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ON THE HOMOTOPY TYPE OF VMO

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Dedicated to Louis Nirenberg on the occasion of his 70th birthday

Let X and Y be two compact smooth manifolds without boundary. Recently Brezis and Nirenberg [1] have developed a topological degree theory for maps belonging to the class $VMO(X,Y)$ (Vanishing Mean Oscillation). This class is strictly larger than the class of continuous maps.

They have also proved a fact, which is closely related to the possibility of defining the degree: the connected components of $VMO(X,Y)$ are the closures of the connected components of $C(X,Y)$ (recall that the space of continuous maps $C(X,Y)$ is dense in $VMO(X,Y)$). Therefore there is a bijection between the 0-homotopy sets $\pi_0(C(X,Y))$ and $\pi_0(VMO(X,Y))$.

Here we generalize this fact proving that the inclusion map $i : C(X,Y) \hookrightarrow VMO(X,Y)$ is a homotopy equivalence: there exists a continuous map $r : VMO(X,Y) \mapsto C(X,Y)$ such that $r \circ i$ is homotopic to the identity on $C(X,Y)$ and $i \circ r$ is homotopic to the identity on $VMO(X,Y)$.

A well known theorem of Whitney asserts that the inclusions $C^k(X,Y) \hookrightarrow C(X,Y)$ are homotopy equivalences for $1 \leq k \leq \infty$ (see [2]). Therefore we can add VMO to Whitney's sequence of inclusions and state that

$$C^\infty(X,Y) \hookrightarrow C^k(X,Y) \hookrightarrow \ldots \hookrightarrow C^1(X,Y) \hookrightarrow C(X,Y) \hookrightarrow VMO(X,Y)$$

is a sequence of homotopy equivalences.

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In particular, all the homotopy, homology and cohomology groups of VMO are isomorphic, via the homomorphisms induced by inclusion, to the homotopy, homology and cohomology groups of $C^\infty(X,Y)$.

1. Basic properties of VMO

In this section the definitions and some useful properties of the spaces BMO and VMO are summarized. See [1] for full details and proofs.

Let X and Y be compact smooth manifolds without boundary. Put a smooth Riemannian structure on X: it induces a volume form σ on X. Let r_0 be the injectivity radius of X and let $B_\varepsilon(x)$ be the geodesic ball in X of radius $\varepsilon < r_0$ centered at x. For every $u \in L^1(X,\mathbb{R}^N)$, set

$$\overline{u}_\varepsilon(x) = \fint_{B_\varepsilon(x)} u(y)\,d\sigma(y) = \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} u(y)\;d\sigma(y)$$

where $|B|$ is the σ-volume of $B \subset X$. Then $\overline{u}_\varepsilon$ is a continuous map.

$BMO(X,\mathbb{R}^N)$ is the space of maps $u \in L^1(X,\mathbb{R}^N)$ such that

$$\|u\|_{BMO} = \sup_{\substack{\varepsilon<r_0\\ x\in X}} \fint_{B_\varepsilon(x)} |u(y)-\overline{u}_\varepsilon(x)|\,d\sigma(y) < \infty.$$

$\|\cdot\|_{BMO}$ is a seminorm on BMO and it is a norm on the quotient space $BMO(X,\mathbb{R}^N)/\mathbb{R}^N$, where $\mathbb{R}^N$ denotes the subspace of constant maps.

$BMO(X,\mathbb{R}^N)/\mathbb{R}^N$ is complete under this norm.

There exists a constant C, depending only on the Riemannian structure of X, such that for every $u \in BMO(X,\mathbb{R}^N)$,

$$\|u\|_{L^1} \le C\|u\|_{BMO} + \left|\int_X u(y)\,d\sigma(y)\right|.$$

This fact is proved in [1], Lemma A.1. It implies that BMO convergence is stronger than L^1 convergence (modulo constants). Therefore we can define a complete norm on $BMO(X,\mathbb{R}^N)$, which induces on $BMO(X,\mathbb{R}^N)/\mathbb{R}^N$ the norm $\|\cdot\|_{BMO}$, by setting

$$|||u|||_{BMO} = \|u\|_{BMO} + \|u\|_{L^1}.$$

$VMO(X,\mathbb{R}^N)$ is the closure of $C(X,\mathbb{R}^N)$ under the $|||\cdot|||_{BMO}$ norm. Assume that F is a closed subset of $\mathbb{R}^N$ and set

$$VMO(X,F) = \left\{u \in VMO(X,\mathbb{R}^N) \mid u(x) \in F \text{ for } \sigma\text{-a.e. } x \in X\right\}.$$

Since L^1 convergence implies a.e. convergence of a subsequence, $VMO(X,F)$ is a closed subset of $VMO(X,\mathbb{R}^N)$. Therefore it is a complete metric space, with the distance induced by the $|||\cdot|||_{BMO}$ norm.

Now we assume that the compact manifold Y is smoothly embedded in $\mathbb{R}^N$. The set $VMO(X,Y)$ and its topology do not depend on the Riemannian structure of X and on the embedding of Y (see [1], Section I.1).

Reamark 1.1. We point out a difference in notation from Brezis and Nirenberg's paper: they consider VMO with only the $\|\cdot\|_{BMO}$ seminorm and therefore they identify maps which differ by a constant. When they want to consider VMO as we do, they denote it by $VMO \cap L^1$. There is no difference if we deal with vector valued maps, but the situation changes if we deal with manifold valued maps. For example our homotopy result cannot be true in general when maps which differ by a constant are identified, as Remark A.6 of [1] shows.

The inclusion map $C(X,Y) \hookrightarrow VMO(X,Y)$ is continuous (see [1], Remark 2), and therefore the uniform topology is stronger than the VMO topology. Moreover, the continuous maps are dense in $VMO(X,Y)$ (see [1], Corollary 4) and therefore $VMO(X,Y)$ is separable.

For $u \in BMO(X,\mathbb{R}^N)$ and $0 < a < r_0$ set

$$M_a(u) = \sup_{\substack{\varepsilon \le a \\ x \in X}} \fint_{B_\varepsilon(x)} |u(y) - \overline{u}_\varepsilon(x)|\, d\sigma(y) \le \|u\|_{BMO}.$$

Then M_a is a seminorm on $BMO(X,\mathbb{R}^N)$, equivalent to $\|\cdot\|_{BMO}$.

The following proposition summarizes some basic properties of BMO and VMO:

Proposition 1.1. (a) *There is a constant A, depending only on the Riemannian structure of X, such that*

$$\|u - \overline{u}_\varepsilon\|_{BMO} \le AM_\varepsilon(u) \quad \forall \varepsilon < r_0,\ \forall u \in BMO(X,\mathbb{R}^N).$$

(b) *If $\mathcal{K}$ is a compact subset of $VMO(X,\mathbb{R}^N)$ then*

$$\lim_{\varepsilon \to 0} M_\varepsilon(u) = 0 \quad \textit{uniformly in } u \in \mathcal{K}.$$

(c) *If $F : \mathbb{R}^N \to \mathbb{R}^M$ is uniformly continuous, then the map*

$$VMO(X,\mathbb{R}^N) \ni u \mapsto F \circ u \in VMO(X,\mathbb{R}^M)$$

is well defined and it is continuous.

All these properties are proved in [1]: see Lemma A.5 for (a), Lemma 4 for (b) and the more general Lemma A.8 for (c).

2. The homotopy equivalence

Theorm 2.1. *The inclusion map $i : C(X,Y) \hookrightarrow VMO(X,Y)$ is a homotopy equivalence.*

Let U be a tubular neighbourhood of Y in $\mathbb{R}^N$. Assume that $\operatorname{dist}(y,Y) < r$ implies $y \in U$. Let $P : U \to Y$ be the projection on the nearest point of Y: we can assume that P is Lipschitz continuous.

If $u \in VMO(X,Y)$, by Proposition 1.1(b) there exists a positive number $\varepsilon(u) < r_0$ such that $M_{\varepsilon(u)}(u) < r/2$.

The function $M_{\varepsilon(u)} : VMO(X,Y) \to \mathbb{R}$ is continuous and therefore we can find a ball in $VMO(X,Y)$ centered at u of radius $\delta(u)$ such that

$$M_{\varepsilon(u)}(v) < r \quad \forall v \in B(u,\delta(u)) \subset VMO(X,Y). \tag{2.1}$$

Being a metric space, $VMO(X,Y)$ is paracompact. Let $\{U_j \mid j \in J\}$ be a locally finite refinement of the open covering $\{B(u,\delta(u)) \mid u \in VMO(X,Y)\}$, where J is a set of indices. Let $\{\varphi_j \mid j \in J\}$ be a partition of unity associated with $\{U_j \mid j \in J\}$:

(i) $\varphi_j : VMO(X,Y) \to [0,1]$ is continuous;
(ii) the support of φ_j is a subset of U_j;
(iii) all but a finite number of the φ_j vanish on some neighbourhood of each $u \in VMO(X,Y)$;
(iv) $\sum_{j\in J} \varphi_j(u) = 1$ for every $u \in VMO(X,Y)$.

Take a family of $u_j \in VMO(X,Y)$, $j \in J$, such that $U_j \subset B(u_j,\delta(u_j))$. Define a function $\overline{\varepsilon} : VMO(X,Y) \to \,]0,\infty[$ as

$$\overline{\varepsilon}(u) = \sum_{j\in J} \varphi_j(u)\varepsilon(u_j).$$

Then $\overline{\varepsilon}$ is continuous because it is, locally, the sum of a finite number of continuous functions.

If $u \in VMO(X,Y)$, let $j_1,\dots,j_k$ be the indices for which $\varphi_{j_i}(u) > 0$. We can assume that $\varepsilon(u_{j_1}) = \max_{i=1,\dots,k} \varepsilon(u_{j_i})$. Now $\overline{\varepsilon}(u)$ is a convex combination of numbers which are not greater than $\varepsilon(u_{j_1})$, therefore $\overline{\varepsilon}(u) \le \varepsilon(u_{j_1})$. Since $u \in \operatorname{supp}\varphi_{j_1} \subset U_{j_1} \subset B(u_{j_1},\delta(u_{j_1}))$, by (2.1) we have

$$M_{\overline{\varepsilon}(u)}(u) \le M_{\varepsilon(u_{j_1})}(u) < r. \tag{2.2}$$

By (2.2), for every $(u,t) \in VMO\times]0,1]$

$$\operatorname{dist}(\overline{u}_{t\overline{\varepsilon}(u)}(x),Y) \le \fint_{B_{t\overline{\varepsilon}(u)}(x)} |\overline{u}_{t\overline{\varepsilon}(u)}(x) - u(y)|\, d\sigma(y) \le M_{\overline{\varepsilon}(u)}(u) < r \quad \forall x \in X.$$

Therefore we can define a map $H : VMO(X,Y) \times [0,1] \to VMO(X,Y)$ as

$$H(u,t) = \begin{cases} P \circ \overline{u}_{t\overline{\varepsilon}(u)} & \text{if } t > 0, \\ u & \text{if } t = 0. \end{cases}$$

In order to prove continuity properties of H, we need a lemma.

LEMMA 2.2. *Assume that* $u^n, u \in VMO(X,Y)$ *and that* $s_n, s \in\ ]0, r_0[$ *for* $n \in \mathbb{N}$. *Then:*

(1) *if* $u^n \to u$ *in* L^1 *and* $s_n \to s$, *then* $\overline{u^n}_{s_n} \to \overline{u}_s$ *uniformly;*
(2) *if* $u^n \to u$ *in* L^1 *and* $s_n \to 0$, *then* $\overline{u^n}_{s_n} \to u$ *in* L^1*;*
(3) *if* $\|u^n - u\|_{BMO} \to 0$ *and* $s_n \to 0$, *then* $\|\overline{u^n}_{s_n} - u\|_{BMO} \to 0$*;*
(4) *if* $u^n \in C(X,Y)$, $u^n \to u$ *uniformly and* $s_n \to 0$, *then* $\overline{u^n}_{s_n} \to u$ *uniformly.*

PROOF. (1) If B_a is the smallest measure of a geodesic ball of radius a in X, then

$$\begin{aligned}
&|\overline{u^n}_{s_n}(x) - \overline{u}_s(x)| \\
&\qquad \le |\overline{u^n}_{s_n}(x) - \overline{u}_{s_n}(x)| + |\overline{u}_{s_n}(x) - \overline{u}_s(x)| \\
&\qquad \le \fint_{B_{s_n}(x)} |u^n(y) - u(y)|\, d\sigma(y) \\
&\qquad\quad + \left| \frac{1}{|B_{s_n}(x)|} \int_{B_{s_n}(x)} u(y)\, d\sigma(y) - \frac{1}{|B_s(x)|} \int_{B_s(x)} u(y)\, d\sigma(y) \right| \\
&\qquad \le \frac{1}{B_{s_n}} \int_X |u^n(y) - u(y)|\, d\sigma(y) + \left| \frac{1}{|B_{s_n}(x)|} - \frac{1}{|B_s(x)|} \right| \int_X |u(y)|\, d\sigma(y) \\
&\qquad\quad + \frac{1}{B_s} \int_{B_{s_n}(x) \triangle B_s(x)} |u(y)|\, d\sigma(y)
\end{aligned}$$

where $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Since $|B_{s_n}(x)| \to |B_s(x)|$ uniformly in $x \in X$, the above inequalities imply assertion (1).

(2) It is a standard fact that $\overline{u}_{s_n} \to u$ in L^1. Moreover, there exists a constant D such that

$$\|\overline{v}_s\|_{L^1} \le D\|v\|_{L^1} \quad \forall v \in L^1(M, \mathbb{R}^N),\ \forall s \in\ ]0, r_0[\,.$$

Therefore the inequalities

$$\|\overline{u^n}_{s_n} - u\|_{L^1} \le \|\overline{u^n}_{s_n} - \overline{u}_{s_n}\|_{L^1} + \|\overline{u}_{s_n} - u\|_{L^1} \le D\|u^n - u\|_{L^1} + \|\overline{u}_{s_n} - u\|_{L^1}$$

imply assertion (2).

(3) By Proposition 1.1(a)

$$\|\overline{u^n}_{s_n} - u\|_{BMO} \le \|\overline{u^n}_{s_n} - u^n\|_{BMO} + \|u^n - u\|_{BMO} \le AM_{s_n}(u^n) + \|u^n - u\|_{BMO}$$

and assertion (3) follows from Proposition 1.1(b).

(4) For every $x \in X$,

$$\begin{aligned}
|\overline{u^n}_{s_n}(x) - u(x)| &\le \fint_{B_{s_n}(x)} |u^n(y) - u(x)|\, d\sigma(y) \\
&\le \|u^n - u\|_\infty + \fint_{B_{s_n}(x)} |u(y) - u(x)|\, dx
\end{aligned}$$

and the uniform continuity of u implies assertion (4).

We are now ready to prove Theorem 2.1. Define $r : VMO(X,Y) \to C(X,Y)$ as $r(u) = H(u,1)$.

Since the VMO topology is stronger than the L^1 topology, and since $\overline{\varepsilon}$ is continuous on $VMO(X,Y)$, by Lemma 2.2(1) the map $VMO(X,Y) \ni u \mapsto u_{\overline{\varepsilon}(u)} \in C(X,U)$ is continuous. Since composition with the Lipschitz continuous map P is continuous from $C(X,U)$ to $C(X,Y)$, r is continuous.

Since the VMO topology is weaker than the uniform topology, but stronger than the L^1 topology, and since $\overline{\varepsilon}$ is continuous on $VMO(X,Y)$, by Lemma 2.2(1) the map $VMO \times]0,1] \ni (u,t) \mapsto \overline{u}_{t\overline{\varepsilon}(u)} \in VMO(X,\mathbb{R}^N)$ is continuous. By Proposition 1.1(c), the map $H : VMO(X,Y) \times]0,1] \to VMO(X,Y)$ is continuous.

The continuity of H on $VMO(X,Y) \times [0,1]$ follows from Lemma 2.2(2) and (3), from the continuity of $\overline{\varepsilon}$ and from Proposition 1.1(c).

Therefore $H : VMO(X,Y) \times [0,1] \to VMO(X,Y)$ is a homotopy between $i \circ r$ and the identity on $VMO(X,Y)$.

Let $\overline{H}$ be the restriction of H to $C(X,Y) \times [0,1]$; $\overline{H}$ is a map from $C(X,Y) \times [0,1]$ to $C(X,Y)$. Since the uniform topology is stronger than the VMO topology, $\overline{\varepsilon}$ is continuous also on $C(X,Y)$. The continuity of $\overline{H}$ on $C(X,Y) \times]0,1]$ follows from this fact, from Lemma 2.2(1) and from the fact that composition with the Lipschitz continuous map P is continuous from $C(X,U)$ to $C(X,Y)$.

The continuity of $\overline{H}$ on $C(X,Y) \times [0,1]$ follows from the above facts and from Lemma 2.2(4).

Therefore $\overline{H} : C(X,Y) \times [0,1] \to C(X,Y)$ is a homotopy between $r \circ i$ and the identity on $C(X,Y)$.

References

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