuous linear functionals on X:*

$$\begin{aligned} &l_f \in \Lambda(Q_f(x_0, \ldots, x_{n-1})),\\ &l_g \in \Lambda(Q_g(x_0, \ldots, x_{n-1})),\\ &l_h \in \Lambda(V_h(x_0, \ldots, x_{n-1})), \end{aligned} \tag{2.6}$$

not all zero, which satisfy the Euler–Lagrange equation

$$l_f + l_g + l_h = 0_{X^*} \tag{2.7}$$

and the Legendre inequality

$$\begin{aligned} \delta^*(l_f | Q_f(x_0, \ldots, x_{n-1})) + \delta^*(l_g | Q_g(x_0, \ldots, x_{n-1}))&\\ + \delta^*(l_h | V_h(x_0, \ldots, x_{n-1})) \le 0.& \end{aligned} \tag{2.8}$$

PROOF. Statement (a) follows from Definitions 1 and 3 exactly in the same way as in the proof of [1, Theorem 2.1]. Statement (b) follows from (a), Definition 4 and [1, Lemma 3.2]. □

It should be noted that in [1, Theorem 2.1], for $n = 2$, an additional assumption (2.9) appears, which in our notation has the form

$$x_1 \in D_f(x_0) \cap D_g(x_0) \cap V_h(x_0). \tag{2.9}$$

This assumption is not needed for the proof of the theorem, but it is added to specify the set of directions x_1 for which the second order conditions can give nontrivial information. Similarly, for an arbitrary n, we should be able to identify some set of sequences $(x_1, \ldots, x_{n-1})$ for which statement (a) of Theorem 5 holds trivially (such sequences should be excluded from verifying necessary conditions of part (b)). Below we give an answer to this question by proving a result analogous to [2, Proposition 5.1].

DEFINITION 6.

(a) A vector $x_n \in X$ is called an *n-order tangent direction* to the relation $f(x) \preceq f(x_0)$ at x_0 with respect to $(x_1, \ldots, x_{n-1})$ if there exist $t_0 > 0$ and a function $r : (0, t_0] \to X$ such that

$$f\left(\sum_{i=0}^{n} t^i x_i + r(t)\right) \preceq f(x_0) \quad \text{for all } t \in (0, t_0] \tag{2.10}$$

and, for every $N \in \mathcal{N}(0_X)$, there exists $t_1 \in (0, t_0]$ such that (2.5) holds.

(b) A vector $x_n \in X$ is called an *n-order tangent direction* to the constraint $g(x) \in -K$ at x_0 with respect to $(x_1, \ldots, x_{n-1})$ if there exist $t_0 > 0$ and a function $r : (0, t_0] \to X$ such that

$$g\left(\sum_{i=0}^{n} t^i x_i + r(t)\right) \in -K \quad \text{for all } t \in (0, t_0] \tag{2.11}$$

and, for every $N \in \mathcal{N}(0_X)$, there exists $t_1 \in (0, t_0]$ such that (2.5) holds.

The set of all n-order tangent directions to $f(x) \preceq f(x_0)$ (respectively, $g(x) \in -K$) at x_0 with respect to $(x_1, \ldots, x_{n-1})$ is denoted by $V_f(x_0, \ldots, x_{n-1})$ (respectively, $V_g(x_0, \ldots, x_{n-1})$).

PROPOSITION 7. *For every sequence $(x_0, \ldots, x_{n-1})$ of points of X, the following relations hold:*

(a) $Q_f(x_0, \ldots, x_{n-1}) \neq \emptyset \;\Rightarrow\; x_{n-1} \in V_f(x_0, \ldots, x_{n-2})$,
(b) $Q_g(x_0, \ldots, x_{n-1}) \neq \emptyset \;\Rightarrow\; x_{n-1} \in V_g(x_0, \ldots, x_{n-2})$,
(c) $V_h(x_0, \ldots, x_{n-1}) \neq \emptyset \;\Rightarrow\; x_{n-1} \in V_h(x_0, \ldots, x_{n-2})$.

Consequently, the necessary conditions in Theorem 5 *are meaningful only if*

$$x_{n-1} \in V_f(x_0, \dots, x_{n-2}) \cap V_g(x_0, \dots, x_{n-2}) \cap V_h(x_0, \dots, x_{n-2}) \tag{2.12}$$

(*otherwise, statement* (a) *of the theorem holds trivially without any optimality assumption at* x_0).

Proof. (a) Define $\widetilde{f}(x) := f(x) - f(x_0)$. Then the condition $f(x) \preceq f(x_0)$ is equivalent to $\widetilde{f}(x) \in -C$, where C is a convex cone with nonempty interior. Therefore, part (a) can be obtained as a special case of part (b) which is proved below.

(b) Suppose that there exists a point $x_n \in Q_g(x_0, \dots, x_{n-1})$. Then, according to Definition 3(b), there exist $v \in \operatorname{int} K$, $N \in \mathcal{N}(x_n)$ and $t_0 > 0$ such that (2.3) holds. Since K is a convex cone, we have

$$-K - t^n K \subset -K \quad \text{for all } t > 0. \tag{2.13}$$

Hence, by putting $r(t) := t^n x_n$, we obtain from (2.3) (where $y = x_n \in N$) and (2.13)

$$g\bigg(\sum_{i=0}^{n-1} t^i x_i + r(t)\bigg) \in -K \quad \text{for all } t \in (0, t_0]. \tag{2.14}$$

Moreover, for every $N_0 \in \mathcal{N}(0_X)$, there exists $t_1 \in (0, t_0]$ such that $t^{-(n-1)} r(t) = t x_n \in N_0$ for all $t \in (0, t_1]$. This, together with (2.14), means that $x_{n-1} \in V_g(x_0, \dots, x_{n-2})$.

(c) Suppose that there exists a point $x_n \in V_h(x_0, \dots, x_{n-1})$. Then we can find $t_0 > 0$ and a function $r : (0, t_0] \to X$ such that

$$h\bigg(\sum_{i=0}^{n-1} t^i x_i + t^n x_n + r(t)\bigg) = 0 \quad \text{for all } t \in (0, t_0], \tag{2.15}$$

and

$$\forall N \in \mathcal{N}(0_X), \quad \exists t_N \in (0, t_0], \quad \forall t \in (0, t_N], \quad t^{-n} r(t) \in N. \tag{2.16}$$

Now, take any $N_0 \in \mathcal{N}(0_X)$. We can choose $t_1 \in (0, t_0]$ and $N_1 \in \mathcal{N}(0_X)$ such that

$$(0, t_1](x_n + N_1) \subset N_0. \tag{2.17}$$

Let $t_2 := t_{N_1}$ be the number selected for $N = N_1$ according to (2.16); we may assume that $t_2 \le t_1$. Putting $\widetilde{r}(t) := t^n x_n + r(t)$, we get from (2.15)–(2.17) that

$$h\bigg(\sum_{i=0}^{n-1} t^i x_i + \widetilde{r}(t)\bigg) = 0 \quad \text{for all } t \in (0, t_0], \tag{2.18}$$

$$t^{-(n-1)} \widetilde{r}(t) = t(t^{-n} \widetilde{r}(t)) = t(x_n + t^{-n} r(t)) \in N_0 \quad \text{for all } t \in (0, t_2]. \tag{2.19}$$

Conditions (2.18) and (2.19) mean that $x_{n-1} \in V_h(x_0, \ldots, x_{n-2})$. □

3. Comparison with the results of Furukawa and Yoshinaga

The authors of [2] have formulated their n-order necessary optimality conditions by using three kinds of n-order variational sets defined in a Banach space. For the first two of these sets, $F(Q; x_0, \ldots, x_{n-1})$ and $V(Q; x_0, \ldots, x_{n-1})$, it is easy to give equivalent definitions which are valid in an arbitrary real topological vector space. This is done below in Definition 8. The third variational set $T(Q; x_0, \ldots, x_{n-1})$ is defined in terms of sequences and will not be considered here.

DEFINITION 8. Let X be a real topological vector space, and let Q be an arbitrary nonempty subset of X. For $x_0 \in Q$, and for any sequence $(x_1, \ldots, x_{n-1})$ of points of X, we define the *variational sets* $F(Q; x_0, \ldots, x_{n-1})$ and $V(Q; x_0, \ldots, x_{n-1})$ as follows:

(a) $x_n \in F(Q; x_0, \ldots, x_{n-1})$ if and only if there exist $N \in \mathcal{N}(x_n)$ and $t_0 > 0$ such that

$$\sum_{i=0}^{n-1} t^i x_i + t^n N \subset Q \quad \text{for all } t \in (0, t_0].$$

(b) $x_n \in V(Q; x_0, \ldots, x_{n-1})$ if and only if there exist $t_0 > 0$ and a function $r : (0, t_0] \to X$ such that

$$\sum_{i=0}^{n} t^i x_i + r(t) \in Q \quad \text{for all } t \in (0, t_0],$$

and for every $N \in \mathcal{N}(0_X)$, there exists $t_1 \in (0, t_0]$ such that $t^{-n} r(t) \in N$ for all $t \in (0, t_1]$.

It is easy to see that both the variational sets are empty if $x_0 \notin \operatorname{cl} Q$, and are equal to X if $x_0 \in \operatorname{int} Q$. Hence, these variational sets can give nontrivial information only if x_0 is a boundary point of Q.

We now give an inductive definition of n-order Neustadt derivatives of an arbitrary mapping between two real topological vector spaces (they are called *variational derivatives* in [4]).

DEFINITION 9. Let $f : X \to U$ where X and U are real topological vector spaces, and let $x_0 \in X$.

(a) Suppose that, for every $x_1 \in X$, there exists an element $f^{(1)}(x_0; x_1) \in U$ such that, for every $N_0 \in \mathcal{N}(0_U)$, there are $N_1 \in \mathcal{N}(x_1)$ and $t_1 > 0$ for which

$$t^{-1}(f(x_0 + ty) - f(x_0) - t f^{(1)}(x_0; x_1)) \in N_0 \quad \text{for all } y \in N_1 \text{ and } t \in (0, t_1].$$

Then we call the mapping $f^{(1)}(x_0;\cdot)$ the *first-order Neustadt derivative* of f at x_0.

(b) Let $n \geq 2$, and suppose that f has the derivatives $f^{(1)}(x_0;\,\cdot\,),\dots,$ $f^{(n-1)}(x_0,\dots,x_{n-2};\,\cdot\,)$, and for every $x_n \in X$, there exists an element $f^{(n)}(x_0,\dots,x_{n-1};x_n) \in U$ such that, for every $N_0 \in \mathcal{N}(0_U)$, there are $N_n \in \mathcal{N}(x_n)$ and $t_n > 0$ for which

$$t^{-n}\bigg(f\bigg(\sum_{i=0}^{n-1} t^i x_i + t^n y\bigg) - f(x_0) - \sum_{i=1}^{n} t^i f^{(i)}(x_0,\dots,x_{i-1};x_i)\bigg) \in N_0 \qquad \text{for all } y \in N_n \text{ and } t \in (0,t_n].$$

Then we call the mapping $f^{(n)}(x_0,\dots,x_{n-1};\,\cdot\,)$ the *n-order Neustadt derivative* of f at x_0 with respect to $(x_1,\dots,x_{n-1})$. In this case, f is called *n-times Neustadt differentiable* at x_0 with respect to $(x_1,\dots,x_{n-1})$.

Let us now return to problem (1.1), and suppose that the assumptions formulated in Section 1 hold. By comparing Definitions 3, 8 and 9, we can obtain the following result.

Theorem 10. *Suppose that g is n-times Neustadt differentiable at x_0 with respect to $(x_1,\dots,x_{n-1})$, and that $g(x_0) \in -K$. Then the following conditions are equivalent:*

$$x_n \in Q_g(x_0,\dots,x_{n-1}), \tag{3.1}$$

$$g^{(n)}(x_0,\dots,x_{n-1};x_n) \in F(-K; g(x_0), g^{(1)}(x_0;x_1),\dots,g^{(n-1)}(x_0,\dots,x_{n-2};x_{n-1})). \tag{3.2}$$

Proof. (3.1)$\Rightarrow$(3.2). Let $x_n \in Q_g(x_0,\dots,x_{n-1})$; then there exist $v \in \operatorname{int} K$, $N \in \mathcal{N}(x_n)$ and $t_0 > 0$ such that (2.3) holds. Since $v \in \operatorname{int} K$, we can find $N_0 \in \mathcal{N}(0_V)$ satisfying

$$v - N_0 - N_0 \subset K. \tag{3.3}$$

It follows from the assumption of n-times Neustadt differentiability of g that there exist $N_n \in \mathcal{N}(x_n)$ and $t_n \in (0,t_0]$ such that $N_n \subset N$ and

$$\begin{aligned} t^n g^{(n)}(x_0,\dots,x_{n-1};x_n) \in{}& g\bigg(\sum_{i=0}^{n-1} t^i x_i + t^n y\bigg) - g(x_0) \\ &- \sum_{i=1}^{n-1} t^i g^{(i)}(x_0,\dots,x_{i-1};x_i) + t^n N_0 \end{aligned} \tag{3.4}$$

for all $y \in N_n$ and $t \in (0, t_n]$. Applying successively conditions (3.4), (2.3) and (3.3), we get

$$
\begin{aligned}
(3.5)\qquad & g(x_0) + \sum_{i=1}^{n} t^i g^{(i)}(x_0, \dots, x_{i-1}; x_i) + t^n N_0 \\
& \subset g(x_0) + \sum_{i=1}^{n-1} t^i g^{(i)}(x_0, \dots, x_{i-1}; x_i) + g\biggl(\sum_{i=0}^{n-1} t^i x_i + t^n y\biggr) \\
& \quad - g(x_0) - \sum_{i=1}^{n-1} t^i g^{(i)}(x_0, \dots, x_{i-1}; x_i) + t^n (N_0 + N_0) \\
& = g\biggl(\sum_{i=0}^{n-1} t^i x_i + t^n y\biggr) + t^n (N_0 + N_0) \\
& \subset -K - t^n (v - N_0 - N_0) \subset -(K + t^n K)
\end{aligned}
$$

for all $y \in N_n$ and $t \in (0, t_n]$. Since K is a convex cone, we have

$$
(3.6)\qquad K + t^n K \subset K \quad \text{for all } t > 0.
$$

Let us define $M := g^{(n)}(x_0, \dots, x_{n-1}; x_n) + N_0 \in \mathcal{N}(g^{(n)}(x_0, \dots, x_{n-1}; x_n))$. Then (3.5) and (3.6) imply

$$
(3.7)\qquad g(x_0) + \sum_{i=1}^{n-1} t^i g^{(i)}(x_0, \dots, x_{i-1}; x_i) + t^n M \subset -K \quad \text{for all } t \in (0, t_n],
$$

which means that condition (3.2) holds.

(3.2)$\Rightarrow$(3.1). If (3.2) holds, then there exist $t_n > 0$ and $M \in \mathcal{N}(g^{(n)}(x_0, \dots, x_{n-1}; x_n))$ such that (3.7) is fulfilled. Then we can find $N_0 \in \mathcal{N}(0_V)$ satisfying

$$
(3.8)\qquad g^{(n)}(x_0, \dots, x_{n-1}; x_n) + N_0 + N_0 \subset M.
$$

It follows from the assumption of n-times Neustadt differentiability of g that there exist $N_n \in \mathcal{N}(x_n)$ and $t_n^* \in (0, t_n]$ such that

$$
\begin{aligned}
(3.9)\quad & g\biggl(\sum_{i=0}^{n-1} t^i x_i + t^n y\biggr) - g(x_0) - \sum_{i=1}^{n-1} t^i g^{(i)}(x_0, \dots, x_{i-1}; x_i) \\
& \qquad \in t^n (g^{(n)}(x_0, \dots, x_{n-1}; x_n) + N_0) \quad \text{for all } y \in N_n \text{ and } t \in (0, t_n^*].
\end{aligned}
$$

Applying successively conditions (3.9), (3.8) and (3.7), we get

$$
\begin{aligned}
(3.10)\qquad & g\biggl(\sum_{i=0}^{n-1} t^i x_i + t^n y\biggr) + t^n N_0 = g(x_0) + \sum_{i=1}^{n-1} t^i g^{(i)}(x_0, \dots, x_{i-1}; x_i) \\
& \quad + g\biggl(\sum_{i=0}^{n-1} t^i x_i + t^n y\biggr) - g(x_0) - \sum_{i=1}^{n-1} t^i g^{(i)}(x_0, \dots, x_{i-1}; x_i) + t^n N_0
\end{aligned}
$$

$$\begin{aligned}&\subset g(x_0)+\sum_{i=1}^{n-1}t^ig^{(i)}(x_0,\dots,x_{i-1};x_i)\\&\quad+t^n(g^{(n)}(x_0,\dots,x_{n-1};x_n)+N_0+N_0)\\&\subset g(x_0)+\sum_{i=1}^{n-1}t^ig^{(i)}(x_0,\dots,x_{i-1};x_i)+t^nM\subset -K\end{aligned}$$

for all $y\in N_n$ and $t\in(0,t_n^*]$. Since K is a cone with nonempty interior, the set $\operatorname{int}K$ is a nonempty cone; hence, there exists a vector $v\in(\operatorname{int}K)\cap N_0$.Then (3.10) implies

$$g\biggl(\sum_{i=0}^{n-1}t^ix_i+t^ny\biggr)+t^nv\in -K\quad\text{for all } y\in N_n \text{ and } t\in(0,t_n^*],$$

which means that $x_n\in Q_g(x_0,\dots,x_{n-1})$. □

Corollary 11. *Suppose that f is n-times Neustadt differentiable at x_0 with respect to $(x_1,\dots,x_{n-1})$. Then the following conditions are equivalent:*

$$x_n\in Q_f(x_0,\dots,x_{n-1}),\tag{3.11}$$

$$\begin{aligned}&f^{(n)}(x_0,\dots,x_{n-1};x_n)\\&\quad\in F(f(x_0)-\operatorname{int}C;f(x_0),f^{(1)}(x_0;x_1),\dots,f^{(n-1)}(x_0,\dots,x_{n-2};x_{n-1})).\end{aligned}\tag{3.12}$$

Proof. From Definition 8(a), it is easy to see that $F(Q;x_0,\dots,x_{n-1})=F(\operatorname{int}Q;x_0,\dots,x_{n-1})$ for an arbitrary set Q. Hence, the set $\operatorname{int}C$ in condition (3.12) may be replaced by C. Now, the equivalence of (3.11) and (3.12) follows from Theorem 10 with g and K replaced by $\widetilde{f}$ and C, respectively, where $\widetilde{f}$ is defined by $\widetilde{f}(x):=f(x)-f(x_0)$. □

Remark 12. If $U=\mathbb{R}$ and $C=\{\lambda\in\mathbb{R}\mid\lambda\ge 0\}$, then condition (3.12) is identical with condition (5.7) in [2].

Gathering the information provided in Theorems 5(a), 10 and in Corollary 11, we obtain the following result.

Theorem 13. *Let x_0 be a weak local minimum point for problem* (1.1). *Then, for every sequence $(x_1,\dots,x_{n-1})$ of elements of X, there is no solution x_n to the system*

$$\begin{aligned}&f^{(n)}(x_0,\dots,x_{n-1};x_n)\\&\quad\in F(f(x_0)-\operatorname{int}C;f(x_0),f^{(1)}(x_0;x_1),\dots,f^{(n-1)}(x_0,\dots,x_{n-2};x_{n-1})),\\&g^{(n)}(x_0,\dots,x_{n-1};x_n)\\&\quad\in F(-K;g(x_0),g^{(1)}(x_0;x_1),\dots,g^{(n-1)}(x_0,\dots,x_{n-2};x_{n-1})),\\&x_n\in V_h(x_0,\dots,x_{n-1}).\end{aligned}$$

Theorem 13 has a form similar to that of [2, Theorem 5.2]. However, the authors of [2] use the "sequential" variational set $T(Q; x_0, \dots, x_{n-1})$ instead of $V(Q; x_0, \dots, x_{n-1})$, for an arbitrary set constraint $x \in Q$ (in our case, $Q = \{x \in X \mid h(x) = 0\}$). Although the set $T(Q; x_0, \dots, x_{n-1})$ is, in general, larger than $V(Q; x_0, \dots, x_{n-1})$, which leads to a formally stronger result in [2], it can be seen that these two sets are equal under sufficiently strong differentiability assumptions in Banach spaces. (See, for example, the proof of [1, Proposition 7.2] which can be repeated as well for the "sequential" definitions.) Therefore, Theorem 13 shows that the n-order necessary optimality conditions of Ben–Tal and Zowe are essentially equivalent to the conditions of Furukawa and Yoshinaga.

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LEON MIKOŁAJCZYK AND MARCIN STUDNIARSKI
Faculty of Mathematics
University of Łódź
ul. S. Banacha 22
90-238 Łódź, POLAND

E-mail address: marstud@math.uni.lodz.pl