Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 15, 2000, 101–113

SOLUTIONS OF IMPLICIT EVOLUTION INCLUSIONS WITH PSEUDO-MONOTONE MAPPINGS

Wenming M. Bian

Dedicated to the memory of Juliusz P. Schauder

Abstract. Existence results are given for the implicit evolution inclusions $(Bx(t))' + A(t,x(t)) \ni f(t)$ and $(Bx(t))' + A(t,x(t)) - G(t,x(t)) \ni f(t)$ with B a bounded linear operator, $A(t,\cdot\,)$ a bounded, coercive and pseudo-monotone set-valued mapping and G a set-valued mapping of non-monotone type. Continuity of the solution set of first inclusion with respect to f is also obtained which is used to solve the second inclusion.

1. Introduction

In this paper, we shall consider existence and continuity problems of solutions for the implicit inclusion

$$\begin{aligned} &\frac{d}{dt}(Bx(t)) + A(t,x(t)) \ni f(t) \quad \text{a.e. on } [0,T], \\ &Bx(0) = Bx_0, \end{aligned} \tag{1.1}$$

and the perturbation problem

$$\begin{aligned} &\frac{d}{dt}(Bx(t)) + A(t,x(t)) - G(t,x(t)) \ni f(t) \quad \text{a.e. on } [0,T], \\ &Bx(0) = Bx_0, \end{aligned} \tag{1.2}$$

2000 *Mathematics Subject Classification.* 34A09, 34A60, 35K22, 47H15.

Key words and phrases. Implicit differential inclusions, pseudo-monotone mappings, solution set, perturbations.

The author thanks Professor J. R. L. Webb for his help.

in an evolution triple (V,H,V^*) with V,H real separable Hilbert spaces. Here B is a linear bounded, symmetric and positive operator from V to V^* and $\inf_{\|u\|_V} \|Bu\|_{V^*} > 0$, $A(t,\cdot)$ is a set-valued, bounded and coercive pseudo-monotone mapping from V to V^*, $f \in L^q(0,T;V^*)$ and G is a set-valued mapping of non-monotone type with values in H. The initial value x_0 is supposed to be in V although it can be in the larger space H. We will prove that these two problems have solutions $x \in L^p(0,T;V)$ with $x' \in L^q(0,T;V^*)$ and the set of all such solutions to (1.1) is continuous with respect to f.

Problems (1.1) and (1.2) allow many special cases that have been studied already. When B is the identity operator on V, (1.1) is the problem considered by the Bian and Webb in [3] (where V can be a reflexive Banach space). When $A(t,x) \equiv A(x)$ and A is a maximal monotone mapping, (1.1) is studied by Barbu and Favini in [2]. When A is monotone and Lipschitz, it is a problem treated by Andrews, Kuttler and Shillor in [1]. When A is monotone and B is the identity operator on V, (1.2) is the problem considered by Migórski in [4]. More further special cases can be found in the references of the papers cited above.

We remark that we work in $L^p(0,T;V)$ and $L^q(0,T;V^*)$ with $p \ge 2, q = p/(p-1)$, and in [1] and [2], the spaces used are $L^2(0,T;V)$ and $L^2(0,T;V^*)$. We also note that, in [1] and [2], the coercivity condition was imposed on the sum $A + \lambda B$ for some $\lambda > 0$ and the assumption $\inf\{\|Bu\| : \|u\| = 1\} > 0$ was not imposed, but in this paper, coercivity condition is made on A (if $p > 2$, these are equivalent). The extra condition we imposed on B makes that the solution x of (1.1) is such that $x' \in L^q(0,T;V^*)$ (particularly if $p = q = 2$, $x' \in L^2(0,T;V)$) and, from this property, the continuity result for (1.1) and the solvability for (1.2) can be derived which are not given in [1], [2] or [3].

2. Preliminaries

In this paper, we always suppose that (V,H,V^*) is an evolution triple with V,H Hilbert spaces, we suppose $p \ge 2$ is a given number and write $q = p/(p-1)$. The scalar product in H and the duality pairing between V and V^* are denoted by $(\cdot,\cdot)$. The space $L^r(0,T;V)$ will be abbreviated as $L^r(V)$ and the duality pairing between $L^p(V)$ and $L^q(V^*)$ will be denoted by $(\!(\cdot,\cdot)\!)$. The set of all bounded linear operators from V to V^* is denoted by $L(V,V^*)$. The norm in a space X is denoted by $\|\cdot\|_X$ except that in $L(V,V^*)$ which will be denoted by $\|\cdot\|$ only. Convergence in the weak topology will be written $x_n \rightharpoonup x$. The space X endowed with the weak topology will be denoted by X_w.

Suppose $N : V \to 2^{V^*}$ is a set-valued mapping. N is said to beof class (S_+) if

$$x_n \rightharpoonup x \quad \text{in } V, \quad u_n \in Nx_n \quad \text{and} \quad \limsup_{n\to\infty} (u_n, x_n - x) \le 0 \tag{2.1}$$

imply $x_n \to x$. N is said to be pseudo-monotone if (2.1) implies that for each $y \in V$, there exists $u = u(y) \in Nx$ such that $(u, x - y) \le \liminf_{n\to\infty} (u_n, x_n - y)$. N is said to be quasi-monotone if $x_n \rightharpoonup x$ in V. It is known that, if the mapping involved is bounded and demicontinuous, monotonicity implies pseudo-monotonicity, pseudo-monotonicity implies quasi-monotonicity, and a mapping of class (S_+) is pseudo-monotone.

Now, we introduce the following conditions regarding B and A.

(H1) $B \in L(V, V^*)$ is symmetric, positive and

$$l := \inf\{\|Bu\| : u \in V, \|u\|_V = 1\} > 0.$$

(H2) $A : [0,T] \times V \to 2^{V^*}$ is measurable with nonempty closed convex values and $v \mapsto A(t,v)$ is pseudo-monotone for every $t \in [0,T]$.

(H3) There exist $b_1 \ge 0, b_2 \in L^q(0,T)$ such that

$$\sup\{\|u\|_{V^*} : u \in A(t,v)\} \le b_1\|v\|_V^{p-1} + b_2(t), \quad \text{for all } v \in V,\ t \in [0,T].$$

(H4) There exist $b_3 \ge 0, b_4 \in L^1(0,T)$ such that

$$\inf_{u\in A(t,v)} (u,v) \ge b_3\|v\|_V^p - b_4(t), \quad \text{for all } v \in V,\ t \in [0,T].$$

We denote by

$$(Lx)(t) = \int_0^t x(s)\,ds, \quad \text{for each } x \in L^r(V),\ r \ge 1,$$
$$\widehat{A}x = \{g \in L^1(V^*) : g(t) \in A(t, x(t)) \text{ a.e.}\}, \quad \text{for each } x \in L^p(V).$$

It is known that, under (H2)–(H4), $\widehat{A}$ is a well-defined bounded mapping from $L^p(V)$ to $L^q(V^*)$ with closed convex values. Moreover, in [3], the authors proved the following results which remains valid if we replace the general triple by a Hilbert space one.

LEMMA 2.1 ([3]). *Suppose* (H2)–(H4) *are satisfied. Then the following assertions hold.*

(i) *For each $f \in L^q(V^*)$ and each $x_0 \in V$, there exists $x \in L^p(V)$ such that*

$$x' \in L^q(V^*), \quad x'(t) + A(t, x(t)) \ni f(t) \text{ a.e.} \quad \textit{and } x(0) = x_0.$$

(ii) *If x_n are functions from $[0,T]$ into V with $x_n \rightharpoonup x$ in $L^q(V^*)$, $Lx_n \rightharpoonup Lx$ in $L^p(V)$ and $z_n \in \widehat{A}Lx_n$, $\limsup ((z_n, Lx_n - Lx)) \le 0$, then there exist $z \in \widehat{A}Lx$, a subsequence $\{z_{n_j}\}$ such that$z_{n_j} \rightharpoonup z$ and $((z_{n_j}, Lx_{n_j})) \to ((z, Lx))$.*

Let $\Lambda : V \to V^*$ be the canonical isomorphism and $\varepsilon > 0$ be given. Under assumption (H1), we see that $\varepsilon\Lambda + B$ is an isomorphism from V to V^*. So we can let

$$\langle u, v\rangle_W := ((\varepsilon\Lambda + B)^{-1}u, v) \quad \text{and} \quad A_\varepsilon(t, v) := A(t, (\varepsilon\Lambda + B)^{-1}v)$$

for all $u, v \in V^*$. Since B is symmetric, $\langle\,\cdot\,,\,\cdot\,\rangle_W$ is an inner product on V^* and the space $W := (V^*, \langle\,\cdot\,,\,\cdot\,\rangle_W)$ is a Hilbert space in which the norm is denoted by $\|\cdot\|_W$.

The following conclusion regarding the equivalence of the two norms on V^* might be known, but for completeness, we give it with proof.

Lemma 2.2. *$\|(\varepsilon\Lambda + B)^{-1}\|^{-1/2}\|v\|_W \le \|v\|_{V^*} \le \|\varepsilon\Lambda + B\|^{1/2}\|v\|_W$ for each $v \in W$.*

Proof. Let $v \in V^*$. Then

$$\|v\|_W^2 = ((\varepsilon\Lambda + B)^{-1}v, v) \le \|(\varepsilon\Lambda + B)^{-1}\|\|v\|_{V^*}^2$$

which implies the first part of our inequalities. Also, there exists $u \in V$, $\|u\|_V = 1$ such that $\|v\|_{V^*} = (u, v)$. Write $z = (\varepsilon\Lambda + B)u \in V^*$. Then

$$\|z\|_W^2 = \langle z, z\rangle_W = (u, z) \le \|z\|_{V^*},$$

and, therefore, we have

$$\begin{aligned}\|v\|_{V^*} &= ((\varepsilon\Lambda + B)^{-1}z, v) = \langle z, v\rangle_W \\ &\le \|v\|_W\|z\|_W \le \|v\|_W\|z\|_{V^*}^{1/2} \\ &\le \|v\|_W\|\varepsilon\Lambda + B\|^{1/2}\|u\|_V^{1/2} = \|\varepsilon\Lambda + B\|^{1/2}\|v\|_W \qquad \square\end{aligned}$$

3. Existence

In this section, we consider the existence of solutions for problem (1.1) and some related second order problems.

Lemma 3.1. *Under assumptions* (H1)–(H4), *suppose* $\varepsilon \in (0, l/(2\|\Lambda\|))$. *Then $A_\varepsilon : [0, T] \times W \to 2^W$ is a measurable mapping with closed convex values, $A_\varepsilon(t, \cdot)$ is pseudo-monotone and, for each $v \in W$ and each $y \in A_\varepsilon(t, v)$, we have*

$$\|y\|_W \le b_1(2/l)^{p-(1/2)}(2\|B\|)^{(p-1)/2}\|v\|_W^{p-1} + (2/l)^{1/2}b_2(t), \tag{3.1}$$

$$\langle y, v\rangle_W \ge b_3k^p(l/2)^{p/2}\|v\|_W^p - b_4(t). \tag{3.2}$$

Proof. First, under our assumptions, we see

$$\|\varepsilon\Lambda + B\| \le \|B\| + \varepsilon\|\Lambda\| \le 2\|B\|, \tag{3.3}$$

$$\|(\varepsilon\Lambda + B)^{-1}\| = \sup_{\|u\|_V = 1} \frac{1}{\|(\varepsilon\Lambda + B)u\|_{V^*}} \le \frac{2}{l}. \tag{3.4}$$

By our assumption (H2) and Lemma 2.2, A_ε is a measurable mapping from $[0,T]\times W$ to 2^W with closed convex values.

Suppose $v_n \rightharpoonup v$ in W, $w_n \in A_\varepsilon(t,v_n)$ and $\limsup_{n\to\infty}\langle w_n, v_n - v\rangle_W \le 0$. Let $x_n = (\varepsilon\Lambda + B)^{-1}v_n, x = (\varepsilon\Lambda + B)^{-1}v$. Then we see that $w_n \in A(t,x_n), x_n \rightharpoonup x$ in V and

$$0 \ge \limsup_{n\to\infty}\langle w_n, v_n - v\rangle_W = \limsup_{n\to\infty}(w_n, x_n - x).$$

Since $A(t,\,\cdot\,)$ is pseudo-monotone, for each $y \in V^*$, there exists $w(y) \in A(t,x)$ such that

$$\begin{aligned}\langle w(y), v - y\rangle_W &= (w(y), x - (\varepsilon\Lambda + B)^{-1}y)\\ &\le \liminf_{n\to\infty}(w_n, x_n - (\varepsilon\Lambda + B)^{-1}y) = \liminf_{n\to\infty}\langle w_n, v_n - y\rangle_W.\end{aligned}$$

This means that $A_\varepsilon(t,\,\cdot\,)$ is pseudo-monotone.

To verify (3.1) and (3.2), we suppose $v \in W$ and let $y \in A(t,(\varepsilon\Lambda + B)^{-1}v)$. Then

$$\|y\|_W^2 = \langle y, y\rangle_W = ((\varepsilon\Lambda + B)^{-1}y, y) \le \|(\varepsilon\Lambda + B)^{-1}\|\|y\|_{V^*}^2.$$

Since $\varepsilon \in (0, l/(2\|\Lambda\|))$, by (3.4), we see $\|(\varepsilon\Lambda + B)^{-1}\| \le 2/l$. So from (H3), Lemma 2.2 and (3.3), it follows

$$\begin{aligned}\|y\|_W &\le b_1\|(\varepsilon\Lambda + B)^{-1}\|^{1/2}\|(\varepsilon\Lambda + B)^{-1}v\|_{V^*}^{p-1} + \|(\varepsilon\Lambda + B)^{-1}\|^{1/2}b_2(t)\\ &\le b_1\|(\varepsilon\Lambda + B)^{-1}\|^{p-(1/2)}\|\varepsilon\Lambda + B\|^{(p-1)/2}\|v\|_W^{p-1} + \|(\varepsilon\Lambda + B)^{-1}\|^{1/2}b_2(t)\\ &\le b_1(2/l)^{p-(1/2)}(2\|B\|)^{(p-1)/2}\|v\|_W^{p-1} + (2/l)^{1/2}b_2(t).\end{aligned}$$

On the other hand, let

$$k = \inf_{\varepsilon>0}\inf_{v\in V^*\setminus\{0\}}\frac{\|(\varepsilon\Lambda + B)^{-1}v\|_V}{\|v\|_{V^*}}.$$

If $k = 0$, then there exist sequences $\{v_n\} \in V^*$ and $\{\varepsilon_n\}$ such that $\|v_n\|_{V^*} = 1$, $\varepsilon_n \to 0$ and $\|(\varepsilon_n\Lambda + B)^{-1}v_n\|_V \to 0$. Writing $u_n = (\varepsilon_n\Lambda + B)^{-1}v_n$, we see

$$1 = \|v_n\|_{V^*} = \|(\varepsilon_n\Lambda + B)u_n\|_{V^*} \le (\varepsilon_n\|\Lambda\| + \|B\|)\|u_n\|_V \to 0$$

which is a contradiction. So $k > 0$ and, by (H4), Lemma 2.2 and (3.3), we have

$$\begin{aligned}\langle y, v\rangle_W &= ((\varepsilon\Lambda + B)^{-1}v, y) \ge b_3\|(\varepsilon\Lambda + B)^{-1}v\|_V^p - b_4(t)\\ &\ge b_3k^p\|v\|_{V^*}^p - b_4(t) \ge b_3k^p\|(\varepsilon\Lambda + B)^{-1}\|^{-p/2}\|v\|_W^p - b_4(t)\\ &\ge b_3k^p(l/2)^{p/2}\|v\|_W^p - b_4(t).\end{aligned}$$ $\square$

The main result of this section is

THEOREM 3.2. *Under the assumptions* (H1)–(H4), *there exists $c > 0$ such that, for each $f \in L^q(V^*)$, problem* (1.1) *has at least one solution $x \in L^p(V)$ with $x' \in L^q(V^*)$ and $\|x\|_{L^p(V)}, \|x'\|_{L^q(V^*)} \le c(1 + \|f\|_{L^q(V^*)})$. If, in addition, $p = 2$, then $x' \in L^2(V)$.*

PROOF. For each $\varepsilon \in (0, l/(2\|\Lambda\|))$, applying Lemma 3.1 and Lemma 2.1(i) in the triple (W, W, W), we see that there exists $x_\varepsilon \in L^p(W)$ with $x_\varepsilon(0) = x_1 := (\varepsilon\Lambda + B)x_0$ and $x'_\varepsilon \in L^q(W)$ such that

$$x'_\varepsilon(t) + A_\varepsilon(t, (\varepsilon\Lambda + B)^{-1}x_\varepsilon(t)) \ni f(t), \quad \text{a.e. } t \in [0, T]. \tag{3.5}$$

Scalar multiplying (3.5) by $x_\varepsilon(t)$ and using the coercivity (3.2) of A_ε, we have

$$\frac{1}{2}\frac{d}{dt}\|x_\varepsilon(t)\|_W^2 + C_1\|x_\varepsilon(t)\|_W^p - b_4(t) \le \|f(t)\|_W\|x_\varepsilon(t)\|_W$$

with $C_1 := (l/2)^{p/2}b_3k^p$. Therefore

$$\frac{1}{2}\|x_\varepsilon(T)\|_W^2 + C_1\|x_\varepsilon\|_{L^p(W)}^p \le \frac{1}{2}\|x_1\|_W^2 + \int_0^T |b_4(t)|dt + \|f\|_{L^q(W)}\|x_\varepsilon\|_{L^p(W)}.$$

Using (3.5) and the growth condition (3.1), we see

$$\|x'_\varepsilon\|_{L^q(W)} \le \|f\|_{L^q(W)} + C_2\|x_\varepsilon\|_{L^p(W)}^{p-1} + C_2$$

with $C_2 > 0$ a constant independent of f and ε. By Lemma 2.2, (3.3) and (3.4), we see

$$\begin{aligned}\|x_1\|_W &\le \|(\varepsilon\Lambda + B)^{-1}\|^{1/2}\|x_1\|_{V^*}\\ &\le (2/l)^{1/2}\|\varepsilon\Lambda + B\|\|x_0\|_V \le 2\|B\|(2/l)^{1/2}\|x_0\|_V.\end{aligned}$$

Similarly, $\|f\|_{L^q(W)} \le 2\|B\|(2/l)^{1/2}\|f\|_{L^q(V^*)}$. So there exists constant $C_3 > 0$, independent of f and ε, such that

$$\|x'_\varepsilon\|_{L^q(W)}, \|x_\varepsilon\|_{L^p(W)} \le C_3(1 + \|f\|_{L^q(V^*)}). \tag{3.6}$$

Let n be so large that $1/n < l/(2\|\Lambda\|)$. Let $\varepsilon = 1/n, y_n = ((1/n)\Lambda + B)^{-1}x_\varepsilon$. Then $y_n \in L^p(V)$, $y'_n = ((1/n)\Lambda + B)^{-1}x'_\varepsilon \in L^q(V) \subset L^q(V^*)$ and there exists $z_n \in L^q(V^*)$ with $z(t) \in A(t, y_n(t))$ a.e. (that is $z_n \in \widehat{A}Ly'_n$) such that

$$y_n(0) = x_0 \quad \text{and} \quad ((1/n)\Lambda + B)y'_n(t)) + z_n(t) = f(t), \quad \text{a.e. on } [0, T]. \tag{3.7}$$

Since (V, H, V^*) is an evolution triple, there exists $\beta > 0$ such that

$$\|u\|_{V^*} \le \beta\|u\|_V \quad \text{and} \quad \|u\|_H \le \beta\|u\|_V \quad \text{for all } u \in V. \tag{3.8}$$

From (3.6), Lemma 2.2, (3.3) and (3.4), it follows that there exist constants $C_4 > 0$, independent of f and ε, such that

$$\begin{aligned}(3.9)\quad \|y_n'\|_{L^q(V^*)} &\le \beta\|y_n'\|_{L^q(V)} \le \beta\|((1/n)\Lambda + B)^{-1}\|\|x_\varepsilon'\|_{L^q(V^*)} \\ &\le C_4(1+\|f\|_{L^q(V^*)}), \\ \|y_n\|_{L^p(V)} &\le \|((1/n)\Lambda + B)^{-1}\|\|x_\varepsilon\|_{L^p(V^*)} \le C_4(1+\|f\|_{L^q(V^*)}).\end{aligned}$$

So we may suppose that $y_n \to y := Ly' + x_0$ in $L^p(0,T;V)$, $y_n' \rightharpoonup y'$ in $L^q(0,T;V^*)$, $z_n \rightharpoonup z$ in $L^q(0,T;V^*)$ and $((1/n)\Lambda+B)y_n' \rightharpoonup (By)'$ in $L^q(0,T;V^*)$ (by passing to subsequences). By (3.7) and noting $y_n(0) - y(0) = x_0 - x_0 = 0$, we have

$$\begin{aligned}\limsup_{n\to\infty} ((z_n, y_n - y)) &= \limsup_{n\to\infty} \int_0^T (-((1/n)\Lambda - B)y_n'(t), y_n(t) - y(t))dt \\ &= -\liminf_{n\to\infty} \int_0^T \frac{1}{2}\frac{d}{dt}(B(y_n(t)-y(t)), y_n(t)-y(t))dt \\ &= -\frac{1}{2}\liminf_{n\to\infty} (B(y_n(T)-y(T)), y_n(T)-y(T)) \le 0.\end{aligned}$$

By Lemma 2.1(ii), $z_n \rightharpoonup z \in \widehat{A}(Ly')$. So $(By)' + z = f$, that is, y is a solution for (1.1). Obviously, $\|y'\|_{L^q(V^*)}, \|y\|_{L^p(V)} \le C_4(1+\|f\|_{L^q(V^*)})$.

If $p = q = 2$, from (3.9), it follows that $\{y_n'\}$ is bounded in $L^2(V)$. So we may suppose $y_n' \rightharpoonup y'$ in $L^2(V)$. This means that $y' \in L^2(V)$. □

REMARK 3.3. In [1] or [2], $l > 0$ is not imposed, but, the boundedness of x (the solution) and x' are not derived there either. The property that $x' \in L^2(0,T;V)$ when $p = q = 2$ is claimed in [1] under other extra assumptions.

COROLLARY 3.4. *Under the assumptions* (H1)–(H4)*, suppose, A is measurable mapping from* $[0.T] \times V$ *to* V^* *with closed convex values and, for each* $t \in [0,T], v \mapsto A(t,v)$ *is quasi-monotone and weakly closed. Then, for each* $f \in L^q(V^*)$*, problem* (1.1) *is almost solvable in the sense that* $f \in \overline{\mathrm{range}(L^*B + L^*\widehat{A}L)}$. *More precisely, if we denote by j the duality map from V to V^*, then for each n, there exists $x_n \in L^p(V)$, $x(0) = x_0$ such that*

$$(3.10)\qquad \frac{d}{dt}(Bx_n(t)) + A(t,x_n(t)) \ni -\frac{1}{n}j(x_n(t)) + f(t), \quad \text{a.e.}$$

and $j(x_n)/n \to 0$ *in* $L^q(V^*)$.

PROOF. For each n, define a mapping $A_n : [0,T] \times V \to V^*$ by

$$A_n(t,v) = \frac{1}{n}j(v) + A(t,v) \quad \text{for } t \in [0,T], v \in V.$$

Since j is single-valued, of class (S_+) and demicontinuous, It can be proved easily that $v \mapsto A_n(t,v)$ is pseudo-monotone and

$$\sup_{u \in A_n(t,v)} \|u\|_{V^*} \le (1+b_1)\|v\|_V^{p-1} + 1 + b_2(t),$$
$$\inf_{u \in A_n(t,v)} (u,v) \ge b_3\|v\|_V^p - b_4(t)$$

for all $v \in V$, $t \in [0,T]$ and $n > 0$. Applying Theorem 3.2, there exists $x_n \in L^p(V)$ satisfying (3.10) for each $n > 0$ and $\|x_n\|_{L^p(V)} \le c$ for some constant c independent of n. As $\|j(x_n(t))\|_{V^*} = \|x_n(t)\|_V$, $\{j(x_n)\}$ is bounded in $L^q(V^*)$. So, $j(x_n)/n \to 0$ in $L^q(V^*)$. □

Now, we consider some second order differential inclusions. The first one is

$$\text{(3.11)} \quad ((Px(t))' + m(x(t)))' + Qx(t) = f(t), \quad m(x(t)) \in N(t,x(t)) \quad \text{a.e.,}$$
$$Px(0) = Px_0, \quad ((Px)' + m(x))(0) = Qx_1, x_0, x_1 \in V.$$

Here, $P, Q \in L(V,V^*)$ are symmetric operators and $(Pu,u) \ge 0$, $(Qu,u) \ge \omega\|u\|_V^p$ for some $\omega > 0$ for all $u \in V$, $\inf_{\|u\|_V} \|Pu\|_{V^*} > 0$, and $N : [0,T] \times V \to 2^{V^*}$ is a set-valued mapping. Its solvability can be obtained directly from Theorem 3.2.

Corollary 3.5. *Suppose $x_0, x_1 \in V$, N satisfies* (H2)–(H4). *Then problem* (3.11) *has at least one solution $x \in L^p(V)$ with $Px' + m(x) \in L^q(V^*)$.*

Proof. Obviously, (3.11) is equivalent to

$$(Bz(t))' + A(t,z(t)) \ni \widehat{f}(t) \quad \text{a.e. and } Bz(0) = Bz_0$$

in the evolution triple (V^2, H^2, V^{*2}) with

$$B = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}, \quad A(t,\cdot\,) = \begin{pmatrix} N(t,\cdot\,) & -Q \\ Q & 0 \end{pmatrix}, \quad \widehat{f} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad z_0 = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}.$$

We take the duality pairing between V^2 and V^{*2} as

$$\langle\langle (u,v),(x,y) \rangle\rangle = \langle u,x \rangle + \langle v,y \rangle \quad \text{for } u,v \in V^*,\ x,y \in V.$$

Here, in order to distinguish the duality pairing different from the points-pairing $(u,v) \in V^2$ or V^{*2}, we use $\langle\,\cdot\,,\cdot\,\rangle$ to stand for the duality pairing between V and V^*. Let $z_n := (x_n,y_n) \in V^2, w_n = (u_n,v_n) \in A(t,z_n)$ such that $z_n \rightharpoonup z = (x,y) \in V^2$ and

$$\limsup_{n\to\infty} \langle\langle (u_n,v_n),(x_n,y_n) - (x,y) \rangle\rangle \le 0.$$

Then $u_n \in N(t, x_n) - Qy_n, v_n = Qx_n$ and $x_n \rightharpoonup x$, $y_n \rightharpoonup y$, $Qx_n \rightharpoonup Qx$, $Qy_n \rightharpoonup Qy$. Since Q is symmetric, we see that

$$\begin{aligned}(3.12)\quad &(\liminf)\limsup_{n\to\infty} \langle\langle (u_n, v_n), (x_n, y_n) - (x^*, y^*)\rangle\rangle \\ &= (\liminf)\limsup_{n\to\infty} \langle u_n + Qy_n, x_n - x^*\rangle + \langle Qy, x^*\rangle - \langle Qx, y^*\rangle,\end{aligned}$$

for all x^*, $y^* \in V$.By taking $x^* = x, y = y^*$ in (3.12), we obtain $\limsup_{n\to\infty} \langle u_n + Qy_n, x_n - x\rangle \le 0$ and, therefore, the pseudo-monotonicity of N implies that, for each $(\widehat{x}, \widehat{y}) \in V^2$, there exists $u^* \in N(t, x)$ such that

$$\langle u^*, x - \widehat{x}\rangle \le \liminf_{n\to\infty} \langle u_n + Qy_n, x_n - \widehat{x}\rangle.$$

Let $\widehat{u} = u^* - Qy, \widehat{v} = Qx$. Then $(\widehat{u}, \widehat{v}) \in A(t, (x, y))$. Using (3.12), we have

$$\begin{aligned}\langle\langle (\widehat{u}, \widehat{v}), (x, y) - (\widehat{x}, \widehat{y})\rangle\rangle &= \langle u^*, x - \widehat{x}\rangle + \langle Qy, \widehat{x}\rangle - \langle Qx, \widehat{y}\rangle \\ &\le \liminf_{n\to\infty} \langle\langle (u_n, v_n), (x_n, y_n) - (\widehat{x}, \widehat{y})\rangle\rangle,\end{aligned}$$

that is, $A(t, \cdot\,)$ is pseudo-monotone. Also, it can be proved easily that the other conditions of Theorem 3.2 are satisfied in the present situation. So, the conclusion follows. □

Theorem 3.6. *Under the assumptions* (H1)–(H4)*, suppose* $P : V \to V^*$ *is a linear, bounded, symmetric and positive operator. Then, for each* $f \in L^q(V^*), x_0, x_1 \in V$*, there exists* $x \in L^p(V)$ *such that*

$$(3.13)\qquad \begin{aligned}(Bx(t))'' + A(t, x'(t)) + Px(t) \ni f(t) \quad \text{a.e.}, \\ Bx(0) = Bx_0, \quad (Bx(0))' = Bx_1.\end{aligned}$$

Proof. Consider the problem

$$(3.14)\qquad (By(t))' + A(t, y(t)) + PLy(t) \ni f(t) \quad \text{a.e.}, \quad By(0) = Bx_1.$$

Let $\widehat{P}$ be the realization of P. By our assumptions on P, $\widehat{P}L$ is continuous and positive from $L^p(V)$ to $L^q(V^*)$. So $L^*(\widehat{A} + \widehat{P}L)L$ is pseudo-monotone and satisfies the same coercive and growth conditions as $L^*\widehat{A}L$. Using almost the same method as used in Theorem 3.2 (just replace $\widehat{A}$ by $\widehat{A}+\widehat{P}L$), problem (3.14) has a solution y. Obviously, $x = Ly + x_0$ is a solution of (3.13). □

4. Continuity

Now, we denote the solution set of problem (1.1) by

$$\begin{aligned}S(f) = \{&x \in W(0, T) : x \text{ is a solution of (1.1)}, \\ &\|x\|_{L^p(V)},\ \|x'\|_{L^q(V^*)} \le c(1 + \|f\|_{L^q(V^*)})\}\end{aligned}$$

and consider its continuity with respect to f. Here c is the constant obtained in Theorem 3.2 and $W(0,T)=\{x\in L^p(V): x'\in L^q(V^*)\}$. Recall that is compact, then $W(0,T)\hookrightarrow L^p(H)$ compactly.

THEOREM 4.1. *Under the assumptions* (H1)–(H4), *$S(f)$ is a bounded weakly closed subset of $W(0,T)$. If, in addition, $V\hookrightarrow H$ compactly, then $f\mapsto S(f)$ is upper semicontinuous as a set-valued mapping from $L^q(H)_w$ to both $W(0,T)_w$ and $L^p(H)$.*

PROOF. Suppose $f\in L^q(V^*)$ and $x_n\in S(f)$ with $x_n\rightharpoonup x$ in $W(0,T)$. Then $x_n\rightharpoonup x$ in $L^p(V), x_n'\rightharpoonup x'$ in $L^q(V^*)$ and there exist $z_n\in S^q_{A(\,\cdot\,,x_n(\,\cdot\,))}$ such that

$$(Bx_n(t))'+z_n(t)=f(t)\quad \text{a.e..}$$

Multiplying both sides by x_n-x, we have

$$\begin{aligned}((Bx(t))',x_n(t)-x(t))+\frac{1}{2}\frac{d}{dt}(Bx_n(t)-Bx(t),x_n(t)-x(t))&\\ +(z_n(t),x_n(t)-x(t))=(f(t),x_n(t)-x(t))&\end{aligned}$$

and, therefore

$$\begin{aligned}\limsup_{n\to\infty}\langle\langle z_n,x_n-x\rangle\rangle&=\limsup_{n\to\infty}\langle\langle f-(Bx)',x_n-x\rangle\rangle\\ &\quad+\frac{1}{2}\limsup_{n\to\infty}[-(B(x_n(T)-x(T)),x_n(T)-x(T)]\le 0.\end{aligned}\tag{4.1}$$

Applying Lemma 2.1(ii) to the sequence $\{x_n'\}$, we see that there exist a subsequence $\{z_{n_j}\}$ and a point $z\in S^q_{A(\,\cdot\,,x(\,\cdot\,))}$ such that $z_{n_j}\rightharpoonup z$ in $L^q(V^*)$. Hence $(Bx_{n_j})'=f-z_{n_j}\rightharpoonup f-z$. Since $(Bx_n)'\rightharpoonup(Bx)'$, we see $(Bx)'+z=f$, that is. $x\in S(f)$. This proves the closedness. Obviously, $S(f)$ is a bounded subset.

Now, suppose $V\hookrightarrow H$ compactly. If S is not u.s.c. from $L^q(H)_w$ to $W(0,T)_w$ or $L^p(H)$, then there exist $f_n\rightharpoonup f$ in $L^q(H)$, $x_n\in S(f_n)$ and a neighbourhood $\mathcal{V}$ of $S(f)$in $W(0,T)_w$ or $L^p(H)$ with $x_n\notin\mathcal{V}$ for all $n>0$. Since $\{f_n\}$ is boundedin $L^q(V^*)$, we see that $\{x_n\}$ is bounded in $W(0,T)$. We may suppose(by passing to subsequences) that

$$x_n\rightharpoonup x\quad\text{in } L^p(V),\quad x_n'\rightharpoonup x'\quad\text{in } L^q(V^*)$$

for some $x\in W(0,T)$ and, therefore, $Bx_n\rightharpoonup Bx,(Bx_n)'\rightharpoonup(Bx)'$ in $L^q(V^*)$. The continuous embedding of $W(0,T)$ into $C(0,T;H)$ implies $x(0)=x_0$. Since $W(0,T)\hookrightarrow L^p(H)$ compactly, we may suppose $x_n\to x$ in $L^p(H)$. Therefore

$$\langle\langle f_n,x_n-x\rangle\rangle=\langle\langle f_n,x_n-x\rangle\rangle_H\to 0\quad\text{as } n\to\infty.$$

Here, $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_H$ stands for the duality pairing between $L^p(H)$ and $L^q(H)$. Let $z_n\in S^q_{A(\,\cdot\,,x_n(\,\cdot\,))}$ be the functions such that $(Bx_n)'(t)+z_n(t)=f_n(t)$ a.e. So, using the same method as used to obtain (4.1), we have

$$\limsup_{n\to\infty} ((z_n, x_n - x)) = \limsup_{n\to\infty} \bigg[((f_n - (Bx)', x_n - x)) \\ - \frac{1}{2}(Bx_n(T) - Bx(T), x_n(T) - x(T)) \bigg] \leq 0.$$

Applying Lemma 2.1(ii) to the sequence $\{x_n'\}$, we see that there exist a subsequence $\{z_{n_j}\}$ and $z \in S^q_{A(\,\cdot\,,x(\,\cdot\,))}$ such that $z_{n_j} \rightharpoonup z$ in $L^q(V^*)$ and $((z_{n_j}, x_{n_j} - x)) \to 0$. So $(Bx(t))' + z(t) = f(t)$ a.e. which implies $x \in S(f) \subset \mathcal{V}$. In case $\mathcal{V}$ is a neighbourhood of $S(f)$in $W(0,T)_w$, this has contradicted the assumption that $x_n \notin \mathcal{V}$ for all n. In case $\mathcal{V}$ is a neighbourhood of $S(f)$ in $L^p(H)$, the compact embedding of $W(0,T)$ into $L^p(H)$ implies that we can suppose (by passing to a further sequence) $x_{n_j} \to x$ in $L^p(H)$ which also contradicts our assumption.□

5. Perturbation problem

In this section, we consider the solvability of (1.2).

Theorem 5.1. *Under the assumptions* (H1)–(H4), *let $V \hookrightarrow H$ compactly and, for each $f \in L^q(H)$, problem* (1.1) *has a unique solution. Suppose $G : [0,T] \times H \to 2^H$ is a measurable set-valued mapping with closed convex values, $v \mapsto G(t,v)$ is u.s.c. as a mapping from H into H_w. If there exist $d_1 \in L^q(H), d_2, d_3 > 0$ such that either*

$$\|G(t,u)\|_H := \sup\{\|v\|_H : v \in G(t,u)\} \leq d_1(t) \quad \text{for all } t \in [0,T],\ u \in H \tag{5.1}$$

or

$$(Bu,u) \geq d_2\|v\|_H^2, \quad \|G(t,u)\|_H \leq d_3\|u\|_H^{p-1} + d_1(t) \tag{5.2}$$

for all $t \in [0,T]$, $u \in H$, then, for each $x_0 \in V$ and each $f \in L^q(V^*)$, problem (1.2) *has solutions.*

Proof. First, we suppose (5.1) is satisfied. Let x_f be the unique solution of problem (1.1) and let

$$F(g) = S^1_{G(\,\cdot\,,x_{f+g}(\,\cdot\,))} = \{z \in L^1(H) : z(t) \in G(t, x_{f+g}(t)) \text{ a.e.}\},$$
$$D = \{x \in L^q(H) : \|x(t)\|_H \leq d(t)\}.$$

Then, our assumptions imply that F is a well-defined mapping from D into itself with closed convex values.

Let $(g_n, z_n) \in \mathrm{Graph}(F)$ and $g_n \rightharpoonup g, z_n \rightharpoonup z$ in $L^q(H)$. By Theorem 4.1, $x_{g_n+f} \to x_{g+f}$ in $L^p(H)$ and, therefore, $x_{g_n+f}(t) \to x_{g+f}(t)$ in H a.e. (by passing to a subsequence). Since $G(t,\,\cdot\,)$ is u.s.c., we see

$$w\text{-}\limsup_{n\to\infty} G(t, x_{g_n+f}(t)) \subset G(t, x_{g+f}(t))$$

for almost all t. Invoking Theorem 4.2 of [5], we have

$$z \in w\text{-}\limsup_{n\to\infty} F(g_n) \subset S^1_{w\text{-}\limsup_{n\to\infty} G(\,\cdot\,, x_{g_n+f}(\,\cdot\,))} \subset S^1_{G(\,\cdot\,, x_{g+f}(\,\cdot\,))} = F(g).$$

So $(g, z) \in \operatorname{Graph} F$, that is F is closed under the weak topology. Since D is weakly compact, we see F is weakly upper semicontinuous under the weak topology. Since D is convex, from Kakutani's fixed point theorem, it follows that F has fixed point, say g. Obviously, x_{g+f} is a solution of (1.2).

Now, suppose (5.2) is satisfied. We claim that there exists $M > 0$ such that

$$\|x(t)\|_H \le M \quad \text{for each } t \in [0, T] \text{ and each solution } x \text{ of (1.2).} \tag{5.3}$$

In fact, let x be a solution to (1.2). Then there exist $g_1 \in L^q(V^*), g_2 \in L^q(H)$ such that $g_1(t) \in A(t, x(t)), g_2(t) \in G(t, x(t))$ a.e. and $(Bx(t))' + g_1(t) - g_2(t) = f(t)$ a.e. Therefore, by (5.3) and Young's inequality, for each $\varepsilon > 0$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(Bx(t), x(t))^2 \ &+ (g_1(t), x(t)) \\
&= (g_2(t), x(t)) + (f(t), x(t)) \\
&\le (d_3\|x(t)\|_H^{2/q} + d_1(t))\|x(t)\|_H + \|f(t)\|_{V^*}\|x(t)\|_V \\
&\le \frac{1}{\varepsilon^q q}(d_3\|x(t)\|_H^{2/q} + d_1(t))^q \\
&\quad + \frac{\varepsilon^p}{p}\|x(t)\|_H^p + \frac{1}{\varepsilon^q q}\|f(t)\|_{V^*}^q + \frac{\varepsilon^p}{p}\|x(t)\|_V^p.
\end{aligned}$$

Noting (5.2), (H4) and (3.8), we obtain

$$\begin{aligned}
\frac{1}{2}d_2\|x(t)\|_H^2 \ &+ b_3\int_0^t \|x(s)\|_V^p\,ds \\
&\le \frac{1}{2}(Bx_0, x_0) + \int_0^t b_4(s)\,ds + \frac{2^q d_3^q}{\varepsilon^q q}\int_0^t \|x(s)\|_H^2\,ds \\
&\quad + \frac{1}{\varepsilon^q q}\int_0^t (2^q d_1^q(s) + \|f(s)\|_{V^*}^q)^q\,ds + \frac{\varepsilon^p}{p}(\beta^p + 1)\int_0^t \|x(s)\|_V^p\,ds.
\end{aligned}$$

Choosing $\varepsilon = [(pb_3)/(\beta^p + 1)]^{1/p}$, and by Gronwall's Inequality, we see that *a priori* estimates (5.3) hold. Let

$$\begin{aligned}
G_1(t, x) &= G(t, x) && \text{if } \|x\|_H \le M, \\
G_1(t, x) &= G(t, Mx/\|x\|_H) && \text{if } \|x\|_H > M.
\end{aligned}$$

Then G_1 is an upper semicontinuous mapping from $[0, T] \times H$ into H with closed convex values and $\|G_1(t, x)\|_H \le d_1(t) + d_3 M^{2/q}$. Applying the conclusion obtained in the first case, we see that there exists $x \in W(0, T)$ such that

$$Bx(0) = Bx_0 \quad \text{and} \quad (Bx(t))' + A(t, x(t)) - G_1(t, x(t)) \ni f(t) \quad \text{a.e..}$$

Using the same method as the one used to obtain (5.3), we can prove that $\|x(t)\|_H \leq M$ on $[0,T]$ and, therefore, $G_1(t,x(t)) = G(t,x(t))$ a.e. Hence, x is a solution of (1.1). $\square$

REFERENCES

[1] K. T. ANDREWS, K. L. KUTTLER AND M. SHILLOR, *Second order evolution equations with dynamic boundary conditions*, J. Math. Anal. Appl. **197** (1996), 781–795.

[2] V. BARBU AND A. FAVINI, *Existence for an implicit nonlinear differential equation*, Nonlinear Anal. **32** (1998), 33–40.

[3] W. M. BIAN AND J. R. L. WEBB, *Solutions of nonlinear evolution inclusions*, Nonlinear Anal. **37** (1999), 915–932.

[4] S. MIGÓRSKI, *On an existence result for nonlinear evolution inclusions*, Proc. Edinburgh Math. Soc. (2) **39** (1996), 133–141.

[5] N. S. PAPAGEORGIOU, *Convergence theorems for Banach spaces valued integrable multifunctions*, Internat. J. Math. Math. Sci. **10** (1987), 433–442.

Manuscript received November 2, 1999

WENMING M. BIAN
Department of Mathematics
University of Glasgow
Glasgow G12 8QW, UNITED KINGDOM

E-mail address: wb@maths.gla.ac.uk