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ON THE NUMBER OF INTERIOR MULTIPEAK SOLUTIONS FOR SINGULARLY PERTURBED NEUMANN PROBLEMS

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1. Introduction

In this paper, we will estimate the number of the solutions with exactly k interior local maximum points for the following singularly perturbed problem:

$$
\begin{cases} -\varepsilon^2\Delta u + u = u^{p-1} & y \text{ in } \Omega, \\ u > 0 & y \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & y \text{ on } \partial\Omega, \end{cases} \tag{1.1}
$$

where ε is a small positive number, Ω is a bounded domain in $\mathbb{R}^N$ with C^1-boundary, n is the unit outward normal of $\partial\Omega$ at $y, 1 < p < (N+2)/(N-2)$ if $N \geq 3$ and $1 < p < \infty$ if $N = 2$.

Much work has been done on (1.1) in the past several years. In [17], [18], Ni and Takagi proved that the least energy solution of (1.1) has exactly one local maximum point x_ε which lies in $\partial\Omega$, and x_ε tends to a point x_0 which attains the maximum of the mean curvature function of $\partial\Omega$. Since then, many authors have constructed solutions for (1.1) with their local maximum points lying in the boundary of Ω. See [2], [5], [8], [12], [15], [21], [22]. Recently, Wei [23], Kowalczyk [13], Bates and Fusco [3] considered the existence of solutions for (1.1),

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with their local maximum points tending to some designated points in the interior of Ω.

In [8], it is proved that for each integer $k \geq 1$, (1.1) has at least one solution u_ε such that u_ε has exactly k local maximum points lying on the boundary, provided ε is small enough. The aim of this paper is to prove that (1.1) always has a solution u_ε such that u_ε has exactly k local maximum points lying in Ω and to estimate the number of such solutions.

In the following, we call a solution u_ε of (1.1) an interior k-peak solution if u_ε has exactly k local maximum points lying inside Ω.

Before we introduce the main results, we give some notation. Let $U(y)$ be the unique positive solution (see [14]) of

$$\begin{cases} -\Delta u + u = u^{p-1} & y \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \\ u(0) = \max_{y\in\mathbb{R}^N} u(y). \end{cases}$$

It is well known (see [11]) that $U(y)$ is radially symmetric about the origin, decreasing and

$$\lim_{|y|\to\infty} U(y)e^{|y|}|y|^{(N-2)/2} = c_0 > 0.$$

Define

$$\langle u, v\rangle_\varepsilon = \int_\Omega \varepsilon^2 Du\cdot Dv + uv, \quad \text{for all } u, v \in H^1(\Omega), \tag{1.2}$$

$$\|u\|_\varepsilon = \langle u, u\rangle_\varepsilon^{1/2}. \tag{1.3}$$

For any $z \in \mathbb{R}^N$, $\varepsilon > 0$, let

$$U_{\varepsilon,z}(y) =: U\left(\frac{y-z}{\varepsilon}\right).$$

We denote $P_{\varepsilon,\Omega}v$ the solution of the following problem

$$\begin{cases} -\varepsilon^2\Delta u + u = |v|^{p-2}v & y \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & y \text{ on } \partial\Omega. \end{cases}$$

By maximum principle, we know $P_{\varepsilon,\Omega}U_{\varepsilon,z} > 0$. For any $x_i \in \Omega$, $i = 1, \dots, k$, define

$$E_{\varepsilon,x,k} = \biggl\{ v \in H^1(\Omega) : \langle P_{\varepsilon,\Omega}U_{\varepsilon,x_i}, v\rangle_\varepsilon = 0, \\ \left\langle \frac{\partial P_{\varepsilon,\Omega}U_{\varepsilon,x_i}}{\partial x_{ij}}, v\right\rangle_\varepsilon = 0,\ i = 1, \dots, k,\ j = 1, \dots, N \biggr\}.$$

Let σ_k denote the group of k permutations. We also let

$$D_k = \underbrace{\Omega \times \ldots \times \Omega}_{k} \setminus \bigcup_{i \neq j} \{|x_i - x_j| < d\}, \tag{1.4}$$

$$A_k = D_k / \sigma_k. \tag{1.5}$$

The main results of this paper are the following

THEOREM 1.1. *For each fixed positive integer $k \geq 2$, there exists an $\varepsilon_0 = \varepsilon_0(k)$, such that for each $\varepsilon \in (0, \varepsilon_0]$,* (1.1) *has at least* $\mathrm{Cat}_{A_k} A_k$ *solutions of the form*

$$u_\varepsilon = \sum_{i=1}^{k} \alpha_{\varepsilon i} P_{\varepsilon,\Omega} U_{\varepsilon, x_{\varepsilon i}} + v_\varepsilon$$

where, as $\varepsilon \to 0$,

$$\alpha_{\varepsilon i} \to 1, \quad i = 1, \ldots, k, \tag{1.6}$$

$$\frac{|x_{\varepsilon i} - x_{\varepsilon j}|}{\varepsilon} \to \infty, \quad \textit{for all } i \neq j, \tag{1.7}$$

$$x_{\varepsilon i} \to x_i \in \Omega, \tag{1.8}$$

$$v_\varepsilon \in E_{\varepsilon, x_\varepsilon, k}, \quad \|v\|_\varepsilon^2 = o(\varepsilon^N). \tag{1.9}$$

THEOREM 1.2. *There exists an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$,* (1.1) *has at least* $\mathrm{Cat}_\Omega \Omega$ *solutions of the form*

$$u_\varepsilon = \alpha_\varepsilon P_{\varepsilon,\Omega} U_{\varepsilon, x_\varepsilon} + v_\varepsilon,$$

where as $\varepsilon \to 0$, $\alpha_\varepsilon \to 1$, $x_\varepsilon \to x \in \Omega$, $\|v\|_\varepsilon^2 = o(\varepsilon^N)$ and $v_\varepsilon \in E_{\varepsilon, x_\varepsilon, 1}$.

Denote

$$V_k = \left(\underbrace{\mathbb{R}^N \times \ldots \times \mathbb{R}^N}_{k} \setminus \bigcup_{i \neq j} \{|x_i - x_j| \leq d\} \right) \Big/ \sigma_k.$$

Let $x_0 \in \Omega$ and let $\delta > 0$ be so small that $B_\delta(x_0) \subset \Omega$. We also let

$$V_k' = \left(\underbrace{B_\delta(x_0) \times \ldots \times B_\delta(x_0)}_{k} \setminus \bigcup_{i \neq j} \{|x_i - x_j| \leq d\} \right) \Big/ \sigma_k,$$

Then V_k and V_k' are homotopically equivalent. So

$$\mathrm{Cat}_{A_k} A_k \geq \mathrm{Cat}_{V_k}(V_k') = \mathrm{Cat}_{V_k}(V_k) \geq \mathrm{cuplength}(V_k) + 1.$$

For the estimate of the cuplength of the space V_k, the readers can refer to [6]. For the case $k = 2$, V_2 is homotopically equivalent to the projection space $\mathbb{R}P^{N-1}$. Thus $\mathrm{Cat}_{A_k} A_k \geq \mathrm{cuplength}(\mathbb{R}P^{N-1}) + 1 = N$. So (1.1) has at least N interior two-peak solutions for every domain.

The technique developed in this paper can also be used to discuss the following Neumann problem in exterior domains:

$$\begin{cases} \varepsilon^2 \Delta u + u = u^{p-1} & y \text{ in } \Omega_1, \\ u > 0 & y \text{ in } \Omega_1, \\ \dfrac{\partial u}{\partial n} = 0 & y \text{ on } \partial\Omega_1, \\ u \to 0 & \text{as } |y| \to \infty, \end{cases} \tag{1.10}$$

where Ω_1 is an exterior domain in $\mathbb{R}^N$.

Let $R > 0$ be a large constant such that $\mathbb{R}^N \setminus \Omega_1 \subset B_R(0)$. We have

THEOREM 1.3. *There is an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$,*

(i) (1.10) *has at least* $\mathrm{Cat}_{\Omega_1}(\Omega_1, B_R(0))$ *solutions of the form*

$$u_\varepsilon = \alpha_\varepsilon P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon} + v_\varepsilon, \tag{1.11}$$

where $v_\varepsilon \in E_{\varepsilon,x_\varepsilon}$, and

$$\alpha_\varepsilon \to 1, \quad d(x_\varepsilon, \partial\Omega)/\varepsilon \to \infty, \quad \|v_\varepsilon\|_\varepsilon^2 = o(\varepsilon^N), \quad \text{as } \varepsilon \to 0; \tag{1.12}$$

(ii) *if $\mathbb{R}^N \setminus \Omega_1$ is convex,* (1.10) *does not have solution of the form* (1.11).

The method in [8] is still valid for the exterior Neumann problem. So we see that there is no difference between the interior Neumann problem and the exterior Neumann problem if we construct solutions with all the peaks lying on the boundary. Our results here show whether the exterior Neumann problem has interior single peak solutions depends on the topology of the domain, while the interior Neumann problem always has at least one interior single peak solution. The results for the existence of interior single peak solutions for both the interior and exterior Neumann problems are very similar to those for Dirichlet problems. However, the existence results for multipeak solutions between Dirichlet problems and Neumann problems are totally different, because for the Dirichlet problem, the existence of multipeak solutions depends on the topology of the domain. See [7], [9], [10] for existence results for Dirichlet problems.

This paper is arranged as follows. In Section 2, we will present some basic estimates needed in the proof of the main results. Section 3 is devoted to the proof of Theorems 1.1 and 1.2, and Theorem 1.3 is proved in Section 4.

2. Basic estimates

In this section, we develop a simple and direct method to get all the basic estimates needed in the proof of the main results. So we are able to avoid using the viscosity solution method of [19], [23] to prove these estimates which are essential to characterize the locations of the peaks of the solutions for (1.1). It is worth pointing out that the method used in this section works for bounded domain problems and exterior domain problems, while the viscosity solution method of [19], [23] seems only applicable to bounded domain problems since it depends heavily on the comparison theorems for the elliptic equations.

In the following, Ω is either a bounded domain or an exterior domain in $\mathbb{R}^N$. From now on, we always assume that $x \in \Omega$ and $d(x, \partial\Omega)/\varepsilon \geq M$ for some large constant $M > 0$. Let $\varphi_{\varepsilon,x} = U_{\varepsilon,x} - P_{\varepsilon,\Omega} U_{\varepsilon,x}$. Then $\varphi_{\varepsilon,x}$ satisfies

$$\begin{cases} -\varepsilon^2 \Delta \varphi_{\varepsilon,x} + \varphi_{\varepsilon,x} = 0 & y \text{ in } \Omega, \\ \dfrac{\partial \varphi_{\varepsilon,x}}{\partial n} = \dfrac{\partial U_{\varepsilon,x}}{\partial n} & y \text{ on } \partial\Omega. \end{cases} \tag{2.1}$$

We denote

$$\tau_{\varepsilon,x} = \int_\Omega U_{\varepsilon,x}^{p-1} \varphi_{\varepsilon,x}.$$

LEMMA 2.1. *For any $\theta > 0$, there are $C_2 > C_1 > 0$, such that*

$$C_1 \varepsilon^N e^{-(2+\theta) d(x,\partial\Omega)/\varepsilon} \leq -\tau_{\varepsilon,x} \leq C_2 \varepsilon^N e^{-(2-\theta) d(x,\partial\Omega)/\varepsilon}. \tag{2.2}$$

PROOF. Multiplying (2.1) by $U_{\varepsilon,x}$ and integrating by parts, we get

$$\tau_{\varepsilon,x} = \varepsilon^2 \int_{\partial\Omega} \frac{\partial U_{\varepsilon,x}}{\partial n} U_{\varepsilon,x} - \varepsilon^2 \int_{\partial\Omega} \frac{\partial U_{\varepsilon,x}}{\partial n} \varphi_{\varepsilon,x}. \tag{2.3}$$

Multiplying (2.1) by $\varphi_{\varepsilon,x}$ and integrating by parts, we obtain

$$\varepsilon^2 \int_{\partial\Omega} \frac{\partial U_{\varepsilon,x}}{\partial n} \varphi_{\varepsilon,x} = \|\varphi_{\varepsilon,x}\|_\varepsilon^2 > 0. \tag{2.4}$$

Combining (2.3) and (2.4), we have

$$\begin{aligned} \tau_{\varepsilon,x} <& \varepsilon^2 \int_{\partial\Omega} \frac{\partial U_{\varepsilon,x}}{\partial n} U_{\varepsilon,x} \\ \leq& -c'\varepsilon \int_{\partial\Omega \cap B_{2\varepsilon\theta}(q)} \left(\frac{\varepsilon}{|y-x|} \right)^{N-2} e^{-2|y-x|/\varepsilon} \\ &+ O\big(\varepsilon^N e^{-(2+2\theta) d(x,\partial\Omega)/\varepsilon}\big) \\ \leq& -c'\varepsilon \int_{\partial\Omega \cap B_{\varepsilon\theta}(q)} \left(\frac{\varepsilon}{|y-x|} \right)^{N-2} e^{-2|y-x|/\varepsilon} \\ &+ O\big(\varepsilon^N e^{-(2+2\theta) d(x,\partial\Omega)/\varepsilon}\big) \\ \leq& -\varepsilon^N c_0 \left[\frac{\varepsilon}{d(x,\partial\Omega) + \varepsilon\theta} \right]^{N-2} e^{-2(d(x,\partial\Omega)+\varepsilon\theta)/\varepsilon}, \end{aligned} \tag{2.5}$$

where $q \in \partial\Omega$ satisfying $|x - q| = d(x, \partial\Omega)$. Since $d(x, \partial\Omega)/\varepsilon > M$, we see that (2.5) implies the left hand side of (2.2).

Let $G(y,x)$ be the Green's function subject to the Neumann boundary condition, that is, $G(y,x)$ satisfies

$$\begin{cases} -\varepsilon^2\Delta G + G = \delta_x & y \text{ in } \Omega, \\ \dfrac{\partial G}{\partial n} = 0 & y \text{ on } \partial\Omega. \end{cases}$$

Then, $|G(y,x)| \le Ce^{-|y-x|/\varepsilon}$ for $y \in \Omega \setminus B_\delta(x)$. We have

$$\varphi_{\varepsilon,x}(y) = \int_{\partial\Omega} \frac{\partial U_{\varepsilon,x}(z)}{\partial n} G(z,y)\,dz.$$

Hence,

$$|\varphi_{\varepsilon,x}(y)| \le Ce^{-(1-\theta)[d(x,\partial\Omega)/\varepsilon + d(y,\partial\Omega)/\varepsilon]}. \tag{2.6}$$

Inserting (2.6) into (2.3), we obtain

$$|\tau_{\varepsilon,x}| \le Ce^{-(2-\theta)d(x,\partial\Omega)/\varepsilon}. \qquad \square$$

Lemma 2.2. *For any $1 < l \le p$, there is a $\sigma > 0$, such that*

$$\int_\Omega \varphi_{\varepsilon,x}^l U_{\varepsilon,x}^{p-l} = O\big(e^{-(2+\sigma)d(x,\partial\Omega)/\varepsilon}\big).$$

Proof. By (2.6), we have

$$\begin{aligned} \int_\Omega \varphi_{\varepsilon,x}^l U_{\varepsilon,x}^{p-l} &\le Ce^{-l(1-\theta)d(x,\partial\Omega)/\varepsilon} \int_\Omega e^{-l(1-\theta)d(y,\partial\Omega)/\varepsilon} e^{-(p-l)|y-x|/\varepsilon} \\ &\le Ce^{-l(1-\theta)d(x,\partial\Omega)/\varepsilon} \\ &\quad \cdot \int_\Omega e^{-[\min(l,p-l)-2\theta](d(y,\partial\Omega)/\varepsilon + |y-x|/\varepsilon)} e^{-\theta|y-x|/\varepsilon} \\ &\le Ce^{(-l(1-\theta)-[\min(l,p-l)-2\theta])d(x,\partial\Omega)/\varepsilon} \\ &= O\big(e^{-(2+\sigma)d(x,\partial\Omega)/\varepsilon}\big), \end{aligned} \tag{2.7}$$

since $l > 1$ and $p > 2$. $\square$

Lemma 2.3. *There is a $\sigma > 0$, such that*

$$\int_\Omega U_{\varepsilon,x_1} U_{\varepsilon,x_2}^{p-2} \varphi_{\varepsilon,x_2} = O\bigg(\varepsilon^N e^{-(1+\sigma)|x_1-x_2|/\varepsilon} + \sum_{j=1}^2 \varepsilon^N e^{-(2+\sigma)d(x_j,\partial\Omega)/\varepsilon}\bigg).$$

Proof. Let $l =: \min\{|x_1-x_2|, 2d(x_i,\partial\Omega),\, i=1,2\}$. We have

$$\begin{aligned} \int_\Omega U_{\varepsilon,x_1} U_{\varepsilon,x_2}^{p-2} \varphi_{\varepsilon,x_2} &= \int_{\Omega\setminus(B_{l/2}(x_1)\cup B_{l/2}(x_2))} U_{\varepsilon,x_1} U_{\varepsilon,x_2}^{p-2} \varphi_{\varepsilon,x_2} \\ &\quad + \int_{B_{l/2}(x_1)} U_{\varepsilon,x_1} U_{\varepsilon,x_2}^{p-2} \varphi_{\varepsilon,x_2} + \int_{B_{l/2}(x_2)} U_{\varepsilon,x_1} U_{\varepsilon,x_2}^{p-2} \varphi_{\varepsilon,x_2}. \end{aligned} \tag{2.8}$$

On the other hand, it follows from (2.6) that

$$\begin{aligned}(2.9)\qquad &\int_{\Omega\setminus(B_{l/2}(x_1)\cup B_{l/2}(x_2))} U_{\varepsilon,x_1}U_{\varepsilon,x_2}^{p-2}\varphi_{\varepsilon,x_2}\\ &\quad\le Ce^{-(1-\theta)l/(2\varepsilon)}\int_{\Omega\setminus(B_{l/2}(x_1)\cup B_{l/2}(x_2))} U_{\varepsilon,x_1}U_{\varepsilon,x_2}^{p-2}\\ &\quad\le Ce^{-p(1-\theta)l/(2\varepsilon)}\\ &\quad= O\bigg(\varepsilon^N e^{-(1+\sigma)|x_1-x_2|/\varepsilon}+\sum_{j=1}^{2}\varepsilon^N e^{-(2+\sigma)d(x_j,\partial\Omega)/\varepsilon}\bigg).\end{aligned}$$

$$\begin{aligned}(2.10)\qquad &\int_{B_{l/2}(x_1)} U_{\varepsilon,x_1}U_{\varepsilon,x_2}^{p-2}\varphi_{\varepsilon,x_2}\\ &\quad\le Ce^{-(1-\theta)(p-1)l/(2\varepsilon)}\int_{B_{l/2}(x_1)} e^{-(1-\theta)(|y-x_1|+d(y,\partial\Omega))/\varepsilon}\\ &\quad\le Ce^{-(1-\theta)(p-1)l/(2\varepsilon)}\int_{B_{l/2}(x_1)} e^{-(1-\theta)d(x_1,\partial\Omega)/\varepsilon}\\ &\quad= O\bigg(\varepsilon^N e^{-(1+\sigma)|x_1-x_2|/\varepsilon}+\sum_{j=1}^{2}\varepsilon^N e^{-(2+\sigma)d(x_j,\partial\Omega)/\varepsilon}\bigg).\end{aligned}$$

$$\begin{aligned}(2.11)\qquad &\int_{B_{l/2}(x_2)} U_{\varepsilon,x_1}U_{\varepsilon,x_2}^{p-2}\varphi_{\varepsilon,x_2}\le Ce^{-(1-\theta)l/(2\varepsilon)}\\ &\quad\cdot\bigg(\int_{B_{(1-\sigma)l/2}(x_2)} U_{\varepsilon,x_1}U_{\varepsilon,x_2}^{p-2}+\int_{\{(1-\sigma)l/2\le|y-x_2|\le l/2\}} U_{\varepsilon,x_1}U_{\varepsilon,x_2}^{p-2}\bigg)\\ &\le Ce^{-(1-\theta)l/(2\varepsilon)}\big(e^{-(1-\theta)(1+\sigma)l/(2\varepsilon)}+e^{-(1-\theta)(1+(p-2)(1-\sigma))l/(2\varepsilon)}\big)\\ &= O\bigg(\varepsilon^N e^{-(1+\sigma)|x_1-x_2|/\varepsilon}+\sum_{j=1}^{2}\varepsilon^N e^{-(2+\sigma)d(x_j,\partial\Omega)/\varepsilon}\bigg).\end{aligned}$$

Combining (2.8)–(2.11), we obtain the desired result. □

LEMMA 2.4. *Suppose that there are $M > 0$ and $\eta > 0$, such that $|x_j| \le M$ and $d(x_j,\partial\Omega) > \delta > 0$, then*

$$(2.12)\qquad \int_\Omega U_{\varepsilon,x_1}^{p-1}\varphi_{\varepsilon,x_2} = O\bigg(\varepsilon^N\sum_{j=1}^{2} e^{-(2+\sigma)d(x_j,\partial\Omega)/\varepsilon}+\varepsilon^N e^{-(1+\sigma)|x_1-x_2|/\varepsilon}\bigg).$$

PROOF. Similar to Lemma (2.1), we have

$$(2.13)\qquad \int_\Omega U_{\varepsilon,x_1}^{p-1}\varphi_{\varepsilon,x_2} = \varepsilon^2\int_{\partial\Omega}\frac{\partial U_{\varepsilon,x_2}}{\partial n}U_{\varepsilon,x_1} - \varepsilon^2\int_{\partial\Omega}\frac{\partial U_{\varepsilon,x_1}}{\partial n}\varphi_{\varepsilon,x_2}.$$

From (2.13), we see that if $d(x_1,\partial\Omega)\ne(1+o(1))d(x_2,\partial\Omega)$, or $d(S_1,S_2)\ge\delta>0$, where

$$S_i = \{q : q\in\partial\Omega,\ |x_i-q| = d(x,\partial\Omega)\},$$

then

$$\int_\Omega U^{p-1}_{\varepsilon,x_1}\varphi_{\varepsilon,x_2} = O\biggl(\varepsilon^N\sum_{j=1}^2 e^{-(2+\sigma)d(x_j,\partial\Omega)/\varepsilon} + \varepsilon^N e^{-(1+\sigma)|x_1-x_2|/\varepsilon}\biggr). \tag{2.14}$$

But if $d(x_1,\partial\Omega)=(1+o(1))d(x_2,\partial\Omega)$ and $d(S_1,S_2)=o(1)$, then it is not difficult to deduce that $|x_1-x_2|=o(1)$. Thus it follows from (2.13) that

$$\begin{aligned}\int_\Omega U^{p-1}_{\varepsilon,x_1}\varphi_{\varepsilon,x_2} &= O\bigl(\varepsilon^N e^{-(1-\theta)d(x_1,\partial\Omega)/\varepsilon}e^{-(1-\theta)d(x_2,\partial\Omega)/\varepsilon}\bigr)\\ &= O\bigl(\varepsilon^N e^{-(1+\sigma)|x_1-x_2|/\varepsilon}\bigr)\\ &= O\bigl(\varepsilon^N\sum_{j=1}^2 e^{-(2+\sigma)d(x_j,\partial\Omega)/\varepsilon} + \varepsilon^N e^{-(1+\sigma)|x_1-x_2|/\varepsilon}\bigr).\end{aligned}$$

Thus we have completed the proof of this lemma. □

Define

$$\tau'_{\varepsilon,x} = \varepsilon^2\int_{\partial\Omega} P_{\varepsilon,\Omega}U_{\varepsilon,x}\frac{\partial^2 U_{\varepsilon,x}}{\partial r^2},$$

where $r=|y-x|$.

LEMMA 2.5. *For any $\theta>0$, there are $c_1>c_0>0$, such that*

$$c_0\varepsilon^{N-1}e^{-(2+\theta)d(x,\partial\Omega)/\varepsilon}\le\tau'_{\varepsilon,x}\le c_1\varepsilon^{N-1}e^{-(2-\theta)d(x,\partial\Omega)/\varepsilon}.$$

PROOF. Since $|\varphi_{\varepsilon,x}(y)\le Ce^{(1-\theta)|y-x|/\varepsilon}$ we have

$$|P_{\varepsilon,\Omega}U_{\varepsilon,x}|\le Ce^{-(1-\theta)|y-x|/\varepsilon},\quad \text{for all } y\in\partial\Omega,$$

and thus

$$\tau'_{\varepsilon,x}\le c_1\varepsilon^{N-1}e^{-(2-\theta)d(x,\partial\Omega)/\varepsilon}.$$

Next, we claim that there is a $c_0>0$, such that

$$P_{\varepsilon,\Omega}U_{\varepsilon,x}\ge c_0e^{-(1+\theta)|y-x|/\varepsilon},\quad \text{for all } y\in\partial\Omega\cap B_\sigma(q), \tag{2.15}$$

where $q\in\partial\Omega$ satisfies $|q-x|=d(x,\partial\Omega)$. Clearly, (2.15) implies

$$\tau'_{\varepsilon,x}\ge c_0\varepsilon^{N-1}e^{-(2+\theta)d(x,\partial\Omega)/\varepsilon}.$$

Now we prove (2.15). Denote $\psi=P_{\varepsilon,\Omega}U_{\varepsilon,x}-c_0U_{\varepsilon,x}$. Let ψ_j, $j=1,2$, be the solution of the following problems respectively

$$\begin{cases}-\varepsilon^2\Delta\psi+\psi=(1-c_0)U^{p-1}_{\varepsilon,x} & y \text{ in } \Omega,\\ \dfrac{\partial\psi}{\partial n}=-\eta c_0\dfrac{\partial U_{\varepsilon,x}}{\partial n} & y \text{ on } \partial\Omega,\end{cases}$$

$$\begin{cases}-\varepsilon^2\Delta\psi+\psi=0 & y \text{ in } \Omega,\\ \dfrac{\partial\psi}{\partial n}=-(1-\eta)c_0\dfrac{\partial U_{\varepsilon,x}}{\partial n} & y \text{ on } \partial\Omega,\end{cases}$$

where η is a smooth function compactly contained in $B_{2\sigma}(q)$ and $\eta = 1$ if $y \in B_\sigma(q)$. Then,

$$\psi = \psi_1 + \psi_2.$$

It follows from the maximum principle that $\psi_1 \geq 0$. On the other hand, we have

$$\psi_2(y) = -\int_{\partial\Omega} (1-\eta) c_0 \frac{\partial U_{\varepsilon,x}(z)}{\partial n} G(z,y)\, dz.$$

Since $\eta = 1$ for $y \in B_\sigma(q)$, we see from the above relation that

$$|\psi_2(y)| \leq C e^{-(1+\sigma)d(x,\partial\Omega)/\varepsilon}, \quad \text{for all } y \in \partial\Omega \cap B_\sigma(q).$$

As a result,

$$\psi(y) \geq -C e^{-(1+\sigma)d(x,\partial\Omega)/\varepsilon}, \quad \text{for all } y \in \partial\Omega \cap B_\sigma(q).$$

Hence,

$$P_{\varepsilon,\Omega} U_{\varepsilon,x} \geq c_0 U_{\varepsilon,x} - C e^{-(1+\sigma)d(x,\partial\Omega)/\varepsilon} \geq c_0 e^{-(1+\theta)d(x,\partial\Omega)/\varepsilon},$$

for all $y \in \partial\Omega \cap B_\sigma(q)$, and the result follows. □

LEMMA 2.6. *We have*

$$\int_\Omega U_{\varepsilon,x}^{p-2} \frac{\partial U_{\varepsilon,x}}{\partial x_i} \varphi_{\varepsilon,x} = -\varepsilon^2 \int_{\partial\Omega} P_{\varepsilon,\Omega} U_{\varepsilon,x} \frac{\partial}{\partial n}\left(\frac{\partial U_{\varepsilon,x}}{\partial x_i}\right) + O\big(\varepsilon^{N-1} e^{-\mathrm{pd}(x,\partial\Omega)/\varepsilon}\big).$$

Moreover, if $\partial\Omega \cap \partial B_{d(x,\partial\Omega)}(x)$ contains exactly one point q, then

$$\sum_{i=1}^N \int_\Omega U_{\varepsilon,x}^{p-2} \frac{\partial U_{\varepsilon,x}}{\partial x_i} \varphi_{\varepsilon,x} \nu_i \geq c_0 \varepsilon^{N-1} e^{-(2+\theta)d(x,\partial\Omega)/\varepsilon},$$

for any $\theta > 0$, where ν is the outward unit normal of $\partial\Omega$ at q.

PROOF. Multiplying (2.1) by $\frac{\partial U_{\varepsilon,x}}{\partial x_i}$ and integrating by parts, we get

$$\begin{aligned}
\int_\Omega U_{\varepsilon,x}^{p-2} \frac{\partial U_{\varepsilon,x}}{\partial x_i} \varphi_{\varepsilon,x} &= \varepsilon^2 \int_{\partial\Omega} \frac{\partial \varphi_{\varepsilon,x}}{\partial n} \frac{\partial U_{\varepsilon,x}}{\partial x_i} - \varepsilon^2 \int_{\partial\Omega} \varphi_{\varepsilon,x} \frac{\partial}{\partial n}\left(\frac{\partial U_{\varepsilon,x}}{\partial x_i}\right) \\
&= \varepsilon^2 \int_{\partial\Omega} P_{\varepsilon,\Omega} U_{\varepsilon,x} \frac{\partial}{\partial n} \frac{\partial U_{\varepsilon,x}}{\partial x_i} \\
&\quad + \varepsilon^2 \int_{\partial\Omega} \frac{\partial U_{\varepsilon,x}}{\partial n} \frac{\partial U_{\varepsilon,x}}{\partial x_i} - \varepsilon^2 \int_{\partial\Omega} U_{\varepsilon,x} \frac{\partial}{\partial n}\left(\frac{\partial U_{\varepsilon,x}}{\partial x_i}\right).
\end{aligned} \tag{2.16}$$

But

$$\begin{aligned}
&\varepsilon^2 \int_{\partial\Omega} \frac{\partial U_{\varepsilon,x}}{\partial n} \frac{\partial U_{\varepsilon,x}}{\partial x_i} - \varepsilon^2 \int_{\partial\Omega} U_{\varepsilon,x} \frac{\partial}{\partial n}\left(\frac{\partial U_{\varepsilon,x}}{\partial x_i}\right) \\
&\quad = \int_\Omega \varepsilon^2 \left[\Delta U_{\varepsilon,x} \frac{\partial U_{\varepsilon,x}}{\partial x_i} - U_{\varepsilon,x} \Delta\left(\frac{\partial U_{\varepsilon,x}}{\partial x_i}\right)\right] \\
&\quad = -(p-2) \int_\Omega U_{\varepsilon,x}^{p-1} \frac{\partial U_{\varepsilon,x}}{\partial x_i} = (p-2) \int_{\mathbb{R}^N \setminus \Omega} U_{\varepsilon,x}^{p-1} \frac{\partial U_{\varepsilon,x}}{\partial x_i} \\
&\quad = O\big(\varepsilon^{N-1} e^{-\mathrm{pd}(x,\partial\Omega)/\varepsilon}\big).
\end{aligned} \tag{2.17}$$

Combining (2.16) and (2.17) yields

$$\int_\Omega U_{\varepsilon,x}^{p-2}\frac{\partial U_{\varepsilon,x}}{\partial x_i}\varphi_{\varepsilon,x} = -\varepsilon^2\int_{\partial\Omega} P_{\varepsilon,\Omega}U_{\varepsilon,x}\frac{\partial}{\partial n}\bigg(\frac{\partial U_{\varepsilon,x}}{\partial x_i}\bigg) + O\big(\varepsilon^{N-1}e^{-\mathrm{pd}(x,\partial\Omega)/\varepsilon}\big). \tag{2.18}$$

Suppose $\partial\Omega\cap\partial B_{d(x,\partial\Omega)}(x)$ contains exactly one point q. Since in a small neighbourhood of q,

$$\frac{\partial}{\partial n}\bigg(\frac{x_i-y_i}{|y-x|}\bigg) = -\frac{\langle n,\nu\rangle}{|y-x|} + \frac{\langle n,y-x\rangle\langle \nu,y-x\rangle}{|y-x|^3} = o(1),$$

and thus

$$\begin{aligned}\sum_{i=1}^N\frac{\partial}{\partial n}\bigg(\frac{\partial U_{\varepsilon,x}}{\partial x_i}\bigg)\nu_i &= \sum_{i=1}^N \frac{\partial^2 U_{\varepsilon,x}}{\partial r^2}\bigg\langle\frac{y-x}{|y-x|},n\bigg\rangle\frac{x_i-y_i}{|y-x|}\nu_i \\ &\quad+\sum_{i=1}^N\frac{\partial U_{\varepsilon,x}}{\partial r}\frac{\partial}{\partial n}\bigg(\frac{x_i-y_i}{|y-x|}\bigg)\nu_i \\ &\le -c_0\varepsilon^{N-1}e^{-(1+2\theta)d(x,\partial\Omega)/\varepsilon},\end{aligned}$$

which, together (2.15) and (2.18), gives the result. □

3. Interior Neumann Problem

Let

$$I(u) = \frac{1}{2}\int_\Omega(\varepsilon^2|Du|^2+u^2) - \frac{1}{p}\int_\Omega|u|^p, \quad u\in H^1(\Omega). \tag{3.1}$$

For fixed integer $k>0$, let

$$\alpha = (\alpha_1,\dots,\alpha_k)\in R^k, \tag{3.2}$$

$$x=(x_1,\dots,x_k)\in R^{kN}, \quad x_i\in R^N,\ i=1,\dots,k. \tag{3.3}$$

Define

$$\begin{aligned}D_{k,\delta} &= \{x: x\in\Omega,\ |x_i-x_j|\ge 2\delta,\ i\ne j,\ d(x_i,\partial\Omega)\ge\delta,\ i=1,\dots,k\},\\ M_{\varepsilon,\delta} &= \{(\alpha,x,v)\colon |\alpha_i-1|\le\delta,\ i=1,\dots,k;\ x\in D_{k,\delta},\\ &\qquad v\in E_{\varepsilon,x,k},\ \|v\|_\varepsilon\le\delta\varepsilon^{N/2}\}.\end{aligned}$$

Let

$$J(\alpha,x,v) = I\bigg(\sum_{i=1}^k\alpha_i P_{\varepsilon,\Omega}U_{\varepsilon,x_i}+v\bigg), \quad (\alpha,x,v)\in M_{\varepsilon,\delta}. \tag{3.4}$$

It is well known that if $\delta>0$ is small enough, $(\alpha,x,v)\in M_{\varepsilon,\delta}$ is a critical point of $J(\alpha,x,v)$ if and only if $u=\sum_{i=1}^k\alpha_i P_{\varepsilon,\Omega}U_{\varepsilon,x_i}+v$ is a positive critical point of $I(u)$. See [20]. So we just need to estimate the number of critical points $(\alpha, x, v) \in M_{\varepsilon,\delta}$ for $J(\alpha, x, v)$, that is, to find $(\alpha, x, v) \in M_{\varepsilon,\delta}$ and A_l, B_{li}, $i = 1, \dots, k$, such that

$$\frac{\partial J(\alpha, x, v)}{\partial x_{li}} = \sum_{j=1}^{N} B_{lj} \left\langle \frac{\partial^2 P_{\varepsilon,\Omega} U_{\varepsilon,x_l}}{\partial x_{li} \partial x_{lj}}, v \right\rangle_{\varepsilon}, \quad i = 1, \dots, N, \ l = 1, \dots, k, \tag{3.5}$$

$$\frac{\partial J(\alpha, x, v)}{\partial \alpha_l} = 0, \quad l = 1, \dots, k, \tag{3.6}$$

$$\frac{\partial J(\alpha, x, v)}{\partial v} = \sum_{l=1}^{k} A_l P_{\varepsilon,\Omega} U_{\varepsilon,x_l} + \sum_{l=1}^{k} \sum_{j=1}^{N} B_{lj} \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_l}}{\partial x_{lj}}. \tag{3.7}$$

We first reduce the problem of finding a critical point for $J(\alpha, x, v)$ to that of finding a critical point for a function defined in a finite dimensional domain. We will proceed in a similar way as [4], [8].

PROPOSITION 3.1. *There are $\varepsilon_0 > 0$ and $\delta > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, there is a unique C^1-map $(\alpha_\varepsilon(x), v_\varepsilon(x)) \colon D_{k,\delta} \to \mathbb{R}^k \times H^1(\Omega)$, satisfying $v_\varepsilon(x) \in E_{\varepsilon,x,k}$ and*

$$\frac{\partial J(\alpha_\varepsilon, x, v_\varepsilon)}{\partial \alpha_l} = 0, \quad l = 1, \dots, k, \tag{3.8}$$

$$\left\langle \frac{\partial J(\alpha_\varepsilon, x, v_\varepsilon)}{\partial v}, \omega \right\rangle_{\varepsilon} = 0, \quad \text{for all } \omega \in E_{\varepsilon,x,k}. \tag{3.9}$$

Besides, if $k \geq 2$, then for $l = 1, \dots, k$

$$|\alpha_{\varepsilon l} - 1| = O\bigg(\sum_{j=1}^{k} e^{-(1+\sigma) d(x_j, \partial\Omega)/\varepsilon} + \sum_{i \neq j} e^{-(1+\sigma)|x_i - x_j|/(2\varepsilon)} \bigg), \tag{3.10}$$

$$\|v_\varepsilon\|_\varepsilon = O\bigg(\varepsilon^{N/2} \bigg(\sum_{j=1}^{k} e^{-(1+\sigma) d(x_j, \partial\Omega)/\varepsilon} + \sum_{i \neq j} e^{-(1+\sigma)|x_i - x_j|/(2\varepsilon)} \bigg) \bigg), \tag{3.11}$$

and if $k = 1$, then

$$|\alpha_{\varepsilon l} - 1| \varepsilon^{N/2} + \|v_\varepsilon\|_\varepsilon = O\big(\varepsilon^{N/2} e^{-(1+\sigma) d(x, \partial\Omega)/\varepsilon} \big), \tag{3.12}$$

where σ is some positive constant. Moreover,

$$v_\varepsilon(\sigma_k x) = v_\varepsilon(x), \quad \sigma_k \alpha_\varepsilon(\sigma_k x) = \alpha_\varepsilon(x). \tag{3.13}$$

PROOF. The proof of the existence part is standard. See [4], [8], and also [1], [20]. The estimates (3.10) and (3.11) follows from the same procedure as in Proposition 2.3 of [8] and Lemmas 2.1 and 2.2. Finally, (3.13) is a direct consequence of the fact $J(\alpha, x, v) = J(\sigma_k \alpha, \sigma_k x, v)$ and the uniqueness of $(\alpha_\varepsilon(x), v_\varepsilon(x))$ satisfying (3.6) and (3.7). We thus omit the details. □

Let $(\alpha_\varepsilon(x), v_\varepsilon(x))$ be the function attained in Proposition 3.1. Define

$$K([x]) = J(\alpha_\varepsilon(x), x, v_\varepsilon(x)), \quad [x] \in D_{k,\delta}/\sigma_k. \tag{3.14}$$

Then from (3.13) we see $K([x])$ is well defined in $D_{k,\delta}/\sigma_k$.

LEMMA 3.2. *$D_{k,\delta}$ is a covering space of $D_{k,\delta}/\sigma_k$.*

PROOF. For any $[x] \in D_{k,\delta}/\sigma_k$, we have $|x_i - x_j| \geq 2\delta$, for all $i \neq j$. So we can choose $\gamma > 0$ small enough, such that $B_\gamma(x_i) \cap B_\gamma(x_j) = \emptyset$. Suppose that there are $y, z \in B_\gamma(x_1) \times \ldots \times B_\gamma(x_k)$ with $[y] = [z]$. Then $y = \sigma_k z$, and thus $|y - z| > 4\delta$. This is a contradiction. $\square$

It follows from the lifting path theorem that $x \in D_{k,\delta}$ is a critical point of $J(\alpha_\varepsilon(x), x, v_\varepsilon(x))$ if and only if $[x] \in D_{k,\delta}/\sigma_k$ is a critical point of $K([x])$.

PROOF OF THEOREM 1.1. From Proposition 3.1, for any $x \in D_{\varepsilon,R}$, we have

$$\begin{aligned} &J(\alpha_\varepsilon(x), x, v_\varepsilon(x)) \\ &\quad = J(1,x,0) + O\left[\varepsilon^N\left(\sum_{i=1}^k e^{-(2+\sigma)d(x_i,\partial\Omega)/\varepsilon} + \sum_{i\neq j} e^{-(1+\sigma)|x_i-x_j|/\varepsilon}\right)\right]. \end{aligned} \tag{3.15}$$

But in view of lemmas 2.3 and 2.4, we have

$$\begin{aligned} J(1,x,0) = {}&\varepsilon^N\left(\frac{1}{2} - \frac{1}{p}\right)kA + \frac{1}{2}\sum_{i=1}^k \tau_{\varepsilon,x_i} - \int_\Omega \sum_{j=1}^{k-1}\left(\sum_{i=j+1}^k U_{\varepsilon,x_i}\right)^{p-1} U_{\varepsilon,x_j} \\ &+ O\left[\varepsilon^N\left(\sum_{i=1}^k e^{-(2+\sigma)d(x_i,\partial\Omega)/\varepsilon} + \sum_{i\neq j} e^{-(1+\sigma)|x_i-x_j|/\varepsilon}\right)\right], \end{aligned} \tag{3.16}$$

where $A = \int_{R^N} U^p$. Inserting (3.16) into (3.15), we obtain

$$\begin{aligned} &J(\alpha_\varepsilon(x), x, v_\varepsilon(x)) \\ &\quad = \varepsilon^N\left(\frac{1}{2} - \frac{1}{p}\right)kA + \frac{1}{2}\sum_{i=1}^k \tau_{\varepsilon,x_i} - \int_\Omega \sum_{j=1}^{k-1}\left(\sum_{i=j+1}^k U_{\varepsilon,x_i}\right)^{p-1} U_{\varepsilon,x_j} \\ &\quad + O\left[\varepsilon^N\left(\sum_{i=1}^k e^{-(2+\sigma)d(x_i,\partial\Omega)/\varepsilon} + \sum_{i\neq j} e^{-(1+\sigma)|x_i-x_j|/\varepsilon}\right)\right]. \end{aligned} \tag{3.17}$$

Define

$$c_{\varepsilon,k} = k\varepsilon^N\left(\frac{1}{2} - \frac{1}{p}\right)A - e^{-3\delta/\varepsilon}.$$

Then we see from (3.17) that

$$J(\alpha_\varepsilon(x), x, v_\varepsilon(x)) < c_{\varepsilon,k},$$

if $d(x_i, \partial\Omega) = \delta$ for some i, or $|x_i - x_j| = 2\delta$ for some $i \neq j$. That is

$$K([x]) < c_{\varepsilon,k}, \quad \text{for all } [x] \in \partial(D_{\delta,k}/\sigma_k).$$

So from the Ljusternik–Schnirelman theory, we have

$$\#\{[x] : DK([x]) = 0, K([x]) \geq c_{\varepsilon,k}\} \geq \mathrm{Cat}_{D_{\delta,k}/\sigma_k}(\{K([x]) \geq c_{\varepsilon,k}\}). \tag{3.18}$$

On the other hand, it is easy to check from (3.17) that

$$D_{4\delta,k}/\sigma_k \subset \{K([x]) \geq c_{\varepsilon,k}\}. \tag{3.19}$$

Combining (3.18) and (3.19), we obtain

$$\#\{[x] : DK([x]) = 0, K([x]) \geq c_{\varepsilon,k}\} \geq \mathrm{Cat}_{D_{\delta,k}/\sigma_k}(D_{4\delta,k}/\sigma_k) = \mathrm{Cat}_{A_k} A_k.$$

So we have completed the proof of Theorem 1.1. □

PROOF OF THEOREM 1.2. Let $(\alpha_\varepsilon(x), v_\varepsilon(x))$ be the map obtained in Proposition 3.1. Define

$$K_1(x) = J_\varepsilon(\alpha_\varepsilon(x), x, v_\varepsilon(x)), \quad \text{for all } x \in \Omega_\delta,$$

Then

$$K_1(x) = \left(\frac{1}{2} - \frac{1}{p}\right)\varepsilon^N A + \frac{1}{2}\tau_{\varepsilon,x} + o(\tau_{\varepsilon,x}). \tag{3.20}$$

Let

$$c_{\varepsilon,1} = \varepsilon^N \left(\frac{1}{2} - \frac{1}{p}\right) A - e^{-3\delta/\varepsilon}.$$

It is not difficult to see that $K_1(x) < c_{\varepsilon,1}$ if $x \in \partial\Omega_\delta$ and $\Omega_{4\delta} \subset \{K_1(x) \geq c_{\varepsilon,1}\}$. So the result follows from the Ljusternik–Schnirelman theory. □

REMARK 3.3. The idea to prove Theorem 1.1 can also be used to estimate the number of boundary k-peak solutions for (1.1). So we see that the number of boundary k-peak solutions for (1.1) is at least $\mathrm{Cat}_{A'_k}(A'_k)$, where

$$\begin{aligned} A'_k = \{(x_1, \ldots, x_k) : x_j \in \partial\Omega, H(x_j) \geq \max_{\partial\Omega} H(x) - \delta,& \\ |x_i - x_j| \geq \delta,\ i, j = 1, \ldots, k,\ i \neq j\}/\sigma_k,& \end{aligned}$$

where $H(x)$ is the mean curvature function of $\partial\Omega$.

REMARK 3.4. By (3.20), it is easy to see that if $x_0 \in \Omega$ is a strictly local maximum point of the function $d(x, \partial\Omega)$, we can construct a solution of the form $u_\varepsilon = \alpha_\varepsilon P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon} + v_\varepsilon$ satisfying $\alpha_\varepsilon \to 1$, $x_\varepsilon \to x_0$, $v_\varepsilon \in E_{\varepsilon,x_\varepsilon}$ and $\|v_\varepsilon\|_\varepsilon^2 = o(\varepsilon^N)$ as $\varepsilon \to 0$.

It is also interesting to characterize the location of the peaks of the interior k-peak solutions for (1.1). This is not easy if $k \geq 3$, because it is very difficult to control the distances between different peaks. On the other hand, consider the following problem

$$\max\{K(x) : x \in D_{k,\delta}\}. \tag{3.21}$$

From (3.17), we see that $K(x)$ is decreasing if one of x_j moves toward $\partial\Omega$ or $|x_i - x_j| \to 0$ for some $i \neq j$. This implies that the maximum x_ε of (3.21) is an interior point in $D_{k,\delta}$ and hence a critical point of $K(x)$. As a result, (1.1) always has a interior k-peak solution

$$u_\varepsilon = \sum_{j=1}^{k} \alpha_{\varepsilon j} P_{\varepsilon,\Omega} U_{\varepsilon,x_{\varepsilon j}} + v_\varepsilon,$$

such that $x_\varepsilon = (x_{\varepsilon 1}, \dots, x_{\varepsilon k})$ is a maximum of problem (3.21). Moreover, from (3.17), we see that as $\varepsilon \to 0$, x_ε tends to a point which is a maximum point of the following function

$$\min\{2d(x_i, \partial\Omega),\ |x_i - x_j|,\ i,j = 1,\dots,k,\ i \neq j\}. \tag{3.22}$$

In order to locate the maximum of (3.22), we only need to put k disjoint open balls $B_\eta(x_i)$ in Ω, and try to make η as large as possible. Let S be the set of all the points $(x_1, \dots, x_k)$ such that x_j, $j = 1, \dots, k$, is the center of $B_\eta(x_j)$ that makes η attain its maximum. Then the maximum point of (3.22) is contained in S.

Before we close this section, we discuss briefly the location of the peak of the interior single peak solution.

Proposition 3.5. *Suppose that $u_\varepsilon = \alpha_\varepsilon P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon} + v_\varepsilon$ is an interior single peak solution for* (1.1)*, satisfying $\alpha_\varepsilon \to 1$, $v_\varepsilon \in E_{\varepsilon,x_\varepsilon}$, $\|v_\varepsilon\|_\varepsilon = o(\varepsilon^{N/2})$ and $x_\varepsilon \to x \in \Omega$ as $\varepsilon \to 0$, then x satisfies*

$$\int_{\partial\Omega} \frac{y - x}{|y - x|}\, d\mu = 0,$$

where $d\mu$ is a measure on $\partial\Omega$, which is one of the weak limits of the sequence

$$\frac{P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon} \dfrac{\partial^2 U_{\varepsilon,x_\varepsilon}}{\partial^2 r}}{\displaystyle\int_{\partial\Omega} P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon} \frac{\partial^2 U_{\varepsilon,x_\varepsilon}}{\partial^2 r}}.$$

Proof. Since $u_\varepsilon = \alpha_\varepsilon P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon} + v_\varepsilon$ is solution of (1.1), we know

$$|\alpha_\varepsilon - 1| + \varepsilon^{-N/2} \|v_\varepsilon\|_\varepsilon = O\big(e^{-(1+\sigma)d(x,\partial\Omega)/\varepsilon}\big). \tag{3.23}$$

See the proof of Proposition 3.1. On the other hand, we have

$$\left\langle \alpha_\varepsilon P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon} + v_\varepsilon, \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon}}{\partial x_j} \right\rangle = \int_\Omega \big(\alpha_\varepsilon P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon} + v_\varepsilon\big)^{p-1} \frac{\partial P_{\varepsilon,\Omega} U_{\varepsilon,x_\varepsilon}}{\partial x_j},$$

which, in view of (3.23), is equivalent to

$$\int_\Omega U_{\varepsilon,x_\varepsilon}^{p-2} \frac{\partial U_{\varepsilon,x_\varepsilon}}{\partial x_j} \varphi_{\varepsilon,x} = O\big(\varepsilon^{N-1} e^{-(2+\sigma)d(x,\partial\Omega)/\varepsilon}\big) \tag{3.24}$$

By Lemma 2.6, we deduce from the above relation that

$$\int_{\partial\Omega} P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon}\frac{\partial^2 U_{\varepsilon,x_\varepsilon}}{\partial^2 r}\frac{y_j - x_{\varepsilon j}}{|y_j - x_{\varepsilon j}|} + o\biggl(\int_{\partial\Omega} P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon}\frac{\partial^2 U_{\varepsilon,x_\varepsilon}}{\partial^2 r}\biggr) = O\bigl(\varepsilon^{N-1}e^{-(2+\sigma)d(x_\varepsilon,\partial\Omega)/\varepsilon}\bigr),$$

where $r = |y-x|$. Using Lemma 2.5, we obtain

$$\frac{\displaystyle\int_{\partial\Omega} P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon}\frac{\partial^2 U_{\varepsilon,x_\varepsilon}}{\partial^2 r}\frac{y_j - x_{\varepsilon j}}{|y_j - x_{\varepsilon j}|}}{\displaystyle\int_{\partial\Omega} P_{\varepsilon,\Omega}U_{\varepsilon,x_\varepsilon}\frac{\partial^2 U_{\varepsilon,x_\varepsilon}}{\partial^2 r}} = o(1),$$

Thus,

$$\int_{\partial\Omega}\frac{y_j - x_j}{|y-x|}\,d\mu = 0.$$

This completes the proof of the lemma. □

REMARK 3.6. It is easy to see that the support of the measure μ contains in

$$\partial\Omega \cap \{y : |y-x| = d(x,\partial\Omega)\},$$

and if $\partial\Omega \cap \{y : |y-x| = d(x,\partial\Omega)\} = \{x_0\}$, then $\mu = \delta_{x_0}$. So if $x \in \Omega$ is a point such that there is an interior single peak solution for (1.1) with its peak near x, then $\partial\Omega \cap \{y : |y-x| = d(x,\partial\Omega)\}$ contains at least two points. The result in Proposition 3.5 is similar to that in [23]. It is interesting to consider whether the measure defined here and the measure defined in [23] are same.

4. Exterior Neumann Problem

The aim of this section is to prove Theorem 1.3. Since most of the calculations are similar to those in Section 3, we merely sketch the proof. First, we need the following lemma.

LEMMA 4.1. *Let X be a topological space and let A_1, A_2 and Y be closed subsets of X satisfying $A_1 \subset Y \subset A_2$. Suppose that A_2 can be deformed into A_1, that is, there is a continuous map $H(x,t) : A_2 \times [0,1] \to A_2$ such that $H(x,0) = x$, $H(x,1) \subset A_1$, $H(x,t) = x$, for all $x \in A_1$, $t \in [0,1]$, then*

$$\mathrm{Cat}_X(X,Y) \geq \mathrm{Cat}_X(X,A_1).$$

PROOF. Suppsoe that $\mathrm{Cat}_X(X,Y) = m$. Then there are $X_0,\dots,X_m$, and continuous maps $h_0(x,t),\dots,h_m(x,t)$, such that

$$X = \bigcup_{l=0}^{m} X_l,$$

(i) $h_l(0,x) = x$, $l = 0,\dots,m$;

(ii) $h_l(1,x)=x_l$, for all $x\in X_l$, $l=1,\dots,m$;
(iii) $h_0(1,X_0)\subset Y$ and $h_0(x,t)=x$, for all $t\in[0,1]$, $x\in Y\cap X_0$.

Define

$$h_0^*(t,x)=\begin{cases} h_0(2t,x) & t\in[0,1/2],\ x\in X_0,\\ H(2t-1,h_0(1,x)) & t\in[1/2,1],\ x\in X_0.\end{cases}$$

Since $h_0(1,X_0)\subset Y\subset A_2$, we see that $h_0^*(t,x)$ is well defined in $[0,1]\times X_0$ and is continuous. It is easy to see that $h_0^*(0,x)=x$, $h_0^*(1,X_0)\subset A_1$. Moreover, for any $x\in A_1\cap X_0$ and $t\in[0,1]$, we have

$$h_0^*(t,x)=\begin{cases} h_0(2t,x)=x & \text{for all } t\in[0,1/2],\\ H(2t-1,h_0(x,1))=H(2t-1,x)=x & \text{for all } t\in[1/2,1].\end{cases}$$

Thus, by the definition of the relative category, we have $\mathrm{Cat}_X(X,A_1)\le m$. □

Proof of Theorem 1.3. (i) We define $D_T=B_T(0)\cap\Omega_{1\delta}$, where $T>0$ is a large constant. Let

$$c'_{\varepsilon,2}=\varepsilon^N(A-e^{-d_2/\varepsilon}),\quad c'_{\varepsilon,1}=\varepsilon^N(A-e^{-d_1/\varepsilon}),$$

where $d_2>0$ is a large constant with $T>2d_2$, and $d_1>0$ is a small constant with $d_1<\delta$. Let

$$J_2(\alpha,x,v)=:I(\alpha P_{\varepsilon,\Omega}U_{\varepsilon,x}+v).$$

Step 1. It is standard to prove that there is an $\varepsilon_0>0$, such that for each $\varepsilon\in(0,\varepsilon_0]$, there is a C^1-map $(\alpha_\varepsilon(x),v_\varepsilon(x)):D_T\to\mathbb{R}^+\times H^1(\Omega)$, such that $v_\varepsilon(x)\in E_{\varepsilon,x,1}$ and

$$\frac{\partial J_2}{\partial\alpha}=0 \tag{4.1}$$

$$\left\langle\frac{\partial J_2}{\partial v},\omega\right\rangle=0,\quad\text{for all }\omega\in E_{\varepsilon,x,1}. \tag{4.2}$$

Moreover, for some $\sigma>0$,

$$|\alpha_\varepsilon-1|\varepsilon^{N/2}+\|v_\varepsilon\|_\varepsilon=O\big(\varepsilon^{N/2}e^{-(1+\sigma)d(x,\partial\Omega_1)/\varepsilon}\big).$$

Step 2. Define

$$K_2(x)=I(\alpha_\varepsilon(x)P_{\varepsilon,\Omega}U_{\varepsilon,x}+v_\varepsilon(x)),\quad x\in D_T.$$

Then it follows from Lemma 2.1 that

$$\begin{aligned} K_2(x)&<c'_{\varepsilon,1},\quad\text{if } x\in\partial\Omega_{1\delta},\\ K_2(x)&>c'_{\varepsilon,2},\quad\text{if } |x|=T.\end{aligned}$$

Combining Steps 1–2, we conclude

$$\#\{x:DK_2(x)=0,\ x\in D_T,\ c'_{\varepsilon,1}<K_2(x)\le c'_{\varepsilon,2}\}\ge\mathrm{Cat}_{D_T}(K_{2,c_{\varepsilon,2}},K_{2,c_{\varepsilon,1}}),$$

where $K_{2,c} = \{x : K_2(x) > c\}$. But it is easy to check that $K_{2,c_{\varepsilon,1}} \subset D_T$, and

$$\{x : T/2 \le |x| \le T\} \subset K_{2,c_{\varepsilon,2}} \subset \{x : d_2/2 \le |x| \le T\}.$$

So by Lemma 4.1, we have

$$\#\{x : DK_2(x) = 0, x \in D_T,\ c'_{\varepsilon,1} < K_2(x) \le c'_{\varepsilon,2}\} \\ \ge \mathrm{Cat}_{D_T}(D_T, \{x : T/2 \le |x| \le T\})$$

and the result follows.

(ii) As in the proof of Proposition 3.5, we have

$$\int_{\partial\Omega_1} P_{\varepsilon,\Omega_1} U_{\varepsilon,x_\varepsilon} \frac{\partial^2 U_{\varepsilon,x_\varepsilon}}{\partial r^2} \frac{y_j - x_{\varepsilon j}}{|y_j - x_{\varepsilon j}|} + o\bigg(\int_{\partial\Omega_1} P_{\varepsilon,\Omega_1} U_{\varepsilon,x_\varepsilon} \frac{\partial^2 U_{\varepsilon,x_\varepsilon}}{\partial r^2}\bigg) \\ = O\big(e^{-(2+\sigma)d(x_\varepsilon,\partial\Omega_1)/\varepsilon}\big). \tag{4.3}$$

Since $\mathbb{R}^N \setminus \Omega_1$ is convex, we know that $\partial\Omega_1 \cap B_{d(x_\varepsilon,\partial\Omega)}(x_\varepsilon)$ contains exactly one point q, and $\langle (y - x_\varepsilon)/|y_j - x_{\varepsilon j}|, n\rangle \ge \beta > 0$ for y in a small neighbourhood of q, where n is the outward unit normal of $\partial\Omega_1$ at q. So, (4.3), together with Lemma 2.5, implies

$$e^{-(2-\theta)d(x_\varepsilon,\partial\Omega)/\varepsilon} \le O\big(e^{-(2+\sigma)d(x_\varepsilon,\partial\Omega)/\varepsilon}\big),$$

for each $\theta > 0$. This is a contradiction. □

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