Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 14, 1999, 1–38

EFFECT OF THE DOMAIN GEOMETRY ON THE EXISTENCE OF MULTIPEAK SOLUTIONS FOR AN ELLIPTIC PROBLEM

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Abstract. In this paper, we construct multipeak solutions for a singularly perturbed Dirichlet problem. Under the conditions that the distance function $d(x, \partial\Omega)$ has k isolated compact connected critical sets $T_1, \ldots, T_k$ satisfying $d(x, \partial\Omega) = c_j = \text{const.}$, for all $x \in T_j$, $\min_{i \neq j} d(T_i, T_j) > 2 \max_{1 \leq j \leq k} d(T_j, \partial\Omega)$, and the critical group of each critical set T_i is nontrivial, we construct a solution which has exactly one local maximum point in a small neighbourhood of T_i, $i = 1, \ldots, k$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$. Consider

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

where ε is a small positive number, $2 < p < 2N/(N-2)$ if $N \geq 3$ and $2 < p < \infty$ if $N = 2$.

In the past few years, a lot of work has been done on the existence and multiplicity of the solutions for (1.1). First, Ni and Wei [27] proved that as $\varepsilon \to 0$,

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1991 *Mathematics Subject Classification.* 35J65, 35B25, 58E05.
Key words and phrases. Peaks, nonlinear, small diffusion, Conley index, Clarke derivative.
This research was supported by ARC.

the least energy solution has exactly one local maximum point and this local maximum point tends to a point which attains the global maximum of the distance function $d(x, \partial\Omega)$. From then, the effect of the distance function on the existence of single peak solution, that is, solution which has exactly one local maximum point, has been extensively studied and single peak solutions with the peak near various kind of critical points of the distance function have been constructed. See [21], [23], [28], [32], [33]. On the other hand, if the distance function has several critical points, there are a few works concerning the construction of multipeak solutions (see [6], [7], [18]).

There is another direction to study the multiplicity of the single peak solutions and the existence of multipeak solution: the effect of the domain topology. In [2], Benci and Cerami proved that (1.1) has at least $\mathrm{Cat}_\Omega(\Omega)$ single peak solution if $\varepsilon > 0$ is small enough. Later, Benci, Cerami and Pasasseo proved in [3] that if Ω is not contractible, the number of the solutions of (1.1) is at least $\mathrm{Cat}_\Omega(\Omega) + 1$. Concerning the effect of the domain topology on the existence of multipeak solution, we proved in [17] that if the homology of the domain is nontrivial, then for any positive integer k, (1.1) has at least one k-peak solution provided $\varepsilon > 0$ is small enough. The method in [17] can also be modified to show that there is a two-peak solution for small $\varepsilon > 0$ if Ω is not contractible. See also [14] for an early result on the existence of two-peak solutions.

The aim of this paper is to construct multipeak solutions for (1.1) provided that $d(x, \partial\Omega)$ has several critical points whose critical groups are nontrivial. We first give some definitions and recall some basic results.

Let $f(x)$ be a Lipschitz continuous function defined on $\mathbb{R}^N$. The Clarke derivative of f is defined as follows (see [10]):

$$\partial f(x) = \{\alpha \in \mathbb{R}^N : f^0(x, v) \geq \langle \alpha, v\rangle, \text{ for all } v \in \mathbb{R}^N\},$$

where

$$f^0(x, v) = \overline{\lim_{h\to 0, \lambda \to 0^+}} \frac{f(x + h + \lambda v) - f(x + h)}{\lambda}.$$

A point x_0 is called a critical point of f if $0 \in \partial f(x_0)$. Let T be an isolated connected critical set of f in the following sense: $0 \in \partial f(x)$, $f(x) = c$ for each $x \in T$, and f has no other critical point in a small neighbourhood $\mathcal{U}$ of T. Then we define the critical group of f on T as follows:

$$C_q(f, T) = H^q(f^c \cap \mathcal{U}, (f^c \setminus T) \cap \mathcal{U}),$$

where $q = 0, 1, \ldots$, $f^c = \{x : f(x) \leq c\}$.

We stress here that all the cohomologies in this paper are with the coefficients in the same field.

The distance function $d(x, \partial\Omega)$ is Lipschitz continuous and its Clarke derivative is

$$\partial d(x, \partial\Omega) = -\mathrm{co}\,(\Pi_{\partial\Omega}(x) - \{x\}), \tag{1.2}$$

where $\Pi_{\partial\Omega}(x) = \{y : y \in \partial\Omega,\ |y - x| = d(x, \partial\Omega)\}$ and $\mathrm{co}\, S$ denotes the convex hull of the set S (see [10]).

Let $U(y)$ be the unique positive solution (see [22]) of

$$\begin{cases} -\Delta u + u = u^{p-1} & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \\ u(0) = \max_{y\in\mathbb{R}^N} u(y). \end{cases}$$

It is well known that $U(y)$ is radially symmetric about the origin, decreasing and

$$\lim_{|y|\to\infty} U(y) e^{|y|} |y|^{(N-1)/2} = c_0 > 0.$$

Define $\langle u, v\rangle_\varepsilon = \int_\Omega \varepsilon^2 Du \cdot Dv + uv$, for all $u, v \in H_0^1(\Omega)$, $\|u\|_\varepsilon = \langle u, u\rangle_\varepsilon^{1/2}$. For any $z \in \mathbb{R}^N$, $\varepsilon > 0$, let $U_{\varepsilon,z}(y) =: U((y - z)/\varepsilon)$.

We denote by $P_{\varepsilon,\Omega} v$ the solution of the following problem:

$$\begin{cases} -\varepsilon^2 \Delta u + u = |v|^{p-2} v & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases}$$

By the maximum principle, we know $P_{\varepsilon,\Omega} U_{\varepsilon,z} > 0$.

For any $x_j \in \Omega$, $j = 1, \dots, k$, define

$$E_{\varepsilon,x,k} = \bigg\{ v \in H_0^1(\Omega) : \langle P_{\varepsilon,\Omega} U_{\varepsilon,x_j}, v\rangle_\varepsilon = \left\langle \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_j}}{\partial x_{ji}}, v \right\rangle_\varepsilon = 0, \\ j = 1, \dots, k,\ i = 1, \dots, N \bigg\}.$$

The main results of this paper are the following.

Theorem 1.1. *Suppose that $k \geq 1$ is an integer. Let T_j be an isolated critical set of the distance function $d(x, \partial\Omega)$ with $d(x, \partial\Omega) = c_j$ for all $x \in T_j$, and D_j be a small neighbourhood of T_j such that $d(x, \partial\Omega)$ has no critical point in $D_j \backslash T_j$, $j = 1, \dots, k$. Suppose that the critical group $C(d(x, \partial\Omega), T_j)$ is nontrivial for $j = 1, \dots, k$. If*

$$\min_{i\neq j} d(D_i, D_j) > 2 \max_{1\leq j\leq k} \max_{x\in D_j} d(x, \partial\Omega), \tag{1.3}$$

then there is an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, (1.1) has a solution of the form

$$u_\varepsilon = \sum_{j=1}^k \alpha_{\varepsilon,j} P_{\varepsilon,\Omega} U_{\varepsilon,x_{\varepsilon,j}} + v_\varepsilon, \tag{1.4}$$

where $v_\varepsilon \in E_{\varepsilon,x_\varepsilon,k}$*, and as* $\varepsilon \to 0$*,*

$$\alpha_{\varepsilon,j} \to 1,\ x_{\varepsilon,j} \to x_j \in T_j,\ \|v_\varepsilon\|_\varepsilon = o(\varepsilon^{N/2}),\ j = 1,\dots,k. \tag{1.5}$$

Usually, we call the solution of the form (1.4) satisfying (1.5) a k-peak solution. Apart from the above existence result, we have the following nonexistence result.

THEOREM 1.2. *Suppose that* Ω *is strictly convex. Then for each integer* $k \geq 2$*, there is an* $\varepsilon_0 > 0$*, such that for each* $\varepsilon \in (0, \varepsilon_0]$*,* (1.1) *does not have any* k*-peak solution.*

REMARK 1.3. It can be proved that a domain Ω is strictly convex if and only if for each point $x \in \partial\Omega$, $T_x \cap (\overline{\Omega} \setminus \{x\}) = \emptyset$, where T_x is the tangent plane of $\partial\Omega$ at x.

REMARK 1.4. In [33], Wei proved Theorem 1.2 for the case $k = 2$ and asked whether the conclusion was still true for $k \geq 3$.

Using (1.2), we can check that the peak of any single peak solution must converge to a critical point of the distance function as $\varepsilon \to 0$. See the discussion in the beginning of Section 3. So it is natural to ask what kind of critical point of the distance function can generate a single peak solution with its peak near this critical point. The example given in Section 4 shows that some of the critical points of the distance function may not generate a single peak solution. On the other hand, for an isolated strict local maximum point, or a critical point such that the distance function is differentiable on the boundary of a small neighbourhood of this point and the degree of $Dd(x, \partial\Omega)$ in this small neighbourhood is not zero, then there is a single peak solution with its peak nearby (see [6], [18], [21], [23], [28], [32], [34]). In Section 3, we prove some results on the nontriviality of the critical group. These results enable us to conclude that the critical groups of local maximum points, or some kind of saddle points, or critical point with nonzero degree are nontrivial. We also give in Section 3 an example where the isolated critical point of the distance function is not the kind of saddle point defined in [9], and the distance function is not differentiable on the boundary of any small neighbourhood of this point, but whose corresponding critical group is nontrivial. This example also shows that the structure of $d(x, \partial\Omega)$ in a small neighbourhood of an isolated critical point may be very complicated even if its corresponding critical group is nontrivial.

As we shall see, to construct a single peak solution for (1.1) with its peak near a designated critical point x_0 of $d(x, \partial\Omega)$ essentially reduces to find a condition under which x_0 is stable subject to suitable small perturbations (see [23] for the precise definition). Thus we need to discuss the stability problem for the critical point of a function $f(x)$. In order to make it possible to glue together the single peak solutions, we also need to study whether (x_1, x_2) is a stable critical point of $f_1(x_1) + f(x_2)$ if x_i is a stable critical point of $f_i(x_i)$, $i = 1, 2$. In the smooth case, we know that if the corresponding critical group is nontrivial, then the critical point is stable. Moreover, $C(f_1 + f_2, (x_1, x_2))$ is nontrivial and thus (x_1, x_2) is a stable critical point of $f_1(x_1) + f(x_2)$, if $C(f_i, x_i)$ is nontrivial, $i = 1, 2$. In the smooth case, it is also well known that the corresponding critical groups of strict local maximum points, strict local minimum points, nondegenerate critical points, critical points with nonzero degree are nontrivial. For the distance function, we know it has no minimum point inside Ω. Wei in [34] gave the definition for a critical point of $d(x, \partial\Omega)$ to be nondegenerate. However, we prove in Section 4 that under Wei's definition, a nondegenerate critical point of $d(x, \partial\Omega)$ must be a strict local maximum point. On the other hand, the disadvantage to use classical degree theory to deal with the stability problem is that in many cases, it may not be possible to find a small neighbourhood for the isolated critical point such that $d(x, \partial\Omega)$ is differentiable on the boundary of this neighbourhood and $d(x, \partial\Omega)$ has no other critical point in this neighbourhood. For these reasons, to deal with the stability problem for a critical point of the distance function, which is not a local maximum point, we need to generalize the degree theory and the critical group theory to the nonsmooth problems. Our result in Section 3 shows that as in the smooth case, the nonzero degree implies the nontriviality of the critical groups. This is one of the reasons we use critical groups to deal with the stability problem for the critical points. There is another advantage to use the critical groups in the nonsmooth problems. To calculate the degree of the Clarke derivative of a Lipschitz continuous function, one needs to choose a smooth approximating vector field and calculate the degree of this smooth vector field, while the calculation of the critical groups does not involve the choice of a smooth approximating vector field. So in practice, the critical group is calculable. In addition, we can also construct an example of a critical set which has zero degree but whose critical group is non-zero. See Section 3.

In [21], Grossi and Pistoia gave the definition for a critical value to be topologically nontrivial. But it is not clear whether $c_1 + c_2$ is a critical value of $f_1(x_1) + f_2(x_2)$ topologically nontrivial if c_i is a critical value of $f_i(x_i)$, $i = 1, 2$, which is topologically nontrivial. Thus it seems difficult to glue together the single peak solutions constructed in [21]. Another disadvantage of using this idea to construct single peak solution with its peak near a designated critical point is that extra conditions on this critical point are needed.

The basic idea to construct multipeak solutions with their peaks close to some specific critical points is to glue some single peak solutions together. To make this idea work, usually it is required that different critical points be suitably separated. In [6], [7], [18], condition (1.3) is imposed so the contribution from the interaction between different peaks is negligible. It seems that (1.3) is almost necessary for Theorem 1.1 to hold, at least this is true if $d(P_j,\partial\Omega) = d(P_i,\partial\Omega)$, $i \neq j$. In [6], [7], reduction methods were used to construct k-peak solutions with the peaks near the local maximum points or saddle points (in the strong sense) $P_1,\dots,P_k$ of $d(x,\partial\Omega)$. But in these two papers, it is required that

$$d(P_j,\partial\Omega) = d(P_i,\partial\Omega), \quad i,j = 1,\dots,k. \tag{1.6}$$

In [18], del Pino, Felmer and Wei used a variational method to construct a k-peak solution with its peaks close to some local maximum points $P_1,\dots,P_k$ without assuming (1.6). But it seems that the variational method is not easy to use to construct multipeak solutions with the peaks near some critical points which are not local maximum points. In this paper, we still use the reduction method to construct multipeak solution for (1.1). Without condition (1.6), the first small order term of the energy of the single peak solution with its peak near P_i is of different order from that of the single peak solution with its peak near P_j. To glue together single peak solutions involving different small order terms needs a lot of work. The new idea here is to split the small part v_ε into k different part, each of which is very close to the corresponding small part of a single peak solution. By this result, we are able to prove that the reduced problem is just a small perturbation of that corresponding to a single peak solution, provided (1.3) holds. In Section 4, we will give an example showing that the nontriviality of the critical group is almost necessary to obtain the result of Theorem 1.1. Theorem 1.1 is new even if all the critical points are saddle points in the strong sense. The nontriviality of $C(d,x_i)$, $i = 1,\dots,k$, is the weakest condition we know to guarantee the existence of a k-peak solution which has exactly one peak near each x_i. Our methods have a number of advantages over other work. Firstly, it applies to critical points which are not saddle points in the sense of Rabinowitz. Moreover, unlike a number of other recent works [2], [14], [17], we can also control the location of the peaks of the solutions.

Note that to obtain our results, we construct the Conley index, critical groups and degree of an isolated critical point of a locally Lipschitz function. This seems of independent interest.

The method to prove Theorem 1.1 can be used to obtain the same result for the Dirichlet problem in exterior domains under the condition that the distance function has some critical points with nontrivial critical group. Unlike the bounded domain case, the distance function on exterior domain does not always have a critical point. However we proved in [16] that the Dirichlet problem in an exterior domain always has a two peak solution and if the domain is the complement of a bounded convex set, then the peaks of any two peak solution move to infinity. For other results on the Dirichlet problem in exterior domains, the readers can refer to [4], [5], [26].

We can also use this technique to glue together a boundary peak solution and an interior peak solution for the singularly perturbed Neumann problem. This is discussed briefly in Section 5.

This paper is organised as follows. In Section 2, we reduce the problem of finding a multipeak solution for (1.1) to a finite dimensional problem. The main ingredient of Section 2 is to split the small term into different parts. This technique is important in gluing different single peak solutions involving different order small terms. Section 3 contains the discussion of the Conley index of the generalized gradient of a Lipschitz function and its critical group. Theorems 1.1 and 1.2 are proved in Section 4. Some basic estimates are presented in the appendix.

2. Reduction of the problem

Let

$$I(u) = \frac{1}{2}\int_\Omega (\varepsilon^2|Du|^2 + u^2) - \frac{1}{p}\int_\Omega |u|^p, \quad u \in H_0^1(\Omega). \tag{2.1}$$

For fixed integer $k > 0$, let

$$\alpha(k) = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k, \tag{2.2}$$

$$x = (x_1, \dots, x_k) \in \mathbb{R}^{kN},\ x_i \in \mathbb{R}^N,\ i = 1, \dots, k. \tag{2.3}$$

Define $D_{k,\delta} = \{x : x_i \in \Omega,\ d(x_i, \partial\Omega) \geq \delta,\ i = 1, \dots, k,\ |x_i - x_j| \geq 2\delta,\ i \neq j\}$, and $M_{\varepsilon,\delta} = \{(\alpha(k), x, v) : |\alpha_i - 1| \leq \delta,\ i = 1, \dots, k;\ x \in D_{k,\delta}, v \in E_{\varepsilon,x,k},\ \|v\|_\varepsilon \leq \delta\varepsilon^{N/2}\}$. Let

$$J(\alpha(k), x, v) = I\biggl(\sum_{i=1}^k \alpha_i P_{\varepsilon,\Omega}U_{\varepsilon,x_i} + v\biggr), \quad (\alpha, x, v) \in M_{\varepsilon,\delta}. \tag{2.4}$$

It is well known that if $\delta > 0$ is small enough, $(\alpha(k), x, v) \in M_{\varepsilon,\delta}$ is a critical point of $J(\alpha(k), x, v)$ if and only if $u = \sum_{i=1}^k \alpha_i P_{\varepsilon,\Omega}U_{\varepsilon,x_i} + v$ is a positive critical point of $I(u)$ (see [30]). So we just need to find $(\alpha(k), x, v) \in M_{\varepsilon,\delta}$ and A_l, B_{li}, $l = 1, \dots, k$, $i = 1, \dots, N$, such that

$$\frac{\partial J(\alpha(k), x, v)}{\partial x_{li}} = \sum_{j=1}^N B_{lj}\Bigl\langle \frac{\partial^2 P_{\varepsilon,\Omega}U_{\varepsilon,x_l}}{\partial x_{li}\partial x_{lj}}, v\Bigr\rangle_\varepsilon, \quad i = 1, \dots, N,\ l = 1, \dots, k, \tag{2.5}$$

$$\frac{\partial J(\alpha(k), x, v)}{\partial \alpha_l} = 0, \qquad l = 1, \dots, k, \tag{2.6}$$

$$\frac{\partial J(\alpha(k), x, v)}{\partial v} = \sum_{l=1}^k A_l P_{\varepsilon,\Omega}U_{\varepsilon,x_l} + \sum_{l=1}^k\sum_{j=1}^N B_{lj}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_l}}{\partial x_{lj}}. \tag{2.7}$$

The aim of this section is to reduce the problem of finding a critical point for $J(\alpha(k), x, v)$ to that of finding a critical point for a function defined in a finite dimensional domain. The proof of the existence of $(\alpha(k), v)$ satisfying (2.6) and (2.7) for each fixed x is quite standard. The crucial part of this section is to split the small term v into different parts.

PROPOSITION 2.1. *There are $\varepsilon_0 > 0$ and $\delta > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, there is a unique C^1-map $(\alpha_\varepsilon(k,x), v_{\varepsilon,k}(x)) : D_{k,\delta} \to R^k \times H_0^1(\Omega)$, satisfying $v_{\varepsilon,k}(x) \in E_{\varepsilon,x,k}$, (2.6) and (2.7). Besides, if $k \geq 2$ and $l = 1, \ldots, k$, then*

$$|\alpha_{\varepsilon,l} - 1| = O\bigg(\sum_{j=1}^{k} e^{-(1+\sigma)d(x_j,\partial\Omega)/\varepsilon} + \sum_{i\neq j} e^{-(1+\sigma)|x_i - x_j|/2\varepsilon}\bigg), \tag{2.8}$$

$$\|v_{\varepsilon,k}\|_\varepsilon = O\bigg(\varepsilon^{N/2}\bigg(\sum_{j=1}^{k} e^{-(1+\sigma)d(x_j,\partial\Omega)/\varepsilon} + \sum_{i\neq j} e^{-(1+\sigma)|x_i - x_j|/2\varepsilon}\bigg)\bigg), \tag{2.9}$$

and if $k = 1$, then

$$|\alpha_{\varepsilon,l} - 1|\varepsilon^{N/2} + \|v_{\varepsilon,1}\|_\varepsilon = O(\varepsilon^{N/2} e^{-(1+\sigma)d(x,\partial\Omega)/\varepsilon}), \tag{2.10}$$

where σ is some positive constant. Moreover, for $l = 1, \ldots, k$, $i = 1, \ldots, N$, we have

$$\varepsilon A_l, B_{li} = O\bigg(\varepsilon \sum_{j=1}^{k} e^{-(2-\theta)d(x_j,\partial\Omega)/\varepsilon} + \varepsilon \sum_{h\neq j} U\bigg(\frac{|x_h - x_j|}{\varepsilon}\bigg)\bigg). \tag{2.11}$$

PROOF. The proof of the existence part is standard (see [6], [15], and also [1], [30]). The estimates (2.8) and (2.9) or (2.10) follow from the same procedure as in Proposition 2.3 of [15] and Lemmas A.l and A.2. Finally, we can solve an appropriate system as in [30, pp. 22–23], to get (2.11). We thus omit the details. □

We also need the following pointwise estimate for $v_{\varepsilon,k}(x,y)$.

LEMMA 2.2. *Let $v_{\varepsilon,k}(x,y)$ be the map obtained in Proposition 2.1. For any small $\theta > 0$, there is a $\sigma > 0$ such that*

$$v_{\varepsilon,k}(x,y) = O\bigg(e^{-\theta\sigma/\varepsilon} \sum_{j=1}^{k} U^{1-\theta}\bigg(\frac{|y - x_j|}{\varepsilon}\bigg)\bigg).$$

Proof. For fixed x, let $w(y) = v_{\varepsilon,k}(x, \varepsilon y)$. By (2.7), we know that $v_{\varepsilon,k}(x,y)$ satisfies

$$\begin{aligned}\left\langle \frac{\partial J}{\partial v}, \varphi \right\rangle &= \sum_{j=1}^{k} A_j \langle P_{\Omega_\varepsilon} U_{\varepsilon,x_j}, \varphi\rangle + \sum_{j=1}^{k}\sum_{i=1}^{N} B_{ij} \left\langle \frac{\partial P_{\Omega_\varepsilon} U_{\varepsilon,x_j}}{\partial x_{ji}}, \varphi \right\rangle \\ &= \sum_{j=1}^{k} A_j \int_\Omega U_{\varepsilon,x_j}^{p-1}\varphi + \sum_{j=1}^{k}\sum_{i=1}^{N} (p-1) B_{ij} \int_\Omega U_{\varepsilon,x_j}^{p-2} \frac{\partial U_{\varepsilon,x_j}}{\partial x_{ji}}\varphi.\end{aligned}$$

Thus, w satisfies

$$\begin{aligned}-\Delta w + w &= \left(\sum_{j=1}^{k} \alpha_j P_{\Omega_\varepsilon} U_{x_j/\varepsilon} + w \right)^{p-1} - \sum_{j=1}^{k} \alpha_j U_{x_j/\varepsilon}^{p-1} \\ &\quad + \sum_{j=1}^{k} A_j U_{x_j/\varepsilon}^{p-1} + \sum_{j=1}^{k}\sum_{i=1}^{N}(p-1) B_{ij} U_{x_j/\varepsilon}^{p-2} \frac{\partial U_{x_j/\varepsilon}}{\partial x_{ij}} =: F(y),\end{aligned}$$

$y \in \Omega_\varepsilon =: \{y : \varepsilon y \in \Omega\}$; $w = 0$, $y \in \partial\Omega_\varepsilon$. Moreover, we have the following estimate

$$\|w\| = O\left(\sum_{j=1}^{k} e^{-(1+\sigma) d(x_j,\partial\Omega)/\varepsilon} + \sum_{i\neq j} e^{-(1+\sigma)|x_i - x_j|/2\varepsilon} \right),$$

and

$$|F(y)| \le C(1 + |w|^{p-2})|w| + |g(y)|,$$

where

$$\begin{aligned}g(y) = C\Bigg(&\sum_{j=1}^{k} |P_{\Omega_\varepsilon} U_{x_j/\varepsilon} - U_{x_j/\varepsilon}| + \sum_{j\neq i} U_{x_i/\varepsilon}^{(p-1)/2} U_{x_j/\varepsilon}^{(p-1)/2} \\ &+ \sum_{j=1}^{k} A_j U_{x_j/\varepsilon}^{p-1} + \sum_{j=1}^{k}\sum_{i=1}^{N}(p-1) B_{ij} U_{x_j/\varepsilon}^{p-2} \frac{\partial U_{x_j/\varepsilon}}{\partial x_{ij}} \Bigg).\end{aligned}$$

Choose $q > N/2$ satisfying $q(p-2) < 2N/(N-2)$. Thus, for any $z \in \Omega_\varepsilon$, we have

$$\int_{B_1(z)} (1 + |w|^{p-2})^q \le C',$$

and

$$\begin{aligned}|g|_{L^q(B_1(z))} &\le C \sum_{j=1}^{k} |P_{\Omega_\varepsilon} U_{x_j/\varepsilon} - U_{x_j/\varepsilon}|_{L^q(B_1(z))} \\ &\quad + C\left[\sum_{j=1}^{k} e^{-(2-\theta) d(x_j,\partial\Omega)/\varepsilon} + \sum_{h\neq j} U\left(\frac{|x_h - x_j|}{\varepsilon} \right) \right] \\ &\le C \sum_{j=1}^{k} e^{-d(x_j,\partial\Omega)/\varepsilon} + C \sum_{i\neq j} e^{-(1+\sigma)|x_i - x_j|/2\varepsilon}.\end{aligned}$$

So it follows from Theorem 8.17 in [20] that

$$\begin{aligned}|w(y)| &\le C|w|_{L^{2N/(N-2)}(B_1(z))} + C|g|_{L^q(B_1(z))}\\ &\le C\sum_{j=1}^{k} e^{-d(x_j,\partial\Omega)/\varepsilon} + C\sum_{i\neq j} e^{-(1+\sigma)|x_i-x_j|/2\varepsilon}, \quad \text{for all } y \in B_{1/2}(z).\end{aligned}$$

Especially, we see

$$|v_{\varepsilon,k}(x,y)| = O\bigg(e^{-\theta l/\varepsilon}\sum_{j=1}^{k} U^{1-\theta}\bigg(\frac{|y-x_j|}{\varepsilon}\bigg)\bigg), \quad y \in \bigcup_{j=1}^{k} B_l(x_j),$$

where $l = \min(d(x_j,\partial\Omega)/4, |x_i - x_j|/8)$.

On the other hand, for $\theta > 0$ small, and $y \in \Omega \setminus \bigcup_{j=1}^{k} B_l(x_j)$, we have $|P_{\varepsilon,\Omega}U_{\varepsilon,x_j}(y)| \le Ce^{-|y-x_j|/\varepsilon} = o(1)$. As a result,

$$F\bigg(\frac{y}{\varepsilon}\bigg) = o(1)v_{\varepsilon,k} + O\bigg(\sum_{j=1}^{k} e^{-(p-1)|y-x_j|/\varepsilon}\bigg),$$

where $o(1) \to 0$ as $\varepsilon \to 0$.

Let $\eta = \sum_{j=1}^{k} e^{-(1-\theta/2)|y-x_j|/\varepsilon}$. By direct calculation, we see $-\varepsilon^2\Delta\eta + \eta \ge c'\eta$, where $c' > 0$ is a constant depending on θ. Let $\omega = \eta \pm v_{\varepsilon,k}$. Since

$$-\varepsilon^2\Delta v_{\varepsilon,k} + v_{\varepsilon,k} = F\bigg(\frac{y}{\varepsilon}\bigg) = o(1)v_{\varepsilon,k} + O\bigg(\sum_{j=1}^{k} e^{-(p-1)|y-x_j|/\varepsilon}\bigg)$$

for any $y \in \Omega \setminus \bigcup_{j=1}^{k} B_l(x_j)$, we have

$$\begin{aligned}-\varepsilon^2\Delta\omega + (1-o(1))\omega &= -\varepsilon^2\Delta\eta + (1-o(1))\eta - O\bigg(\sum_{j=1}^{k} e^{-(p-1)|y-x_j|/\varepsilon}\bigg)\\ &\ge c_0\sum_{j=1}^{k} e^{-(1-\theta/2)|y-x_j|/\varepsilon} - O\bigg(\sum_{j=1}^{k} e^{-(p-1)|y-x_j|/\varepsilon}\bigg) > 0,\end{aligned}$$

where $c_0 > 0$ is a constant depending on θ. Moreover, for any $y \in \partial\Omega$ or $y \in \partial B_l(x_j)$, $|v_{\varepsilon,k}(y)| \le \eta(y)$. So it follows from the comparison theorem that

$$|v_{\varepsilon,k}(x,y)| \le \eta(y), \quad y \in \Omega \setminus \bigcup_{j=1}^{k} B_l(x_j),$$

and hence the result. □

For each $k \ge 2$ and $x = (x_1,\ldots,x_k)$, by Proposition 2.1, we can determine a map $(\alpha_\varepsilon(k,x), v_{\varepsilon,k}(x))$ satisfying (2.6) and (2.7). Let $1 \le m \le k$, $x' = (x_1,\ldots,x_m)$ and $x'' = (x_{m+1},\ldots,x_k)$. Our next result shows that as $\min_{1\le i\le m<m+1\le j\le k}|x_i - x_j|$ becomes large, $(\alpha_\varepsilon(k,x), v_{\varepsilon,k}(x))$ will eventually split into two parts. To be more precisely, we have

Proposition 2.3. *For each $k \geq 2$ and $x = (x_1, \dots, x_k)$, the map*

$$(\alpha_\varepsilon(k,x), v_{\varepsilon,k}(x))$$

obtained in Proposition 2.1 *satisfies*

$$\sum_{j=1}^{m} |\alpha_{\varepsilon,j}(k,x) - \alpha_{\varepsilon,j}(m,x')| + \sum_{j=m+1}^{k} |\alpha_{\varepsilon,j}(k,x) - \alpha_{\varepsilon,j}(k-m,x'')| = O(U^{(1+\sigma)/2}(\eta/\varepsilon)),$$

$$\|v_{\varepsilon,k}(x) - v_{\varepsilon,m}(x') - v_{\varepsilon,k-m}(x'')\|_\varepsilon = O(\varepsilon^{N/2} U^{(1+\sigma)/2}(\eta/\varepsilon)),$$

where $\eta = \min_{1 \leq i \leq m < m+1 \leq j \leq k} |x_i - x_j|$.

Proof. By (2.6), for any $1 \leq i \leq m$, we have

$$\left\langle \sum_{j=1}^{k} \alpha_{\varepsilon,j}(k,x) P_{\varepsilon,\Omega} U_{\varepsilon,x_j}, P_{\varepsilon,\Omega} U_{\varepsilon,x_i} \right\rangle_\varepsilon - \int_\Omega \left(\sum_{j=1}^{k} \alpha_{\varepsilon,j}(k,x) P_{\varepsilon,\Omega} U_{\varepsilon,x_j} + v_{\varepsilon,k}(x) \right)^{p-1} P_{\varepsilon,\Omega} U_{\varepsilon,x_i} = 0 \tag{2.12}$$

and

$$\left\langle \sum_{j=1}^{m} \alpha_{\varepsilon,j}(m,x') P_{\varepsilon,\Omega} U_{\varepsilon,x_j}, P_{\varepsilon,\Omega} U_{\varepsilon,x_i} \right\rangle_\varepsilon - \int_\Omega \left(\sum_{j=1}^{m} \alpha_{\varepsilon,j}(m,x') P_{\varepsilon,\Omega} U_{\varepsilon,x_j} + v_{\varepsilon,m}(x') \right)^{p-1} P_{\varepsilon,\Omega} U_{\varepsilon,x_i} = 0. \tag{2.13}$$

Let

$$G(x) = \sum_{j=1}^{m} \alpha_{\varepsilon,j}(m,x') P_{\varepsilon,\Omega} U_{\varepsilon,x_j} + v_{\varepsilon,m}(x') + v_{\varepsilon,k-m}(x'').$$

Since

$$\langle P_{\varepsilon,\Omega} U_{\varepsilon,x_j}, P_{\varepsilon,\Omega} U_{\varepsilon,x_i} \rangle_\varepsilon = O(\varepsilon^N U(\eta/\varepsilon))$$

and (by Lemma 2.2)

$$\int_\Omega v_{\varepsilon,k-m}^{p-1}(x'') P_{\varepsilon,\Omega} U_{\varepsilon,x_i} = O(\varepsilon^N U(\eta/\varepsilon))$$

for $j > m$ and $1 \leq i \leq m$, we have

$$\begin{aligned} &\left\langle \sum_{j=1}^{m} (\alpha_{\varepsilon,j}(k,x) - \alpha_{\varepsilon,j}(m,x')) P_{\varepsilon,\Omega} U_{\varepsilon,x_j}, P_{\varepsilon,\Omega} U_{\varepsilon,x_i} \right\rangle_\varepsilon + O(\varepsilon^N U(\eta/\varepsilon)) \\ &\quad = \int_\Omega \left(\sum_{j=1}^{k} \alpha_{\varepsilon,j}(k,x) P_{\varepsilon,\Omega} U_{\varepsilon,x_j} + v_{\varepsilon,k}(x) \right)^{p-1} P_{\varepsilon,\Omega} U_{\varepsilon,x_i} \\ &\qquad - \int_\Omega \left(\sum_{j=1}^{m} \alpha_{\varepsilon,j}(m,x') P_{\varepsilon,\Omega} U_{\varepsilon,x_j} + v_{\varepsilon,m}(x') \right)^{p-1} P_{\varepsilon,\Omega} U_{\varepsilon,x_i} \end{aligned} \tag{2.14}$$

$$
\begin{aligned}
&= \int_\Omega \bigg(\sum_{j=1}^m \alpha_{\varepsilon,j}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + v_{\varepsilon,k}(x)\bigg)^{p-1} P_{\varepsilon,\Omega}U_{\varepsilon,x_i} \\
&\quad - \int_\Omega G^{p-1}(x)P_{\varepsilon,\Omega}U_{\varepsilon,x_i} + O(\varepsilon^N U(\eta/\varepsilon)) \\
&= (p-1)\int_\Omega G^{p-2}(x)\sum_{j=1}^m(\alpha_{\varepsilon,j}(k,x)-\alpha_{\varepsilon,j}(m,x'))P_{\varepsilon,\Omega}U_{\varepsilon,x_j}P_{\varepsilon,\Omega}U_{\varepsilon,x_i} \\
&\quad + (p-1)\int_\Omega G^{p-2}(x)(v_{\varepsilon,k}(x)-v_{\varepsilon,m}(x')-v_{\varepsilon,k-m}(x''))P_{\varepsilon,\Omega}U_{\varepsilon,x_i} \\
&\quad + O\bigg(\varepsilon^N \sum_{j=1}^m |\alpha_{\varepsilon,j}(k,x)-\alpha_{\varepsilon,j}(m,x')|^{1+\sigma}\bigg) \\
&\quad + O(\|v_{\varepsilon,k}(x)-v_{\varepsilon,m}(x')-v_{\varepsilon,k-m}(x'')\|_\varepsilon^{1+\sigma}) + O(\varepsilon^N U(\eta/\varepsilon)),
\end{aligned}
$$

for some $\sigma > 0$. Noting that

$$
\|P_{\varepsilon,\Omega}U_{\varepsilon,x_i}\|_\varepsilon^2 - (p-1)\int_\Omega (P_{\varepsilon,\Omega}U_{\varepsilon,x_i})^p = -(p-2)\varepsilon^N(A+o(1)),
$$

$$
\langle P_{\varepsilon,\Omega}U_{\varepsilon,x_j}, P_{\varepsilon,\Omega}U_{\varepsilon,x_i}\rangle_\varepsilon = o(1)
$$

and

$$
\int_\Omega G^{p-2}(x)P_{\varepsilon,\Omega}U_{\varepsilon,x_i}P_{\varepsilon,\Omega}U_{\varepsilon,x_j} = o(1) \quad \text{for } i \neq j,
$$

where $o(1) \to 0$ as $\varepsilon \to 0$, we can solve (2.14) to get

$$
\begin{aligned}
\text{(2.15)}\quad &|\alpha_{\varepsilon,j}(k,x)-\alpha_{\varepsilon,j}(m,x')| \\
&\qquad = O(U(\eta/\varepsilon) + \varepsilon^{-N/2}\|v_{\varepsilon,k}(x)-v_{\varepsilon,m}(x')-v_{\varepsilon,k-m}(x'')\|_\varepsilon).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\text{(2.16)}\quad &|\alpha_{\varepsilon,j}(k,x)-\alpha_{\varepsilon,j}(k-m,x'')| \\
&\qquad = O(U(\eta/\varepsilon) + \varepsilon^{-N/2}\|v_{\varepsilon,k}(x)-v_{\varepsilon,m}(x')-v_{\varepsilon,k-m}(x'')\|_\varepsilon).
\end{aligned}
$$

On the other hand, by (2.7), we have

$$
\begin{aligned}
\text{(2.17)}\quad &\bigg\langle \sum_{j=1}^k \alpha_{\varepsilon,j}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + v_{\varepsilon,k}(x), \phi\bigg\rangle_\varepsilon \\
&\quad - \int_\Omega \bigg(\sum_{j=1}^k \alpha_{\varepsilon,j}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + v_{\varepsilon,k}(x)\bigg)^{p-1}\phi = 0, \quad \phi \in E_{\varepsilon,x,k},
\end{aligned}
$$

$$
\begin{aligned}
\text{(2.18)}\quad &\bigg\langle \sum_{j=1}^m \alpha_{\varepsilon,j}(m,x')P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + v_{\varepsilon,m}(x'), \phi\bigg\rangle_\varepsilon \\
&\qquad - \int_\Omega \bigg(\sum_{j=1}^m \alpha_{\varepsilon,j}(m,x')P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + v_{\varepsilon,m}(x')\bigg)^{p-1}\phi = 0,
\end{aligned}
$$

for $\phi \in E_{\varepsilon,x',m}$, and

$$\text{(2.19)}\quad \Big\langle \sum_{j=m+1}^{k} \alpha_{\varepsilon,j}(k-m,x'')P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + v_{\varepsilon,k-m}(x''), \phi \Big\rangle_{\varepsilon} - \int_{\Omega}\Big(\sum_{j=m+1}^{k} \alpha_{\varepsilon,j}(k-m,x'')P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + v_{\varepsilon,k-m}(x'')\Big)^{p-1}\phi = 0,$$

for $\phi \in E_{\varepsilon,x'',k-m}$. Thus,

$$\text{(2.20)}\quad \begin{aligned}&\langle v_{\varepsilon,k}(x) - v_{\varepsilon,m}(x') - v_{\varepsilon,k-m}(x''), \phi\rangle_{\varepsilon}\\ &= \int_{\Omega}\Big(\sum_{j=1}^{k} \alpha_{\varepsilon,j}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + v_{\varepsilon,k}(x)\Big)^{p-1}\phi\\ &\quad - \int_{\Omega}\Big(\sum_{j=1}^{m} \alpha_{\varepsilon,j}(m,x')P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + v_{\varepsilon,m}(x')\Big)^{p-1}\phi\\ &\quad - \int_{\Omega}\Big(\sum_{j=m+1}^{k} \alpha_{\varepsilon,j}(k-m,x'')P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + v_{\varepsilon,k-m}(x'')\Big)^{p-1}\phi,\end{aligned}$$

for all $\phi \in E_{\varepsilon,x,k}$. Let

$$\text{(2.21)}\quad \begin{aligned}G_1(x) &= \sum_{j=1}^{m} \alpha_{\varepsilon,j}(m,x')P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + v_{\varepsilon,m}(x')\\ &\quad + \sum_{j=m+1}^{k} \alpha_{\varepsilon,j}(k-m,x'')P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + v_{\varepsilon,k-m}(x'').\end{aligned}$$

Then, by Lemma 2.2,

$$\text{(2.22)}\quad \begin{aligned}\int_{\Omega} G_1^{p-1}(x)\phi &= \int_{\Omega}\Big(\sum_{j=1}^{m} \alpha_{\varepsilon,j}(m,x')P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + v_{\varepsilon,m}(x')\Big)^{p-1}\phi\\ &\quad + \int_{\Omega}\Big(\sum_{j=m+1}^{k} \alpha_{\varepsilon,j}(k-m,x'')PU_{\varepsilon,x_j} + v_{\varepsilon,k-m}(x'')\Big)^{p-1}\phi\\ &\quad + O(U^{(p-1)/2}(\eta/\varepsilon))\varepsilon^{N/2}\|\phi\|_{\varepsilon}.\end{aligned}$$

Combining (2.20) and (2.22), we obtain

$$\text{(2.23)}\quad \begin{aligned}&\langle v_{\varepsilon,k}(x) - v_{\varepsilon,m}(x') - v_{\varepsilon,k-m}(x''), \phi\rangle_{\varepsilon}\\ &= \int_{\Omega}\Big(\Big(\sum_{j=1}^{k} \alpha_{\varepsilon,j}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_j} + v_{\varepsilon,k}(x)\Big)^{p-1} - G_1^{p-1}(x)\Big)\phi\\ &\quad + O(U^{(p-1)/2}(\eta/\varepsilon))\varepsilon^{N/2}\|\phi\|_{\varepsilon}\\ &= (p-1)\int_{\Omega} G_1^{p-2}(x)(v_{\varepsilon,k}(x) - v_{\varepsilon,m}(x') - v_{\varepsilon,k-m}(x''))\phi\end{aligned}$$

$$
\begin{aligned}
&+(p-1)\int_\Omega G_1^{p-2}(x)\sum_{j=1}^{m}(\alpha_{\varepsilon,j}(k,x)-\alpha_{\varepsilon,j}(m,x'))P_{\varepsilon,\Omega}U_{\varepsilon,x_j}\phi\\
&+(p-1)\int_\Omega G_1^{p-2}(x)\sum_{j=m+1}^{k}(\alpha_{\varepsilon,j}(k,x)-\alpha_{\varepsilon,j}(k-m,x''))P_{\varepsilon,\Omega}U_{\varepsilon,x_j}\phi\\
&+\varepsilon^{N/2}O\bigg(\sum_{j=1}^{m}|\alpha_{\varepsilon,j}(k,x)-\alpha_{\varepsilon,j}(m,x')|^{1+\sigma}\\
&+\sum_{j=m+1}^{k}|\alpha_{\varepsilon,j}(k,x)-\alpha_{\varepsilon,j}(m-k,x'')|^{1+\sigma}\bigg)\|\phi\|_\varepsilon\\
&+O(\|v_{\varepsilon,k}(x)-v_{\varepsilon,m}(x')-v_{\varepsilon,k-m}(x'')\|_\varepsilon^{1+\sigma})\|\phi\|_\varepsilon\\
&+O(U^{(p-1)/2}(\eta/\varepsilon))\varepsilon^{N/2}\|\phi\|_\varepsilon.
\end{aligned}
$$

Since for any $\phi\in E_{\varepsilon,x,k}$, we have

$$
\begin{aligned}
\int_\Omega G_1^{p-2}(x)P_{\varepsilon,\Omega}U_{\varepsilon,x_j}\phi &= \int_\Omega (G_1^{p-2}(x)-(P_{\varepsilon,\Omega}U_{\varepsilon,x_j})^{p-2})P_{\varepsilon,\Omega}U_{\varepsilon,x_j}\phi\\
&\quad+\int((P_{\varepsilon,\Omega}U_{\varepsilon,x_j})^{p-1}-U_{\varepsilon,x_j}^{p-1})\phi=o(1)\varepsilon^{N/2}\|\phi\|_\varepsilon,
\end{aligned}\tag{2.24}
$$

for $j=1,\dots,k$. Consequently, from (2.23), (2.15), (2.16) and (2.24), we obtain

$$
\begin{aligned}
\langle v_{\varepsilon,k}(x)&-v_{\varepsilon,m}(x')-v_{\varepsilon,k-m}(x''),\phi\rangle_\varepsilon\\
&=(p-1)\int_\Omega G_1^{p-2}(x)(v_{\varepsilon,k}(x)-v_{\varepsilon,m}(x')-v_{\varepsilon,k-m}(x''))\phi\\
&\quad+o(\|v_{\varepsilon,k}(x)-v_{\varepsilon,m}(x')-v_{\varepsilon,k-m}(x''))\|_\varepsilon\|\phi\|_\varepsilon)\\
&\quad+O(\varepsilon^{N/2}U^{(p-1)/2}(\eta/\varepsilon)\|\phi\|_\varepsilon),\quad\text{for all }\phi\in E_{\varepsilon,x,k}.
\end{aligned}\tag{2.25}
$$

Choose β_j and γ_{ij} such that

$$
\begin{aligned}
\phi&=v_{\varepsilon,k}(x)-v_{\varepsilon,m}(x')-v_{\varepsilon,k-m}(x'')\\
&\quad+\sum_{j=1}^{k}\beta_jP_{\varepsilon,\Omega}U_{\varepsilon,x_j}+\sum_{j=1}^{k}\sum_{i=1}^{N}\gamma_{ij}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_j}}{\partial x_{ij}}\in E_{\varepsilon,x,k}.
\end{aligned}
$$

Noting that

$$
\langle v_{\varepsilon,m}(x'),P_{\varepsilon,\Omega}U_{\varepsilon,x_j}\rangle,\bigg\langle v_{\varepsilon,m}(x'),\frac{\partial P_{\varepsilon,\Omega U_{\varepsilon,x_j}}}{\partial x_{ij}}\bigg\rangle=O(U(\eta/\varepsilon))
$$

for $j\ge m+1$, we obtain

$$
\beta_j,\gamma_{ij}=O(U(\eta/\varepsilon)).
$$

So we find from (2.25) that

$$
\|\phi\|_\varepsilon^2\le CU^{p-1}(\eta/\varepsilon)
$$

and the result follows. $\square$

As a direct consequence of Proposition (2.3), we have the following expansion:

PROPOSITION 2.4. *Let $(\alpha_\varepsilon(k,x), v_{\varepsilon,k}(x))$ be the map obtained in Proposition 2.3. Then for each fixed i, we have*

$$\begin{aligned}\frac{\partial J}{\partial x_{il}} =&(p-1)\int_\Omega U^{p-2}_{\varepsilon,x_i}\frac{\partial U_{\varepsilon,x_i}}{\partial x_{il}}\varphi_{\varepsilon,x_i}\\ &-(p-1)\sum_{j\neq i}\int_\Omega U^{p-2}_{\varepsilon,x_i}P_{\varepsilon,\Omega}U_{\varepsilon,x_j}\frac{\partial U_{\varepsilon,x_i}}{\partial x_{il}}\\ &+O(\varepsilon^{N-1}e^{-(2+\sigma)d(x_i,\partial\Omega)/\varepsilon})+O\biggl(\varepsilon^{N-1}\sum_{j\neq i}U^{1+\sigma}\Bigl(\frac{|x_i-x_j|}{\varepsilon}\Bigr)\\ &+\varepsilon^{N-1}\sum_{j\neq i}e^{-\sigma d(x_j,\partial\Omega)/\varepsilon}U\Bigl(\frac{|x_i-x_j|}{\varepsilon}\Bigr)\biggr).\end{aligned}\tag{2.26}$$

$$\Bigl\langle\frac{\partial J}{\partial v},PU_{\varepsilon,x_i}\Bigr\rangle=O\biggl(\varepsilon^Ne^{-(2-\theta)d(x_i,\partial\Omega)/\varepsilon}+\varepsilon^N\sum_{j\neq i}U\Bigl(\frac{|x_i-x_j|}{\varepsilon}\Bigr)\biggr).\tag{2.27}$$

PROOF. We assume $i=1$. Let $H(y)=\sum_{j=1}^k\alpha_{\varepsilon,j}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_j}$. Then

$$\begin{aligned}\frac{\partial J}{\partial x_{1l}} =&\int_\Omega\sum_{j=1}^k\alpha_{\varepsilon,j}(k,x)U^{p-1}_{\varepsilon,x_j}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}}\\ &-\int_\Omega\Bigl(H(y)+v_{\varepsilon,k}(x)\Bigr)^{p-1}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}}\\ =&-\int_\Omega\Bigl[(H(y)+v_{\varepsilon,k}(x))^{p-1}-H^{p-1}(y)\\ &-(p-1)H^{p-2}(y)v_{\varepsilon,k}(x)\Bigr]\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}}\\ &-(p-1)\int_\Omega\Bigl(H^{p-2}(y)-(\alpha_{\varepsilon,1}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_1})^{p-2}\Bigr)\\ &\cdot v_{\varepsilon,k}(x)\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}}\\ &-(p-1)\int_\Omega\Bigl(\alpha_{\varepsilon,1}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_1}\Bigr)^{p-2}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}}v_{\varepsilon,k}(x)\\ &-\int_\Omega\Bigl(H^{p-1}(y)-\sum_{j=1}^k\alpha_{\varepsilon,j}(k,x)U^{p-1}_{\varepsilon,x_j}\Bigr)\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}}\\ =:&\,I_1+I_2+I_3+I_4,\end{aligned}\tag{2.28}$$

where I_i, $i=1,2,3,4$, is the natural splitting of the last formula. Denote $\eta=\min_{j\ge2}|x_j-x_1|$. We have

$$I_1 = -\int_{\Omega\cap B_{\eta/2}(x_1)} \bigg[(H(y)+v_{\varepsilon,k}(x))^{p-1} - H^{p-1}(y) \tag{2.29}$$
$$- (p-1)H^{p-2}(y)v_{\varepsilon,k}(x)\bigg] \frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}}$$
$$- \int_{\Omega\setminus B_{\eta/2}(x_1)} \bigg[(H(y)+v_{\varepsilon,k}(x))^{p-1} - H^{p-1}(y)$$
$$- (p-1)H^{p-2}(y)v_{\varepsilon,k}(x)\bigg] \frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}} =: I_{11} + I_{12}.$$

By Lemmas A.2, 2.2 and Proposition 2.3, we have

$$\begin{aligned} |I_{11}| &\le C\varepsilon^{-1}\int_{\Omega\cap B_{\eta/2}(x_1)} |v_{\varepsilon,k}(x)|^2 U_{\varepsilon,x_1}^{p-2} \\ &\le C\varepsilon^{-1}\int_{\Omega\cap B_{\eta/2}(x_1)} |v_{\varepsilon,1}(x_1)|^2 U_{\varepsilon,x_1}^{p-2} \\ &\quad + C\varepsilon^{-1}\int_{\Omega\cap B_{\eta/2}(x_1)} |v_{\varepsilon,k-1}(x'')|^2 U_{\varepsilon,x_1}^{p-2} \\ &\quad + C\varepsilon^{-1}\int_{\Omega\cap B_{\eta/2}(x_1)} |v_{\varepsilon,k}(x) - v_{\varepsilon,1}(x_1) - v_{\varepsilon,k-1}(x'')|^2 U_{\varepsilon,x_1}^{p-2} \\ &\le C\varepsilon^{N-1}e^{-(2+\sigma)d(x_1,\partial\Omega)/\varepsilon} + C\varepsilon^{N-1}U^{1+\sigma}(\eta/\varepsilon). \end{aligned} \tag{2.30}$$

Also, by Lemma 2.2 and Proposition 2.3, we have

$$\begin{aligned} |I_{12}| &\le \varepsilon^{-1}\int_{\Omega\setminus B_{\eta/2}(x_1)} |v_{\varepsilon,k}(x)|^{p-1} U_{\varepsilon,x_1} \\ &\le C\varepsilon^{-1}\int_{\Omega\setminus B_{\eta/2}(x_1)} |v_{\varepsilon,1}(x_1)|^{p-1} U_{\varepsilon,x_1} \\ &\quad + C\varepsilon^{-1}\int_{\Omega\setminus B_{\eta/2}(x_1)} |v_{\varepsilon,k-1}(x'')|^{p-1} U_{\varepsilon,x_1} \\ &\quad + C\varepsilon^{-1}\int_{\Omega\setminus B_{\eta/2}(x_1)} |v_{\varepsilon,k}(x) - v_{\varepsilon,1}(x_1) - v_{\varepsilon,k-1}(x'')|^{p-1} U_{\varepsilon,x_1} \\ &\le C\varepsilon^{N-1}U^{1+\sigma}(\eta/\varepsilon). \end{aligned} \tag{2.31}$$

Combining (2.29), (2.30) and (2.31), we obtain

$$|I_1| \le C\varepsilon^{N-1}e^{-(2+\sigma)d(x_1,\partial\Omega)/\varepsilon} + C\varepsilon^{N-1}\sum_{j=2}^{k} U^{1+\sigma}(\eta/\varepsilon). \tag{2.32}$$

Now we estimate I_2.

$$\begin{aligned} |I_2| \le C\varepsilon^{-1}\int_{\Omega\cap B_{\eta/2}(x_1)} &\Big|H^{p-2}(y) - (\alpha_{\varepsilon,1}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_1})^{p-2}\Big| \\ &\cdot |v_{\varepsilon,k}(x)| P_{\varepsilon,\Omega}U_{\varepsilon,x_1} \end{aligned} \tag{2.33}$$

$$
\begin{aligned}
&\quad + C\varepsilon^{-1}\int_{\Omega\setminus B_{\eta/2}(x_1)}\left|H^{p-2}(y)-(\alpha_{\varepsilon,1}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_1})^{p-2}\right|\\
&\quad \cdot|v_{\varepsilon,k}(x)|P_{\varepsilon,\Omega}U_{\varepsilon,x_1}\\
&\le C\varepsilon^{-1}\int_{\Omega\cap B_{\eta/2}(x_1)}\sum_{j=2}^{k}U_{\varepsilon,x_1}^{p-2}U_{\varepsilon,x_j}|v_{\varepsilon,k}(x)|\\
&\quad + C\varepsilon^{-1}\int_{\Omega\setminus B_{\eta/2}(x_1)}\left(\sum_{j=2}^{k}U_{\varepsilon,x_j}\right)^{p-2}|v_{\varepsilon,k}(x)|U_{\varepsilon,x_1} =: I_{21}+I_{22}.
\end{aligned}
$$

But

$$
\begin{aligned}
I_{21} &\le C\varepsilon^{-1}\int_{\Omega\cap B_{\eta/2}(x_1)}\sum_{j=2}^{k}U_{\varepsilon,x_1}^{p-2}U_{\varepsilon,x_j}|v_{\varepsilon,k}(x)-v_{\varepsilon,1}(x_1)-v_{\varepsilon,k-1}(x'')|\\
&\quad + C\varepsilon^{-1}\int_{\Omega\cap B_{\eta/2}(x_1)}\sum_{j=2}^{k}U_{\varepsilon,x_1}^{p-2}U_{\varepsilon,x_j}|v_{\varepsilon,1}(x_1)|\\
&\quad + C\varepsilon^{-1}\int_{\Omega\cap B_{\eta/2}(x_1)}\sum_{j=2}^{k}U_{\varepsilon,x_1}^{p-2}U_{\varepsilon,x_j}|v_{\varepsilon,k-1}(x'')|\\
&\le C\varepsilon^{-1+N/2}\|v_{\varepsilon,k}(x)-v_{\varepsilon,1}(x_1)-v_{\varepsilon,k-1}(x'')\|_{\varepsilon}e^{-\eta/2\varepsilon}\\
&\quad + |v_{\varepsilon,1}(x_1)|_{\infty}^{\sigma}\int_{\Omega\cap B_{\eta/2}(x_1)}\sum_{j=2}^{k}U_{\varepsilon,x_1}^{p-1-\sigma}U_{\varepsilon,x_j} + C\varepsilon^{N-1}U^{1+\sigma}(\eta/\varepsilon)\\
&= O(\varepsilon^{N-1}e^{-\sigma/\varepsilon}U(\eta/\varepsilon)).
\end{aligned}
\tag{2.34}
$$

Similarly,

$$
\begin{aligned}
I_{22} &\le C\varepsilon^{-1}\int_{\Omega\setminus B_{\eta/2}(x_1)}\left(\sum_{j=2}^{k}U_{\varepsilon,x_j}\right)^{p-2}\\
&\quad \cdot|v_{\varepsilon,k}(x)-v_{\varepsilon,1}(x_1)-v_{\varepsilon,k-1}(x'')|U_{\varepsilon,x_1}\\
&\quad + C\varepsilon^{-1}\int_{\Omega\setminus B_{\eta/2}(x_1)}\left(\sum_{j=2}^{k}U_{\varepsilon,x_j}\right)^{p-2}|v_{\varepsilon,1}(x_1)|U_{\varepsilon,x_1}\\
&\quad + C\varepsilon^{-1}\int_{\Omega\setminus B_{\eta/2}(x_1)}\left(\sum_{j=2}^{k}U_{\varepsilon,x_j}\right)^{p-2}|v_{\varepsilon,k-1}(x'')|U_{\varepsilon,x_1}\\
&= O(\varepsilon^{N-1}e^{-\sigma/\varepsilon}U(\eta/\varepsilon)).
\end{aligned}
\tag{2.35}
$$

Combining (2.33), (2.34) and (2.35), we obtain

$$
I_2 = O(\varepsilon^{N-1}e^{-\sigma/\varepsilon}U(\eta/\varepsilon)). \tag{2.36}
$$

As for the estimate of I_3, by Proposition 2.1, Lemma 2.2 and Proposition 2.3, we have

$$
\begin{aligned}
(2.37)\qquad I_3 = &-(p-1)\alpha_{\varepsilon,1}(k,x)\int_\Omega (P_{\varepsilon,\Omega}U_{\varepsilon,x_1})^{p-2}\frac{\partial\varphi_{\varepsilon,1}}{\partial x_{1l}}v_{\varepsilon,k}(x)\\
&-(p-1)\alpha_{\varepsilon,1}(k,x)\int_\Omega \Big((P_{\varepsilon,\Omega}U_{\varepsilon,x_1})^{p-2}-U^{p-2}_{\varepsilon,x_1}\Big)\frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}}v_{\varepsilon,k}\\
=&\,O\Big(\varepsilon^{-1}\int_\Omega U^{p-2}_{\varepsilon,x_1}|\varphi_{\varepsilon,x_1}||v_{\varepsilon,k}(x)|\Big)\\
=&\,O\Big(\varepsilon^{-1}\int_\Omega U^{p-2}_{\varepsilon,x_1}|\varphi_{\varepsilon,x_1}||v_{\varepsilon,1}(x_1)|+\varepsilon^{-1}\int_\Omega U^{p-2}_{\varepsilon,x_1}|\varphi_{\varepsilon,x_1}||v_{\varepsilon,k-1}(x'')|\\
&+\varepsilon^{-1}\int_\Omega U^{p-2}_{\varepsilon,x_1}|\varphi_{\varepsilon,x_1}||v_{\varepsilon,k}(x)-v_{\varepsilon,1}(x_1)-v_{\varepsilon,k-1}(x'')|\Big)\\
=&\,O(\varepsilon^{N-1}e^{-(2+\sigma)d(x_1,\partial\Omega)/\varepsilon})+O\Big(\varepsilon^{-1}|\varphi_{\varepsilon,x_1}|^\sigma_\infty\sum_{j=2}^k\int_\Omega U^{p-1-\sigma}_{\varepsilon,x_1}U_{\varepsilon,x_j}\Big)\\
&+O\Big(\varepsilon^{-1}\int_\Omega U^{p(p-2)/(p-1)}_{\varepsilon,x_1}|\varphi_{\varepsilon,x_1}|^{p/(p-1)}\Big)\\
&+O\Big(\varepsilon^{-1}\int_\Omega |v_{\varepsilon,k}(x)-v_{\varepsilon,1}(x_1)-v_{\varepsilon,k-1}(x'')|^p\Big)\\
=&\,O(\varepsilon^{N-1}e^{-(2+\sigma)d(x_1,\partial\Omega)/\varepsilon})+O(e^{-\sigma/\varepsilon}\varepsilon^{N-1}U(\eta/\varepsilon)).
\end{aligned}
$$

Finally, we estimate I_4.

$$
\begin{aligned}
(2.38)\qquad I_4 = &\int_\Omega \bigg[H^{p-1}(y)-\bigg(\sum_{j=2}^k\alpha_{\varepsilon,j}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_j}\bigg)^{p-1}\\
&-\alpha_{\varepsilon,1}(k,x)U^{p-1}_{\varepsilon,x_1}\bigg]\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}}\\
&+\int_\Omega\bigg[\bigg(\sum_{j=2}^k\alpha_{\varepsilon,j}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_j}\bigg)^{p-1}\\
&-\sum_{j=2}^k\alpha_{\varepsilon,j}(k,x)U^{p-1}_{\varepsilon,x_j}\bigg]\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}}=:I_{41}+I_{42}.
\end{aligned}
$$

We have

$$
\begin{aligned}
(2.39)\qquad I_{41} = &\int_{\Omega\cap B_{\eta/2(x_1)}}\bigg(H^{p-1}(y)-\bigg(\sum_{j=2}^k\alpha_{\varepsilon,j}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_j}\bigg)^{p-1}\\
&-\alpha_{\varepsilon,1}(k,x)U^{p-1}_{\varepsilon,x_1}\bigg)\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}}\\
&+\int_{\Omega\setminus B_{\eta/2(x_1)}}\bigg(H^{p-1}(y)-\bigg(\sum_{j=2}^k\alpha_{\varepsilon,j}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_j}\bigg)^{p-1}\\
&-\alpha_{\varepsilon,1}(k,x)U^{p-1}_{\varepsilon,x_1}\bigg)\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}}=:I_{43}+I_{44}.
\end{aligned}
$$

But

$$I_{44} = O\Big(\int_{\Omega\setminus B_{\eta/2(x_1)}} \Big[\Big(\sum_{j=2}^{k}\alpha_{\varepsilon,j}(k,x)PU_{\varepsilon,x_j}\Big)^{p-2}P_{\varepsilon,\Omega}U_{\varepsilon,x_1}+U_{\varepsilon,x_1}^{p-1}\Big] \cdot\Big|\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}}\Big|\Big) = O(\varepsilon^{N-1}U^{1+\sigma}(\eta/\varepsilon)). \tag{2.40}$$

By Lemma A.6, we see

$$\begin{aligned} I_{43} =& \int_{\Omega\cap B_{\eta/2(x_1)}}(H^{p-1}(y)-\alpha_{\varepsilon,1}(k,x)U_{\varepsilon,x_1}^{p-1})\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}} \\ &+O(\varepsilon^{N-1}U^{1+\sigma}(\eta/\varepsilon)) \\ =&\int_{\Omega\cap B_{\eta/2(x_1)}}\Big[\Big(\alpha_{\varepsilon,1}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_1}\Big)^{p-1}-\alpha_{\varepsilon,1}(k,x)U_{\varepsilon,x_1}^{p-1}\Big]\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}} \\ &+\int_{\Omega\cap B_{\eta/2(x_1)}}(\alpha_{\varepsilon,1}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_1})^{p-2} \\ &\cdot\sum_{j=2}^{k}\alpha_{\varepsilon,j}(k,x)P_{\varepsilon,\Omega}U_{\varepsilon,x_j}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}}+O(\varepsilon^{N-1}U^{1+\sigma}(\eta/\varepsilon)) \\ =&-(p-1)\int_\Omega U_{\varepsilon,x_1}^{p-2}\varphi_{\varepsilon,x_1}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}} \\ &+\int_\Omega(P_{\varepsilon,\Omega}U_{\varepsilon,x_1})^{p-2}\sum_{j=2}^{k}P_{\varepsilon,\Omega}U_{\varepsilon,x_j}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1l}} \\ &+O(\varepsilon^{N-1}e^{-(2+\sigma)d(x_1,\partial\Omega)/\varepsilon}+\varepsilon^{N-1}U^{1+\sigma}(\eta/\varepsilon)) \\ =&-(p-1)\int_\Omega U_{\varepsilon,x_1}^{p-2}\varphi_{\varepsilon,x_1}\frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}}+\int_\Omega U_{\varepsilon,x_1}^{p-2}\sum_{j=2}^{k}P_{\varepsilon,\Omega}U_{\varepsilon,x_j}\frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}} \\ &+O\Big(\varepsilon^{N-1}e^{-(2+\sigma)d(x_1,\partial\Omega)/\varepsilon}+\varepsilon^{N-1}U^{1+\sigma}(\eta/\varepsilon) \\ &+\varepsilon^{N-1}\sum_{j=2}^{k}e^{-\sigma d(x_j,\partial\Omega)/\varepsilon}U(\eta/\varepsilon)\Big). \end{aligned} \tag{2.41}$$

Combining (2.39), (2.40) and (2.41), we obtain

$$\begin{aligned} I_{41} =& -(p-1)\int_\Omega U_{\varepsilon,x_1}^{p-2}\varphi_{\varepsilon,x_1}\frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}}+\int_\Omega U_{\varepsilon,x_1}^{p-2}\sum_{j=2}^{k}P_{\varepsilon,\Omega}U_{\varepsilon,x_j}\frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}} \\ &+O\Big(\varepsilon^{N-1}e^{-(2+\sigma)d(x_1,\partial\Omega)/\varepsilon}+\varepsilon^{N-1}U^{1+\sigma}(\eta/\varepsilon) \\ &+\varepsilon^{N-1}\sum_{j=2}^{k}e^{-\sigma d(x_j,\partial\Omega)/\varepsilon}U(\eta/\varepsilon)\Big). \end{aligned} \tag{2.42}$$

On the other hand,

$$
\begin{aligned}
(2.43)\qquad I_{42} = & O\bigg(\int_\Omega \sum_{2\le i<j\le k} U_{\varepsilon,x_i}^{(p-1)/2} U_{\varepsilon,x_j}^{(p-1)/2} \bigg|\frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}}\bigg|\bigg) \\
& + O\bigg(\int_\Omega \sum_{j=2}^k U_{\varepsilon,x_j}^{p-2} \varphi_{\varepsilon,x_j} \bigg|\frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}}\bigg|\bigg) \\
= & O(\varepsilon^{N-1} e^{-|x_i-x_1|/2\varepsilon - |x_j-x_1|/2\varepsilon-(p-2-\theta)|x_j-x_i|/\varepsilon}) \\
& + O\bigg(\varepsilon^{N-1}\sum_{j=2}^k e^{-\sigma d(x_j,\partial\Omega)/\varepsilon} U(\eta/\varepsilon)\bigg) \\
= & O\bigg(\varepsilon^{N-1} e^{-(2+\sigma)d(x_1,\partial\Omega)/\varepsilon} + \varepsilon^{N-1} U^{1+\sigma}(\eta/\varepsilon) \\
& + \varepsilon^{N-1}\sum_{j=2}^k e^{-\sigma d(x_j,\partial\Omega)/\varepsilon} U(\eta/\varepsilon)\bigg).
\end{aligned}
$$

Combining (2.38), (2.42) and (2.43), we obtain

$$
\begin{aligned}
(2.44)\qquad I_4 = & -(p-1)\int_\Omega U_{\varepsilon,x_1}^{p-2}\varphi_{\varepsilon,x_1}\frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}} + \int_\Omega U_{\varepsilon,x_1}^{p-2}\sum_{j=2}^k P_{\varepsilon,\Omega}U_{\varepsilon,x_j}\frac{\partial U_{\varepsilon,x_1}}{\partial x_{1l}} \\
& + O\bigg(\varepsilon^{N-1} e^{-(2+\sigma)d(x_1,\partial\Omega)/\varepsilon} + \varepsilon^{N-1} U^{1+\sigma}(\eta/\varepsilon) \\
& + \varepsilon^{N-1}\sum_{j=2}^k e^{-\sigma d(x_j,\partial\Omega)/\varepsilon} U(\eta/\varepsilon)\bigg).
\end{aligned}
$$

So (2.26) follows from (2.32), (2.36), (2.37) and (2.44). We can prove (2.27) in a similar way. □

LEMMA 2.5. *Let A_l, B_{lj} be the constants in Proposition 2.1. Then for each l, we have*

$$
\varepsilon A_l, B_{lj} = O\bigg(\varepsilon e^{-(2-\theta)d(x_l,\partial\Omega)/\varepsilon} + \varepsilon\sum_{i\ne l} U\bigg(\frac{|x_i-x_l|}{\varepsilon}\bigg)\bigg).
$$

PROOF. Without loss of generality, we assume $l=1$. We have

$$
\begin{aligned}
(2.45)\qquad & A_1\langle P_{\varepsilon,\Omega}U_{\varepsilon,x_1}, P_{\varepsilon,\Omega}U_{\varepsilon,x_1}\rangle + \sum_{j=1}^N B_{1j}\bigg\langle \frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1j}}, P_{\varepsilon,\Omega}U_{\varepsilon,x_1}\bigg\rangle \\
& = \bigg\langle\frac{\partial J}{\partial v}, P_{\varepsilon,\Omega}U_{\varepsilon,x_1}\bigg\rangle + O\bigg(\sum_{l=2}^k U\bigg(\frac{|x_1-x_l|}{\varepsilon}\bigg)\bigg).
\end{aligned}
$$

$$(2.46)\quad A_1\Big\langle P_{\varepsilon,\Omega}U_{\varepsilon,x_1},\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1i}}\Big\rangle+\sum_{j=1}^{N}B_{1j}\Big\langle\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1j}},\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_1}}{\partial x_{1i}}\Big\rangle$$
$$=\frac{\partial J}{\partial x_{1i}}+O\Big(\sum_{l=2}^{k}U\Big(\frac{|x_1-x_l|}{\varepsilon}\Big)\Big).$$

Using Proposition (2.4), we can solve the above system to obtain the desired estimates. □

As a direct consequence of Proposition 2.4 and Lemma 2.5, we have

PROPOSITION 2.6. *Let $(\alpha_\varepsilon(k,x),v_{\varepsilon,k}(x))$ be the map obtained in Proposition 2.3. Then for each fixed i, we have*

$$(2.47)\quad \frac{\partial K(x)}{\partial x_{il}}=(p-1)\int_\Omega U_{\varepsilon,x_i}^{p-2}\frac{\partial U_{\varepsilon,x_i}}{\partial x_{il}}\varphi_{\varepsilon,x_i}(p-1)\sum_{j\neq i}\int_\Omega U_{\varepsilon,x_i}^{p-2}P_{\varepsilon,\Omega}U_{\varepsilon,x_j}\frac{\partial U_{\varepsilon,x_i}}{\partial x_{il}}$$
$$+O(\varepsilon^{N-1}e^{-(2+\sigma)d(x_i,\partial\Omega)/\varepsilon})$$
$$+O\Big(\varepsilon^{N-1}\sum_{j\neq i}U^{1+\sigma}\Big(\frac{|x_i-x_j|}{\varepsilon}\Big)$$
$$+\varepsilon^{N-1}\sum_{j\neq i}e^{-\sigma d(x_j,\partial\Omega)/\varepsilon}U\Big(\frac{|x_i-x_j|}{\varepsilon}\Big)\Big).$$

By Proposition 2.6, Lemmas A.3 and A.4, we have

PROPOSITION 2.7. *Suppose that* $\min_{i\neq j}|x_i-x_j|>2\max_{1\le j\le k}d(x_j,\partial\Omega)$. *Then*

$$(2.48)\qquad \frac{\partial K(x)}{\partial x_{ji}}=\int_\Omega\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_j}}{\partial n}\frac{\partial U_{\varepsilon,x_j}}{\partial r}\left(\frac{\int_\Omega\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_j}}{\partial n}\frac{\partial U_{\varepsilon,x_j}}{\partial x_{ji}}}{\int_\Omega\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_j}}{\partial n}\frac{\partial U_{\varepsilon,x_j}}{\partial r}}-\phi_{ji}(x)\right),$$

where $r=|y-x_l|$ and $\phi_{ji}(x)$ satisfies $\phi_{ji}(x)=O(e^{-\sigma d(x_j,\partial\Omega)/\varepsilon})$.

3. Conley index and critical groups

Suppose that $D_1,\dots,D_k$ are disjoint open sets compactly contained in Ω and satisfy $\min_{i\neq j}d(D_i,D_j)>2\max_{1\le j\le k}\max_{x_j\in D_j}d(x_j,\partial\Omega)$. Let $D=D_1\times\dots\times D_k$. In order to prove that (1.1) has a k-peak solution with exactly one peak in D_j, $j=1,\dots,k$, we need to prove that $K(x)$ has a critical point $x\in D$. A sufficient condition to guarantee that $K(x)$ has a critical point $x\in D$ is that the Conley index $h(DK,D)$ is not trivial. Let

$$(3.1)\qquad X_j(x_j)=\left(\frac{\int_\Omega\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_j}}{\partial n}\frac{\partial U_{\varepsilon,x_j}}{\partial x_{j1}}}{\int_\Omega\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_j}}{\partial n}\frac{\partial U_{\varepsilon,x_j}}{\partial r}},\dots,\frac{\int_\Omega\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_j}}{\partial n}\frac{\partial U_{\varepsilon,x_j}}{\partial x_{jN}}}{\int_\Omega\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_j}}{\partial n}\frac{\partial U_{\varepsilon,x_j}}{\partial r}}\right),$$

and $X(x) = (X_1(x_1), \dots, X_k(x_k))$. Since

$$\frac{\dfrac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_j}}{\partial n}\dfrac{\partial U_{\varepsilon,x_j}}{\partial r}}{\displaystyle\int_\Omega \frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_j}}{\partial n}\frac{\partial U_{\varepsilon,x_j}}{\partial r}}$$

is uniformly bounded in $L^1(\partial\Omega)$, we may assume that there is a measure μ on $\partial\Omega$ such that as $\varepsilon \to 0$ (at least for suitable subsequences),

$$\frac{\dfrac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_j}}{\partial n}\dfrac{\partial U_{\varepsilon,x_j}}{\partial r}}{\displaystyle\int_\Omega \frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_j}}{\partial n}\frac{\partial U_{\varepsilon,x_j}}{\partial r}} \rightharpoonup \mu_j.$$

It is easy to check that $\int_{\partial\Omega} d\mu_j = 1$ and $\operatorname{spt}(\mu_j) \subset \Pi_{\partial\Omega}(x_j) = \{y : y \in \partial\Omega, |y - x_j| = d(x_j, \partial\Omega)\}$. Thus, as $\varepsilon \to 0$, the limit of the vector field in (3.1) is

$$\int_\Omega \frac{x_j - y}{|x - y|}\, d\mu_j = -\int_\Omega n(y)\, d\mu_j,$$

where $n(y)$ is the outward unit normal of $\partial\Omega$ at y, since $\operatorname{spt}(\mu_j) \subset \Pi_{\partial\Omega}(x_j)$ and $(y - x_j)/|y - x_j| = n(y)$ for any $y \in \Pi_{\partial\Omega}(x_j)$. From the result in [10], we see $\int_\Omega((x_j - y)/|x_j - y|)d\mu_j \in \partial d(x_j, \partial\Omega)$, the Clarke gradient of $d(x_j, \partial\Omega)$.

Let $o(1)$ denote any vector field whose norm tends to zero as $\varepsilon \to 0$. Since $d(x, \partial\Omega)$ has no critical point on ∂D_j, it is easy to check that there is a $c_0 > 0$, such that $|X_j(x_j)| \geq c_0$ for all $x_j \in \partial D_j$. Using homotopy invariance, we deduce from Proposition 2.7 that $h(DK, D) = h(X + o(1), D) = h(X, D)$.

From the above discussion, we know that the limit of the vector field in (3.1) belongs to the Clarke gradient of the Lipschitz function $d(x_j, \partial\Omega)$. So in order to see the effect of the domain geometry on the Conley index of the vector field in (3.1), it is much more convenient to discuss the Conley index of the Clarke gradient of $d(x_j, \partial\Omega)$.

The aim of this section is to discuss the Conley index of the Clarke gradient of any Lipschitz function. From now on, we assume that $f(x)$ is a locally Lipschitz function on $\mathbb{R}^m$.

We assume that T is a set of critical points of f satisfying that f is constant on T. We also assume that T is isolated in the sense that there is an open neighbourhood W of T such that $0 \notin \partial f(x)$ for all $x \in \overline{W} \setminus T$. It is convenient to add a constant to f so $f(x) = 0$ for $x \in T$.

Since $\partial f(x)$ is set valued, we need to do more work before we can define the Conley index $h(\partial f, W)$. As in [8], for two open sets W_1 and W_2 satisfying

$$T \subset W_2 \subset \overline{W}_2 \subset W_1 \subset \overline{W}_1 \subset W,$$

there is a locally Lipschitz map $V(x) \to \mathbb{R}^m$ such that $\|V(x)\| \leq 1$ for all x and

$$\langle V(x), t(x)\rangle \geq \alpha_1, \text{ for all } x \in \overline{W} \setminus W_1 \text{ and } t(x) \in \partial f(x), \tag{3.2}$$

$$\langle V(x), t(x)\rangle \geq \alpha_2, \text{ for all } x \in \overline{W} \setminus W_2 \text{ and } t(x) \in \partial f(x), \tag{3.3}$$

where α_i is a positive constant depending on W_i. Suppose that $x(t)$ is a solution of $dx(t)/dt = V(x(t))$ satisfying $x(t_1) \in \partial W$ and $x(t_2) \in \partial W_1$ for some t_1 and t_2, then

$$0 < \delta \leq |x(t_1) - x(t_2)| = \left| \int_{t_2}^{t_1} V(x(t))\, dt \right| \leq |t_1 - t_2|.$$

So if the flow $x(t)$ satisfies $x(t) \in W \setminus \overline{W}_1$ for all $t \in (t_1, t_2)$, $x(t_1) \in \partial W$ and $x(t_2) \in \partial W_1$ for some t_1 and t_2, then

$$f(x(t_2)) - f(x(t_1)) = \int_{t_1}^{t_2} \frac{df(x(t))}{dt}\, dt \geq c_0(t_2 - t_1) \geq c_0\delta = \beta > 0,$$

since by [8], $df(x(t))/dt \geq c_0 > 0$ for almost all $t \in [t_1, t_2]$. Now for fixed W_1, we choose W_2 such that $|f(y)| \leq \beta/2$ for all $y \in \overline{W}_2$. For this W, W_1 and W_2, we have a vector field $V(x)$ satisfying 3.2 and 3.3. Define the Conley index $h(-\partial f, W)$ to be $h(-V(x), W)$.

PROPOSITION 3.1. *$h(-\partial f, W)$ is well defined.*

PROOF. To prove that $h(-\partial f, W)$ is well defined, we need to check that W is an isolating neighbourhood of V and $h(-V(x), W)$ is independent of the choice of $V(x)$ satisfying (3.2) and (3.3).

First, we prove that W is an isolating neighbourhood of V. We argue by contradiction. Suppose that there is a solution $x(t)$ of $dx(t)/dt = -V(x(t))$ such that $x(t)$ always stays in $\overline{W}$ and touch ∂W at some time t^*. If $x(t)$ lies in $\overline{W} \setminus W_2$ for all $t \leq t^*$, then by (3.3), we get

$$C \geq f(x(t)) - f(x(t^*)) = \int_{t^*}^{t} \frac{df(x(t))}{dt}\, dt \geq \alpha_1(t^* - t) \to \infty,$$

as $t \to -\infty$. This is impossible. So we can find a $t_3 < t^*$ such that $x(t_3) \in W_2$ and thus $f(x(t_3)) \leq \beta/2$. Choose $t_1 \in (t_3, t^*)$ such that $x(t_1) \in \partial W_1$ and $x(t)$ stays in $\overline{W} \setminus W_1$ for all $t \in (t_1, t^*)$. From the above discussion, we have $f(x(t_1)) - f(x(t^*)) \geq \beta$. As a result,

$$f(x(t^*)) \leq f(x(t_1)) - \beta \leq f(x(t_3)) - \beta = -\beta/2.$$

Similarly, we can find $t_4 > t_2 > t^*$, such that $x(t_4) \in W_2$, $x(t_2) \in \partial W_1$ and $f(x(t^*)) - f(x(t_2)) \geq \beta$. Thus

$$f(x(t_4)) \leq f(x(t_2) \leq f(x(t^*)) - \beta \leq -3\beta/2.$$

This is a contradiction. So we have proved that W is an isolating neighbourhood of V.

Since the class of V satisfying (3.2) and (3.3) is convex, the homotopy invariance of the usual Conley index shows that our definition is independent of the choice of V. □

Remark 3.2. By standard properties of the usual Conley index, $h(-\partial f, W)$ is independent of the choice of W and W_1. Note that our construction seems to bear some relation to Mischaikov's construction of the Conley index for multivalued maps.

Remark 3.3. If $f(x) = f_1(x_1) + f_2(x_2)$, $x_1 \in \mathbb{R}^l$, $x_2 \in \mathbb{R}^{m-l}$, then

$$h(\partial f, W^1 \times W^2) = h(\partial f_1, W^1) \times h(\partial f_2, W^2),$$

where $W^1 \subset \mathbb{R}^l$ and $W^2 \subset \mathbb{R}^{m-l}$. In fact, for each $f_i(x_i)$, $i = 1, 2$, we choose $V_i(x_i)$ satisfying (3.2) and (3.3). By $\partial f(x) \subset \partial f_1(x_1) \times \partial f_2(x_2)$ (in fact, equality holds in this case), we see that $V(x) = (V_1(x_1), V_2(x_2))$ satisfies (3.2) and (3.3). Thus, by the Conley index formula for products (as in [11]),

$$\begin{aligned} h(\partial f, W^1 \times W^2) &= h(V_1(x_1) \times V_2(x_2), W^1 \times W^2) \\ &= h(V_1(x_1), W^1) \times h(V_2(x_2), W^2) = h(\partial f_1, W^1) \times h(\partial f_2, W^2). \end{aligned}$$

Hence by the Künneth theorem for products in cohomology, the homotopy index of f is nontrivial if the cohomology of f_1 and f_2 are nontrivial (for the same field of coefficients). This is where the Conley index method has advantages.

Definition 3.4. Let f satisfy the conditions mentioned in the begining of this section. We define the critical group $C(f, T)$ of f on T to be the cohomology of $h(-V, W)$ for V and W satisfying (3.2) and (3.3). As usual, we choose a field as the coefficients.

Now we want to prove an alternative formula for the critical groups, which in practice is easier to calculate with. Let $\mathcal{U}$ be a neighbourhood of T.

Proposition 3.5. *We have* $C(f, T) = H^*(f^0 \cap \mathcal{U}, (f^0 \setminus T) \cap \mathcal{U})$, *where* $f^c = \{x : f(x) \le c\}$.

Proof. To prove our claim above, we make a special choice of W. We follow the idea in [13]. By using an inductive procedure in the Chang's construction, we can construct $V(x)$ locally Lipschitz on $\mathcal{U} \setminus T$, so that $\|V(x)\| \le 1$ and $\langle V(x), t(x)\rangle > 0$ on $\mathcal{U} \setminus T$ and $\langle V(x), t(x)\rangle \ge \alpha > 0$ on any compact subset S of $\mathcal{U} \setminus T$, for any $t(x) \in \partial f(x)$. Here α depends on S. Choose a small neighbourhood Z of T and suppose that $\delta > 0$ is relatively small.

Let $x(t)$ be a solution of $x'(t) = -V(x(t))$, $x(0) \in Z \setminus T$. If $f(x(0)) \le 0$, then since $f(x(t))$ is strictly decreasing in t, we see there is a positive lower bound for the distance between T and $\{x(t) : t \ge 0\}$. As a result, $df(x(t))/dt \le -\eta < 0$. So we see that $x(t)$ hits $f^{-1}(-\delta)$. Backwards in time, if there is a positive lower bound for the distance between T and $\{x(t) : t \leq 0\}$, then $x(t)$ hits $f^{-1}(\delta)$. So we have proved that either $x(t)$ hits both $f^{-1}(\delta)$ and $f^{-1}(-\delta)$, or the flow goes to T. A similar result is also true for the case $f(x(0)) \geq 0$. Now we form a set W by taking all these flow lines starting in Z and choosing the part of the flow line till it hits $\{x : f(x) = \pm\delta\}$ or hits T, together with T. It is easily seen that W is a neighbourhood of T and contains no other critical points of ∂f. Moreover, $(W, W \cap f^{-1}(-\delta))$ is an index pair in the sense of Conley.

Now we calculate $h(-V(x), W)$. By our construction, we see that the exit set for the flow of $-V$ on W is $W \cap f^{-1}(-\delta)$. Thus,

$$h(-V(x), W) = [W/(W \cap f^{-1}(-\delta))].$$

From the above discussion, we know that $W = f^{\delta} \cap W$ can be deformed into $f^s \cap W$ along the flow line for $-V$ for any $s \in (0, \delta)$. Hence, noting that $f^s \cap W$ decreases to $f^0 \cap W$, we see by results on direct limits for Alexander cohomology (see p. 238 in [25]) that

$$\begin{aligned} H([W/(W \cap f^{-1}(-\delta))]) &= H(W, W \cap f^{-1}(-\delta)) = H(f^0 \cap W, W \cap f^{-1}(-\delta)) \\ &= H\big(f^0 \cap W, f^0 \cap W \setminus T\big), \end{aligned}$$

since $f^0 \cap W \setminus T$ can be deformed into $f^{-1}(-\delta) \cap W$. Thus the result follows.□

By the homotopy invariance of the degree, we can also use the vector field $V(x)$ to define the degree of the Clarke gradient of a Lipschitz function.

Definition 3.6. Let f satisfy the conditions mentioned in the begining of this section. We define $\deg(\partial f, W, 0) = \deg(V, W, 0)$, where V and W satisfy (3.2) and (3.3).

For a smooth function, we have

$$\deg(Df, W, 0) = \sum_{i=0}^{m} (-1)^i \operatorname{rank} C_i(f, x_0),$$

where x_0 is an isolated critical point f and W is a small neighbourhood of x_0. Our next result shows that the above relation is still true for a Lipschitz function.

Proposition 3.7. *Let f be a Lipschitz function and let x_0 be the unique critical point of f in a neighbourhood W of x_0. Then*

$$\deg(\partial f, W, 0) = \sum_{i=0}^{m} (-1)^i \operatorname{rank} C_i(f, x_0).$$

Proof. First, by Rademacher's Theorem, f is differentiable almost everywhere. Besides, by Proposition 2.2.2 in [10], if f is differentiable at x_0, then $Df(x_0) \in \partial f(x_0)$.

Let $\phi \in C_0^\infty(B_1(0))$ be a function with $0 \le \phi \le 1$ and $\int_{R^N} \phi(y)\,dy = 1$. Let $\phi_\tau(x) = \tau^{-N}\phi(\tau^{-1}y)$ for $\tau > 0$ small. Define

$$f_\tau(x) = \int_{R^N} \phi_\tau(y) f(x-y)\,dy.$$

Then by the dominated convergence theorem, we have

$$Df_\tau(x) = \int_{R^N} \phi_\tau(y) D_x f(x-y)\,dy.$$

Let V and W satisfy (3.2) and (3.3). Since $Df(x) \in \partial f(x)$ almost everywhere and $V(x)$ is continuous, we have

$$\langle D_x f(x-y), V(x)\rangle = \langle D_x f(x-y), V(x-y)\rangle + o(1) \ge \alpha/2$$

for almost $y \in B_\tau(x)$ if $\tau > 0$ is small. As a result,

$$\begin{aligned}\langle Df_\tau(x), V(x)\rangle &= \int_{R^N} \phi_\tau(y)\langle D_x f(x-y), V(x)\rangle\,dy \\ &= \int_{B_\tau(x)} \phi_\tau(y)\langle D_x f(x-y), V(x)\rangle\,dy \ge \frac{\alpha}{2} > 0,\end{aligned}$$

So we obtain

$$\deg(\partial f, W, 0) = \deg(V, W, 0) = \deg(Df_\tau, W, 0)$$

and

$$h(\partial f, W) = h(V, W) = h(Df_\tau, W).$$

Now since f_τ is a smooth function, we can argue in exactly the same way as in [13, p. 14] to find that

$$\deg(Df_\tau, W, 0) = \sum_{i=1}^{m} (-1)^i \operatorname{rank} H^i(h(Df_\tau, W)).$$

Thus the result follows. $\square$

Remark 3.8. In [23], Li and Nirenberg used classical degree theory to study the single peak solution. The main result in [23] states that if A is an open set compactly contained in Ω and $d(x,\partial\Omega)$ is differentiable on ∂A, then (1.1) has a single peak solution with its peak inside A provided $deg(Dd(x,\partial\Omega), A, 0) \ne 0$. It is worth pointing out that for an isolated critical point x_0 of $d(x,\partial\Omega)$, it is not always possible to find a small neighbourhood A of x_0 such that x_0 is the unique critical point of $d(x,\partial\Omega)$ in A and $d(x,\partial\Omega)$ is differentiable on ∂A. For example, if Ω is a symmetric dumbell, then $d(x,\partial\Omega)$ is not differentiable on the segment joining the centers of the balls. Thus any A containing one of the center of the ball (the local maximum point of $d(x,\partial\Omega)$), or the middle of the handle (the saddle point of $d(x,\partial\Omega)$) must contains the whole segment joining the centers of the balls if $d(x,\partial\Omega)$ is differentiable on ∂A. Here we generalize the classical degree theory to the Clarke gradient of any Lipschitz function f and the above proposition shows that if $deg(\partial f, W, 0) \neq 0$, then $C(f, x_0)$ is nontrivial.

Now we give some examples where the critical points have nontrivial critical groups.

Proposition 3.9. *Suppose that T is a set of local maximum points of f such that f is constant on T and T is an isolated set of critical points of f. In this case, $C_i(f,T) = H_{N-i}(T)$, where the homology is non-reduced and is Steenrod homology as in* [25]. *In particular, $C_N(f,T)$ is nontrivial.*

Proof. This follows from Theorem 11.15 in [25] and Proposition 3.5 here.□

Proposition 3.10. *Suppose that $x_0 = 0$ is a saddle point in the following sense: there is an integer $l \geq 1$ and an l-dimensional C^1 manifold Y and an $m-l$ dimensional C^1 manifold Z such that $Y \cap Z = \{0\}$, $\mathbb{R}^m = T_{0,Y} \oplus T_{0,Z}$, where $T_{0,Y}$ and $T_{0,Z}$ are the tangent spaces of Y at 0 and Z at 0 respectively, $f(x) > f(0)$ if $x \in B_\delta(0) \cap Z \setminus \{0\}$, $f(x) < f(0)$ if $x \in B_\delta(0) \cap Y \setminus \{0\}$ for some small $\delta > 0$. Then if $x_0 = 0$ is the unique critical point of f in $B_\delta(0)$, $C_l(f, x_0) \neq 0$.*

Proof. It is easy to see that there is a C^1 local diffeomorphism $\phi : \mathbb{R}^m \to \mathbb{R}^m$ such that $\phi(Y)$ and $\phi(Z)$ are subspaces near the origin. Then the proof of the claim is similar to that of Theorem 2.1 in [24]. □

We can relax the above definition for saddle point a little bit so that it is much easier to check in practice.

Proposition 3.11. *Suppose that $x_0 = 0$ is a saddle point in the following sense: there is an integer $l \geq 1$ and an l-dimensional C^1 manifold Y and an $m-l$ dimensional C^1 manifold Z such that $Y \cap Z = \{0\}$, $\mathbb{R}^m = T_{0,Y} \oplus T_{0,Z}$, where $T_{0,Y}$ and $T_{0,Z}$ are the tangent spaces of Y at 0 and Z at 0 respectively, $f(x) \geq f(0)$ if $x \in B_\delta(0) \cap Z$, $f(x) \leq f(0)$ if $x \in B_\delta(0) \cap Y$ for some small $\delta > 0$. Then if $x_0 = 0$ is the unique critical point of f in $B_\delta(0)$, we have $C_l(f, x_0) \neq 0$.*

Proof. We just need to find an l-dimensional manifold Y' and an $m-l$ dimensional manifold Z' such that $Y' \cap Z' = \{0\}$, $\mathbb{R}^m = T_{0,Y'} \oplus T_{0,Z'}$, where $T_{0,Y'}$ and $T_{0,Z'}$ are the tangent spaces of Y' at 0 and Z' at 0 respectively, $f(x) > f(0)$ if $x \in B_\delta(0) \cap Z' \setminus \{0\}$, $f(x) < f(0)$ if $x \in B_\delta(0) \cap Y' \setminus \{0\}$ for some small $\delta > 0$. This claim follows if we can choose a C^1 vector field $V(x)$ satisfying $V(0) = 0$, $DV(0) = 0$, $\langle V(x), t(x) \rangle > 0$ for $x \in B_\delta(0) \setminus \{0\}$ and $t(x) \in \partial f(x)$. In fact, let $Y' = \{x = x(\eta, x_0), x_0 \in Y\}$, where $x(t, x_0)$ is a solution of

$$x'(t) = -V(x(t)), \quad x(0) = x_0$$

and $\eta > 0$ is a small constant, and let $Z' = \{x = x(\eta, x_0), x_0 \in Z\}$, where $x(t, x_0)$ is a solution of

$$x'(t) = V(x(t)), \quad x(0) = x_0$$

and $\eta > 0$ is a small constant. Then by using $DV(0) = 0$, it is easy to check that $T_{0,Z'} = T_{0,Y}$ and $T_{0,Z'} = T_{0,Z}$. Thus Y' and Z' satisfy the requirements.

To prove the existence of such vector field $V(x)$, since 0 is an isolated critical point of f, by using the construction in [8], we know that there is vector field $V'(x)$ which is locally C^1 in $B_\delta(0) \setminus \{0\}$ and satisfying $\|V'(x)\| \le 1$, $\langle V'(x), t(x)\rangle > 0$ for $x \in B_\delta(0) \setminus \{0\}$ and $t(x) \in \partial f(x)$. For each integer $j > 0$, we can find a increasing sequence of $L_j > j$ such that $|DV'(x)| \le L_j$ for $x \in B_\delta(0) \setminus B_{1/j}(0)$. Let $\psi(t)$ is a C^1 function satisfying $\psi(0) = 0$, $\psi(t) > 0$ for $t > 0$ and $\psi'(t) \le 1/L_{j+1}^2$ for $t \le j^{-1}$. Let $V(x) = \psi(|x|)V'(x)$. Then for any $x \in B_\delta(0) \setminus \{0\}$, we can find a j, such that $x \in B_{1/j}(0) \setminus B_{1/(j+1)}(0)$. Thus, $|DV'(x)| \le L_{j+1}$. As a result, we have

$$|DV(x)| \le \psi(|x|)|DV'(x)| + |D\psi(|x|)| \le \frac{|x|}{L_{j+1}^2} L_{j+1} + 1/L_{j+1}^2 = O(|x|).$$

So $V(x)$ is the vector field we need. □

Remark 3.12. According to the above results, it is easy to check that the critical groups of the critical points in examples 1.12, 1.13 and 1.14 given in [21] are nontrivial.

Remark 3.13. In [33], a point $x_0 \in \Omega$ is called a nondegenerate peak point if

$$\int_{\partial\Omega} e^{\langle z - x_0, a\rangle}(z - x_0)\, d\mu_{x_0} = 0$$

and the matrix

$$\bigg(\int_{\partial\Omega} e^{\langle z - x_0, a\rangle}(z_i - x_{0,i})(z_j - x_{0,j})\, d\mu_{x_0}\bigg)$$

is nonsingular, where a is some point in $\mathbb{R}^N$, μ_{x_0} is a weak limit of

$$e^{-|z - x_0|/\varepsilon}\bigg(\int_{\partial\Omega} e^{-|z - x_0|/\varepsilon}\bigg)^{-1}$$

as $\varepsilon \to 0$. It is proved in [28] that a point $x_0 \in \Omega$ is a nondegenerate peak point if and only if $x_0 \in \operatorname{int}(\operatorname{co}\Pi_{\partial\Omega}(x_0))$. Here, we want to point out a nondegenerate peak point is a strictly local maximum point of the distance function. To see this, let e be any unit vector in $\mathbb{R}^N$. We claim that there is a $\delta > 0$, independent of e, such that $\langle y, e\rangle \ge \delta$ for some $y \in \Pi_{\partial\Omega}(x_0)$. We argue by contradiction. Suppose that there is a sequence of unit vector e_i such that $\langle y, e_i\rangle \le o(1)$ for any $y \in \Pi_{\partial\Omega}(x_0)$ as $i \to \infty$. Assume $e_i \to e_0$. Hence, $\langle y, e_0\rangle \le 0$ for any $y \in \Pi_{\partial\Omega}(x_0)$. By translation and rotation, we may assume that $x_0 = 0$ and $e_0 = (0, \ldots, 0, 1)$. Since $0 \in \operatorname{int}(\operatorname{co}\Pi_{\partial\Omega}(0))$, we can find a point $y \in \Pi_{\partial\Omega}(0)$ satisfying $\langle y, e_0\rangle > 0$.

Otherwise, $\mathrm{co}\,\Pi_{\partial\Omega}(0) \subset \{y_N \le 0\}$, which contradicts $0 \in \mathrm{int}(\mathrm{co}\,\Pi_{\partial\Omega}(0))$. Thus our claim follows. So, for $t > 0$ small (independent of e), we have

$$|y - te|^2 = |y|^2 - 2t\langle y, e\rangle + t^2 < |y|^2,$$

which implies $d(te, \partial\Omega) \le |y - te| < |y| = d(0, \partial\Omega)$.

Before we close this section, let us look at the following examples.

EXAMPLE 3.14. Now we give an example where the critical point of the distance function is not the kind of saddle point defined in [9], but the corresponding critical group is nontrivial.

Let $g(x_1, x_2) = 1 + \delta r^2 \cos a\theta$, where $\delta > 0$ is a small constant such that normal line of the surface $x_3 = g(x_1, x_2)$ at $(x_1, x_2, g(x_1, x_2))$ does not intersect with the normal line of this surface at other point within the region $|x_3| \le 2$, a is prime, $r = (x_1^2 + x_2^2)^{1/2}$ and θ is the polar angle of $x_1 - x_2$ plane. Define a domain in $\mathbb{R}^3$ as follows:

$$\Omega = \{(x_1, x_2, x_3) : |x_3| \le g(x_1, x_2), r \le R\},$$

where $R > 0$ is a large constant. Then it is easy to check Ω is invariant under rotation of $2\pi/a$ in $x_1 - x_2$ plane.

Firstly we claim that $x = 0$ is an isolated critical point of $d(x, \partial\Omega)$. In fact, let $x \ne 0$ be a point in a small neighbourhood of the origin. If x is not in the $x_1 - x_2$ plane, then $\Pi_{\partial\Omega}(x)$ contains just one point and thus is not a critical point. If x is in the $x_1 - x_2$ plane, then $\Pi_{\partial\Omega}(x)$ contains exactly two points $y = (y_1, y_2, y_3)$ and $y^* = (y_1, y_2, -y_3)$ with $y_3 = g(y_1, y_2)$ and $y_1^2 + y_2^2 \ne 0$. Direct calculation shows that $Dg(y_1, y_2) \ne 0$ if $y_1^2 + y_2^2 \ne 0$. But $y - x = c_0(Dg(y_1, y_2), 1)$ and $y^* - x = c_0(Dg(y_1, y_2), -1)$. As a result, $y - x \ne -(y^* - x)$ which implies $0 \notin \mathrm{co}(\Pi_{\partial\Omega}(x) - \{x\})$.

Secondly, we show $x = 0$ is not the kind of saddle point defined in [9]. Suppose that X_1 and X_2 are two subspaces in $\mathbb{R}^3$ such that $\mathbb{R}^3 = X_1 \oplus X_2$, $\max_{x \in X_1 \cap \partial B_\tau(0)} d(x, \partial\Omega) < \min_{x \in X_2 \cap B_\tau(0)} d(x, \partial\Omega)$; $\max_{x \in X_1 \cap B_\tau(0)} d(x, \partial\Omega) < \min_{x \in X_2 \cap \partial B_\tau(0)} d(x, \partial\Omega)$. For any $x \in \Omega$ with polar angle θ satisfying $\cos a\theta \le 0$ and $r \ne 0$, it is easy to check $d(x, \partial\Omega) < g(x_1, x_2) \le 1$. On the other hand, for any small $\tau > 0$, there exists $\overline{x} \in X_2$ with its polar angle satisfying $\cos a\overline{\theta} \le 0$ and $|\overline{x}| = \tau$. So we see $d(\overline{x}, \partial\Omega) < 1$. As a result, $\max_{x \in X_1 \cap B_\tau(0)} d(x, \partial\Omega) \ge d(0, \partial\Omega) > d(\overline{x}, \partial\Omega) \ge \min_{x \in X_2 \cap \partial B_\tau(0)} d(x, \partial\Omega)$. This is a contradiction.

In addition, let $x = (x_1, x_2, 0)$ be a point such that $\cos a\theta > c_0 > 0$. It is easy to check $d(x, \partial\Omega) > 1$. This shows that $x = 0$ is not a local maximum point.

Finally, to calculate the Conley index, we see easily that we can choose the approximating vector fields to preserve the symmetry of rotation of $2\pi/a$ in the $x_1 - x_2$ plane. Hence we can apply the results in Remark (ii) on page 672 of [12] (using a remark on page 14 in [13] to remove a side condition). We see that if the Clarke gradient of the distance function had Conley index trivial at 0, then it would also have to have trivial homology with Z_a coefficients on the one dimensional subspace $x_1 = x_2 = 0$, which is the fixed point set of the rotation. However, this is impossible since 0 is a maximum point of the distance function on this space.

Using a similar argument, we can also prove that in this example, $d(x, \partial\Omega)$ does not satisfy the conditions in Proposition 3.11.

Example 3.15. Let Ω_1 be a bounded domain in $x_1 - x_3$ plane such that $x_1 > 0$ if $(x_1, x_3) \in \Omega_1$. Suppose that $x^* = (x_1^*, x_3^*) \in \Omega_1$ is a saddle point of the distance function in the strong sense, that is, for fixed x_1, $d(x, \partial\Omega_1)$ is increasing in $x_3 > x_3^*$ and decreasing in $x_3 < x_3^*$, while for each fixed x_3, $d(x, \partial\Omega_1)$ is increasing in $x_1 < x_1^*$ and decreasing in $x_1 > x_1^*$. The three dimensional domain Ω is obtained by rotating Ω_1 around the x_3-axis. So Ω has an isolated critical set $T = (x_1^* \cos\theta, x_1^* \sin\theta, x_3^*)$, $\theta \in [0, 2\pi]$.

Let $d_0 = d(x^*, \partial\Omega_1)$, $x^* = (x_1^*, x_3^*)$. It is easy to see the pair $(d^{d_0} \cap B_\tau(x^*), (d^{d_0} \setminus \{x^*\}) \cap B_\tau(x^*))$ can be deformed into $(I, \partial I)$, where $I = \{(x_1, x_3^*) : x_1 \in [x_1^* - \tau, x_1^* + \tau]\}$. As a result, we have

$$C(d, T) = H^*(S \times I, S \times \partial I) = H^*(S) \otimes H^*(I, \partial I),$$

where S is the unit circle in $\mathbb{R}^2$. Hence, $C(d, T)$ is nontrivial.

To calculate the degree of Dd, we use the symmetries again. By using a S^1 equivariant approximation and a theorem of Nussbaum [29], we see that the degree of Dd is the same as on the symmetric subspace $x_1 = x_2 = 0$. But the intersection of this subspace with Ω is empty and hence the degree is zero.

We can also calculate the degree of Dd by proving the natural analogue of Proposition 3.7.

Remark 3.16. We can obtain even more complicated four dimensional examples by rotating the sets in Example 3.14 around an axis (much as in Example 3.15).

4. Proof of the main results

In this section, we will use the results in Section 2 to get some existence result for (1.1).

Proof of Theorem 1.1. Let $X_j(x_j)$ be the vector field defined in Section 3. Since it is not clear whether $X_j(x_j)$ converges uniformly in a neighbourhood of ∂D_j, we cannot conclude immediately that $h(X(x_j), D_j) = h(\partial d(x, \partial\Omega), D_j)$. To check $h(X(x_j), D_j) = h(\partial d(x, \partial\Omega), D_j)$, we choose the vector field $V(x_j)$ satisfying (3.2) and (3.3). We claim that $\langle V(x_j), X_j(x_j)\rangle \geq \eta > 0$ in a neighbourhood of ∂D_j. By the homotopy invariance of the Conley index, we then obtain $h(X(x_j), D_j) = h(V(x_j), D_j)$. Thus

$$H^*(h(X(x_j), D_j)) = H^*h(V(x_j), D_j)) = C(d^{c_j} \cap D_j, (d^{c_j} \cap D_j) \setminus T_j).$$

By assumption, $H^*(h(X(x_j), D_j))$ is nontrivial. But

$$h(X, D) = h(X(x_1), D_1) \times \ldots \times h(X(x_k), D_k).$$

Thus it follows from the Künneth formula that $H^*(h(X, D))$ is nontrivial. So $h(X, D)$ is nontrivial and the result follows.

It remains to prove the claim. We argue by contradiction. Suppose that there is a sequence $x_{j,m}$ in a neighbourhood of ∂D_j, such that $\langle V(x_{j,m}), X_j(x_{j,m})\rangle \to -c_0 \le 0$ as $m \to \infty$. Assume $x_{j,m} \to x_0$. Since

$$\mu_m(y) =: \frac{\dfrac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_{j,m}}}{\partial n} \dfrac{\partial U_{\varepsilon,x_{j,m}}}{\partial r}}{\displaystyle\int_\Omega \dfrac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_{j,m}}}{\partial n} \dfrac{\partial U_{\varepsilon,x_{j,m}}}{\partial r}}$$

is uniformly bounded in $L^1(\partial\Omega)$, we may assume that there is a measure μ on $\partial\Omega$ such that as $m \to \infty$, $\mu_m \rightharpoonup \mu$. Thus,

$$\begin{aligned}
\langle V(x_{j,m}), X_j(x_{j,m})\rangle &= \langle V(x_0), X_j(x_{j,m})\rangle + o(1) \\
&= \int_{\partial\Omega} \Big\langle \frac{y - x_0}{|y - x_0|}, V(x_0) \Big\rangle \mu_m(y)\, dy \\
&\quad + \int_{\partial\Omega} \Big\langle \frac{y - x_{j,m}}{|y - x_{j,m}|} - \frac{y - x_0}{|y - x_0|}, V(x_0) \Big\rangle \mu_m(y)\, dy + o(1) \\
&\to \Big\langle \int_{\partial\Omega} \frac{y - x_0}{|y - x_0|}\, d\mu, V(x_0) \Big\rangle = -c_0 \le 0.
\end{aligned}$$

It is easy to check that $\int_{\partial\Omega} d\mu = 1$ and $\mathrm{spt}(\mu) \subset \Pi_{\partial\Omega}(x_0)$. Thus, $\int_{\partial\Omega}(y - x_0)/|y - x_0|\, d\mu \in \partial d(x_0, \partial\Omega)$. This is a contradiction to (3.2). $\square$

PROOF OF THEOREM 1.2. First we claim that in a strictly convex domain, $d(x, \partial\Omega)$ has exactly one critical point. Thus this critical point is the global maximum point of $d(x, \partial\Omega)$. We argue by contradiction. Suppose that $d(x, \partial\Omega)$ has two critical points x_1 and x_2. By translation and rotation, we may assume that $x_1 = 0$, $x_2 = (l, 0, \ldots, 0)$ and $d_1 = d(x_1, \partial\Omega) \le d_2 = d(x_2, \partial\Omega)$. From [10], we know that

$$\partial d(x, \partial\Omega) \subset \mathrm{co}\left\{ \frac{y - x}{|y - x|} : y \in \Pi_{\partial\Omega}(x) \right\},$$

where co denotes the convex hull. Since $0 \in \partial d(x_1, \partial\Omega)$, we see that

$$\Pi_{\partial\Omega}(x_1) \cap \{y = (y_1, \ldots, y_N) \in \mathbb{R}^N,\ y_1 \ge 0\} \ne \emptyset.$$

Take $y \in \Pi_{\partial\Omega}(x_1) \cap \{y = (y_1, \ldots, y_N) \in \mathbb{R}^N, y_1 \ge 0\}$. It is easy to check that $T_y \cap (\overline{\Omega} \setminus \{y\}) \ne \emptyset$. This is a contradiction.

Since Ω is convex, using the moving plane method of Gidas, Ni and Nirenberg, we know that there is a $d_0 > 0$, such that any local maximum point x_0 of the solution of (1.1) satisfies $d(x_0, \partial\Omega) \geq d_0$. Suppose that (1.1) has a solution of the form (1.4) satisfying (1.5). Since there is always a local maximum point in a small neighbourhood of $x_{\varepsilon,j}$, we know that $d(x_{\varepsilon,j}, \partial\Omega) \geq d_0$. Arguing as in [17], we obtain $\min_{i\neq j} |x_{\varepsilon,i} - x_{\varepsilon,j}| \geq d_0$.

Without loss of generality, we assume $d(x_{\varepsilon,1}, \partial\Omega) = \min_{1\leq j\leq k} d(x_{\varepsilon,j}, \partial\Omega)$. We also assume $x_{\varepsilon,j} \to x_j$ as $\varepsilon \to 0$. As in Section 2, we have

$$\frac{\partial K(x_\varepsilon)}{\partial x_{ji}} = 0.$$

Since Ω is convex, it follows from Proposition 2.6 and Lemma A.7 that the above relation is equivalent to

$$\begin{aligned}
0 = {} & (p-1) \int_\Omega U^{p-2}_{\varepsilon,x_{\varepsilon,1}} \frac{\partial U_{\varepsilon,x_{\varepsilon,1}}}{\partial x_{1l}} \varphi_{\varepsilon,x_{\varepsilon,1}} \\
& - (p-1) \sum_{j=2}^k \int_\Omega U^{p-2}_{\varepsilon,x_{\varepsilon,1}} U_{\varepsilon,x_{\varepsilon,j}} \frac{\partial U_{\varepsilon,x_{\varepsilon,1}}}{\partial x_{1l}} \\
& + O(\varepsilon^{N-1} e^{-(2+\sigma)d(x_{\varepsilon,1},\partial\Omega)/\varepsilon}) \\
& + o\bigg(\varepsilon^{N-1} \sum_{j=2}^k U\bigg(\frac{|x_{\varepsilon,1} - x_{\varepsilon,j}|}{\varepsilon}\bigg)\bigg).
\end{aligned} \tag{4.1}$$

Thus, by Lemma A.4, we have

$$\begin{aligned}
0 = {} & \int_\Omega \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_1}}{\partial n} \frac{\partial U_{\varepsilon,x_1}}{\partial r} \bigg(\int_{\partial\Omega} \frac{y - x_1}{|y - x_1|} d\mu + o(1) \bigg) \\
& + c_0 \varepsilon^{N-1} \sum_{j=2}^k U\bigg(\frac{|x_{\varepsilon,1} - x_{\varepsilon,j}|}{\varepsilon}\bigg) \bigg(\frac{x_1 - x_j}{|x_1 - x_j|} + o(1) \bigg) \\
& + O(\varepsilon^{N-1} e^{-(2+\sigma)d(x_{\varepsilon,1},\partial\Omega)/\varepsilon}) + o\bigg(\varepsilon^{N-1} \sum_{j=2}^k U\bigg(\frac{|x_{\varepsilon,1} - x_{\varepsilon,j}|}{\varepsilon}\bigg)\bigg),
\end{aligned} \tag{4.2}$$

where μ is a measure satisfying $\int_\Omega d\mu = 1$ and $\operatorname{spt} \mu \subset \Pi_{\partial\Omega}(x_1)$.

Since $d(x_1, \partial\Omega) = \min_{1\leq j\leq k} d(x_j, \partial\Omega)$ and $\min_{2\leq j\leq k} |x_j - x_1| > 0$, we know x_1 is not the global maximum point of $d(x, \partial\Omega)$ and thus is not a critical point of $d(x, \partial\Omega)$. So

$$\bigg| \int_{\partial\Omega} \frac{y - x_1}{|y - x_1|}\, d\mu \bigg| \geq \delta > 0.$$

By (4.2), we obtain

$$0 = \int_\Omega \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_1}}{\partial n} \frac{\partial U_{\varepsilon,x_1}}{\partial r} \Big(\Big| \int_{\partial\Omega} \frac{y - x_1}{|y - x_1|}\, d\mu \Big|^2 + o(1) \Big)$$
$$+ c_0 \varepsilon^{N-1} \sum_{j=2}^{k} U\Big(\frac{|x_{\varepsilon,1} - x_{\varepsilon,j}|}{\varepsilon} \Big) \Big(\Big\langle \frac{x_1 - x_j}{|x_1 - x_j|}, \int_{\partial\Omega} \frac{y - x_1}{|y - x_1|}\, d\mu \Big\rangle + o(1) \Big)$$
$$+ O(\varepsilon^{N-1} e^{-(2+\sigma) d(x_{\varepsilon,1}, \partial\Omega)/\varepsilon}) + o\Big(\varepsilon^{N-1} \sum_{j=2}^{k} U\Big(\frac{|x_{\varepsilon,1} - x_{\varepsilon,j}|}{\varepsilon} \Big) \Big), \tag{4.3}$$

We claim that

$$\Big\langle \frac{x_1 - x_j}{|x_1 - x_j|}, \int_{\partial\Omega} \frac{y - x_1}{|y - x_1|}\, d\mu \Big\rangle \geq \delta > 0. \tag{4.4}$$

So by Lemma A.3, we see (4.3) is impossible and thus the result follows.

It remains to check (4.4). Without loss of generality, we assume that $x_1 = 0$ and $x_2 = (l, 0, \dots, 0)$. Arguing as in the proof of the uniqueness of critical point for $d(x, \partial\Omega)$, we obtain

$$\Pi_{\partial\Omega}(x_1) \subset \{ y = (y_1, \dots, y_N) \in \mathbb{R}^N, y_1 < 0 \},$$

which clearly implies (4.4). □

Finally, we give an example which shows that if the critical group of an isolated critical point x_0 of $d(x, \partial\Omega)$ is trivial, there may be no positive solution of (1.1) such that one of its local maximum point is close to x_0.

Example 4.1. Let $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = B_1(0) \cap \{ x = (x_1, \dots, x_N) \in \mathbb{R}^N, x_1 \leq 0 \}$, Ω_2 is any open domain such that $\partial\Omega$ is smooth and the reflection of $\Omega \cap \{x_1 \leq t\}$ about $x_1 = t$ is contained in Ω for all $t \leq \delta$, where $\delta > 0$ is a constant. Then $x_0 = 0$ is a critical point of $d(x, \partial\Omega)$ and the corresponding critical group is trivial. By using the moving plane method of Gidas, Ni and Nirenberg, we conclude that any positive solution of (1.1) is increasing in the direction x_1 till $x_1 = \delta$ and thus there is no local maximum point in $\Omega \cap \{x_1 \leq \delta\}$.

5. Remark on the Neumann problem

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. Consider the following Neumann problem:

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{in } \partial\Omega, \end{cases} \tag{5.1}$$

where ε is a small positive number, n is the unit outward normal of $\partial\Omega$ at y, $2 < p < 2N/(N-2)$ if $N \geq 3$ and $2 < p < \infty$ if $N = 2$.

We denote $P_{\varepsilon,\Omega,N}v$ the solution of the following problem:

$$\begin{cases} -\varepsilon^2\Delta u + u = |v|^{p-2}v & y \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & y \in \partial\Omega, \\ u \in H^1(\Omega). \end{cases}$$

By the maximum principle, we know $P_{\varepsilon,\Omega,N}U_{\varepsilon,z} > 0$.

For any $x_i \in \overline{\Omega}$, $i = 1, 2, \ldots, k$, define

$$E^*_{\varepsilon,x,k} = \bigg\{ v \in H^1(\Omega) : \langle P_{\varepsilon,\Omega,N}U_{\varepsilon,x_i}, v\rangle_\varepsilon = 0, \\ \left\langle \frac{\partial P_{\varepsilon,\Omega,N}U_{\varepsilon,x_i}}{\partial \tau_i}, v \right\rangle_\varepsilon = 0,\ i = 1, 2, \ldots, k \bigg\},$$

where τ_i is any unit vector in $\mathbb{R}^N$ if $x_i \in \Omega$; τ_i is any tangent vector of $\partial\Omega$ at x_i.

Let $H(x)$ be the mean curvature function of $\partial\Omega$. We use I or J to denote a finite index set and use $|I|$ to denote the number of points I contains. Using the techniques in Section 2 and the results in Section 3, we can get the following result:

Theorem 5.1. *Let I and J be two finite index sets where one of the sets may be empty. Suppose that $x_i \in \partial\Omega$, $i \in I$, are different critical points of $H(x)$ with $C(H, x_i)$ nontrivial and $z_j \in \Omega$, $j \in J$, are different critical points of $d(x, \partial\Omega)$ with $C(d, z_j)$ nontrivial. If*

$$\min_{i\in I, l,j\in J, l\neq j}(|x_i - z_j|, |z_l - z_j|) > 2\max_{j\in J} d(x_j, \partial\Omega),$$

then there exists an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, (5.1) has at least one solution of the form

$$u_\varepsilon = \sum_{i\in I} \alpha_{\varepsilon i} P_{\varepsilon,\Omega,N}U_{\varepsilon,x_{\varepsilon i}} + \sum_{j\in J} \alpha_{\varepsilon j} P_{\varepsilon,\Omega,N}U_{\varepsilon,z_{\varepsilon j}} + v_\varepsilon \tag{5.2}$$

where, as $\varepsilon \to 0$, $\alpha_{\varepsilon i} \to 1$, $x_{\varepsilon i} \to x_i$, $x_{\varepsilon i} \in \partial\Omega$, $i \in I$, $\alpha_{\varepsilon j} \to 1$, $z_{\varepsilon j} \to z_j$, $j \in J$, $v_\varepsilon \in E^*_{\varepsilon,x_\varepsilon,|I|+|J|}$ and $\|v\|^2_\varepsilon = o(\varepsilon^N)$.

Appendix A. Basic estimates

In this section, we present some basic estimates needed in the proof of the main results. First, let us recall the well known fact on the asymptotic behaviours of $U(y)$:

$$\lim_{|y|\to\infty} |y|^{(N-1)/2}e^{|y|}U(y) = c_0 > 0.$$

From now on, we always assume that $d(x, \partial\Omega)/\varepsilon \geq M$ for some large constant $M > 0$. Let $\varphi_{\varepsilon,x} = U_{\varepsilon,x} - P_{\varepsilon,\Omega}U_{\varepsilon,x}$. Then $\varphi_{\varepsilon,x}$ satisfies

$$\begin{cases} -\varepsilon^2\Delta\varphi_{\varepsilon,x} + \varphi_{\varepsilon,x} = 0 & y \in \Omega, \\ \varphi_{\varepsilon,x} = U_{\varepsilon,x} & y \in \partial\Omega. \end{cases} \tag{A.1}$$

We denote

$$\tau_{\varepsilon,x} = \int_\Omega U_{\varepsilon,x}^{p-1}\varphi_{\varepsilon,x}.$$

We have the following estimate for $\tau_{\varepsilon,x}$.

LEMMA A.1. *For any $\theta > 0$, there exist $C_2 > C_1 > 0$, such that*

$$C_1\varepsilon^N e^{-(2+\theta)d(x,\partial\Omega)/\varepsilon} \le \tau_{\varepsilon,x} \le C_2\varepsilon^N e^{-(2-\theta)d(x,\partial\Omega)/\varepsilon}. \tag{A.2}$$

LEMMA A.2. *There is a $\sigma > 0$, such that*

$$\int_\Omega \varphi_{\varepsilon,x}^2 U_{\varepsilon,x}^{p-2} = O(e^{-\sigma d(x,\partial\Omega)/\varepsilon}\tau_{\varepsilon,x}), \tag{A.3}$$

$$\int_\Omega \varphi_{\varepsilon,x}^p = O\big(e^{-\sigma d(x,\partial\Omega)/\varepsilon}\tau_{\varepsilon,x}\big). \tag{A.4}$$

Define

$$\tau'_{\varepsilon,x} = \varepsilon^2\int_{\partial\Omega}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x}}{\partial n}\frac{\partial U_{\varepsilon,x}}{\partial r},$$

where $r = |y-x|$ and n is the outward unit normal to $\partial\Omega$ at y.

LEMMA A.3. *For any $\theta > 0$, there exist $c_1 > c_0 > 0$, such that*

$$c_0\varepsilon^{N-1}e^{-(2+\theta)d(x,\partial\Omega)/\varepsilon} \le \tau'_{\varepsilon,x} \le c_1\varepsilon^{N-1}e^{-(2-\theta)d(x,\partial\Omega)/\varepsilon}.$$

LEMMA A.4. *We have*

$$\int_\Omega U_{\varepsilon,x}^{p-2}\frac{\partial U_{\varepsilon,x}}{\partial x_i}\varphi_{\varepsilon,x} = -\varepsilon^2\int_{\partial\Omega}\frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x}}{\partial n}\frac{\partial U_{\varepsilon,x}}{\partial x_i} + O(\varepsilon^{N-1}e^{-pd(x,\partial\Omega)/\varepsilon}).$$

Moreover, if $\partial\Omega\cap\partial B_{d(x,\partial\Omega)}(x)$ contain exactly one point q, then

$$\sum_{i=1}^N\int_\Omega U_{\varepsilon,x}^{p-2}\frac{\partial U_{\varepsilon,x}}{\partial x_i}\varphi_{\varepsilon,x}\nu_i \ge c_0\varepsilon^{N-1}e^{-(2+\theta)d(x,\partial\Omega)/\varepsilon},$$

for any $\theta > 0$, where ν is the outward unit normal to $\partial\Omega$ at q and $c_0 > 0$ is a constant.

LEMMA A.5. *For $i \ne j$, we have*

$$\int_\Omega U_{\varepsilon,x_i}^{p-1}\varphi_{\varepsilon,x_j} = O\bigg(\varepsilon^N\sum_{j=1}^N e^{-(2+\sigma)d(x_j,\partial\Omega)/\varepsilon} + \varepsilon^N\sum_{i\ne j}e^{-(1+\sigma)|x_i-x_j|/\varepsilon}\bigg). \tag{A.5}$$

The proof of the above lemmas can be found in [16]. So we omit the proof of these lemmas.

Lemma A.6. *There is a $\sigma > 0$, such that for $i \neq j$,*

$$\int_\Omega U_{\varepsilon,x_i} U_{\varepsilon,x_j}^{p-2} \varphi_{\varepsilon,x_j} = O\biggl(\varepsilon^N e^{-\sigma d(x_j,\partial\Omega)/\varepsilon} \sum_{i\neq j} e^{-|x_i-x_j|/\varepsilon}\biggr).$$

Proof. Choose $\sigma > 0$ small enough so that $p-2-\sigma > 0$. Since $\varphi_{\varepsilon,x_j} \le U_{\varepsilon,x_j}$, we have

$$\begin{aligned}
\int_\Omega U_{\varepsilon,x_i} U_{\varepsilon,x_j}^{p-2} \varphi_{\varepsilon,x_j} &\le C \int_\Omega e^{-|y-x_i|/\varepsilon} U_{\varepsilon,x_j}^{p-2} \varphi_{\varepsilon,x_j} \\
&\le C e^{-|x_j-x_i|/\varepsilon} \int_\Omega e^{|y-x_j|/\varepsilon} U_{\varepsilon,x_j}^{p-2} \varphi_{\varepsilon,x_j} \\
&\le C e^{-|x_j-x_i|/\varepsilon} \int_\Omega U_{\varepsilon,x_j}^{p-2-\sigma} \varphi_{\varepsilon,x_j}^{\sigma} \\
&\le C e^{-|x_j-x_i|/\varepsilon} e^{-\sigma d(x_j,\partial\Omega)/\varepsilon} \int_\Omega U_{\varepsilon,x_j}^{p-2-\sigma} \\
&= O\biggl(\varepsilon^N e^{-\sigma d(x_j,\partial\Omega)/\varepsilon} \sum_{i\neq j} e^{-|x_i-x_j|/\varepsilon}\biggr). \qquad \square
\end{aligned}$$

Lemma A.7. *Suppose that Ω is convex. For $i \neq j$, if $d(x_l,\partial\Omega) \ge \delta > 0$, $l = i,j$, we have*

$$\int_\Omega U_{\varepsilon,x_i}^{p-1} \varphi_{\varepsilon,x_j} = O(\varepsilon^N e^{-(1+\sigma)|x_i-x_j|/\varepsilon}). \tag{A.6}$$

Proof. Let $G(z,y)$ be the Green's function of $-\varepsilon^2\Delta u + u$ on Ω subject to Dirichlet boundary conditions. Then

$$\varphi_{\varepsilon,x_j}(y) = \varepsilon^2 \int_{\partial\Omega} \frac{\partial G(z,y)}{\partial n} U_{\varepsilon,x_j}(z)\,dz,$$

where n is the outward unit normal of $\partial\Omega$ at y. But for any $\theta > 0$ small, we have

$$\varepsilon \biggl|\frac{\partial G(z,y)}{\partial n}\biggr| \le C e^{-(1-\theta)|z-y|/\varepsilon}.$$

Since Ω is convex, we have that there is a $\sigma > 0$ such that $|y-z| + |z-x| \ge (1+\sigma)|x-y|$ for any $z \in \partial\Omega$ and $x,y \in \Omega$ with $d(x,\partial\Omega) \ge \delta$ and $d(y,\partial\Omega) \ge \delta$. As a result,

$$\varphi_{\varepsilon,x_j}(y) \le \varepsilon \int_{\partial\Omega} e^{-(1-\theta)|z-y|/\varepsilon} e^{-|z-x|/\varepsilon}\,dz \le C e^{-(1+\sigma)|x_j-y|/\varepsilon},$$

and the result follows. $\square$

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Manuscript received July 10, 1999

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