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MULTIPLE POSITIVE SOLUTIONS OF A SCALAR FIELD EQUATION IN $\mathbb{R}^n$

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Dedicated to Louis Nirenberg on the occasion of his 70th birthday

Much interest has been paid in recent years to the Kazdan–Warner problem:

$$\begin{cases} -\Delta u + \lambda u = k(x)u^{2^*-1}, \quad u > 0, \quad \text{in } \mathbb{R}^n, \\ u \to 0 \quad \text{at } \infty \end{cases} \tag{0}$$

(see for example [3], [8], [9], [14], [17]–[19], [24] and the references therein). Here, $\lambda \in \mathbb{R}$ is a positive parameter, k is a given smooth function on $\mathbb{R}^n$, $n \geq 3$, and $2^* = 2n/(n-2)$ is the critical Sobolev exponent. Problem (0) has a geometrical relevance, since for $\lambda = 0$ every solution to (0) gives rise, up to a stereographic projection, to a metric g on the sphere whose scalar curvature is proportional to $k(x)$. From the point of view of the Calculus of Variations the interest in the Kazdan–Warner problem is due to the role of the noncompact group of dilations in $\mathbb{R}^n$. This produces quite a large spectrum of phenomena, like concentrations of maps, lack of compactness, failure of the Palais–Smale condition and nonexistence results.

In the spirit of the paper by Coron [11] (see also [2]) one may ask if the coefficient $k(x)$ affects the topology of the energy sublevels. In this paper we give an answer to this question in the subcritical case. Namely, we study the

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"perturbed" problem

$$\begin{cases} -\Delta u + \lambda u = k(x)u^{p-1}, \quad u > 0, \quad \text{in } \mathbb{R}^n, \\ u \in H^1(\mathbb{R}^n), \end{cases} \tag{1}$$

where $p < 2^*$ is close to the critical exponent 2^*. Our aim is to use some variational arguments which are due to Benci and Cerami [5] (see also [6]) in order to relate the topology of the sublevels of the energy functional to the topology of the superlevels $\{z \in \mathbb{R}^n \mid k(z) \geq t\}$ for $t > 0$. Our assumptions on the map k are the following:

(k_1) $k : \mathbb{R}^n \to \mathbb{R}$ is continuous;
(k_2) the limit $k_\infty = \lim_{|z|\to\infty} k(z)$ exists;
(k_3) there exists $z_0 \in \mathbb{R}^n$ such that $k(z_0) > k_\infty^+ = \max\{k_\infty, 0\}$.

Notice that under these assumptions the map k is bounded, and the set

$$M = \{z \in \mathbb{R}^n \mid k(z) = \max_{z\in\mathbb{R}^n} k(z)\}$$

is compact. Writing $M_\delta = \{x \in \mathbb{R}^n \mid d(x, M) \leq \delta\}$ for $\delta > 0$, and denoting by $\operatorname{cat}_{M_\delta}(M)$ the Lusternik–Schnirelman category of the set M in M_δ, we compare the category of some energy sublevels with $\operatorname{cat}_{M_\delta}(M)$. Notice that $\operatorname{cat}_{M_\delta}(M) = \operatorname{cat}(M)$ for M regular and δ small (see Section 1). The first result we obtain is the following.

THEOREM A. *Assume that k satisfies* (k_1), (k_2) *and* (k_3). *Then for every $\delta > 0$ there exists a $p_\delta < 2^*$ such that for $p \in [p_\delta, 2^*[$ problem* (1) *has at least* $\operatorname{cat}_{M_\delta}(M)$ (*weak*) *solutions.*

We point that the solutions in Theorem A are close to the ground state solution. Moreover, they concentrate as $p \to 2^*$, and then they disappear. The further solution of the next theorem has higher energy and it appears when M has a rich topology. It would be of interest to investigate whether this solution survives as $p \to 2^*$.

THEOREM B. *Assume that k satisfies* (k_1), (k_2) *and*

(k_3^*) *there exists a $t \in]k_\infty^+, \max_{\mathbb{R}^n} k[$ such that M is contractible in the set $\{z \in \mathbb{R}^n \mid k(z) \geq t\}$.*

If $\operatorname{cat}_{M_\delta}(M) > 1$ *for some $\delta > 0$, then for p close to 2^*, problem* (1) *has at least* $\operatorname{cat}_{M_\delta}(M) + 1$ *solutions.*

We illustrate Theorem B with a simple example, in which we use some remarks of Section 1.

EXAMPLE. *Assume that the map* k *satisfies* (k_1) *and* (k_2). *Assume also that* $M \subseteq B_R$, *and* $\min_{B_R} k > k_\infty^+$ *for some* $R > 0$. *If* k *has* $s > 1$ *maximum points, then problem* (1) *has at least* $s + 1$ *solutions.*

The blow-up analysis of Section 3 gives more information in the radially symmetric case. In Section 5 we prove as an example the following theorem.

THEOREM C. *Assume that* $k = k(r)$ *is a radially symmetric map satisfying* (k_1), (k_2) *and* (k_3). *Assume also that* $\max_{r\ge 0} k(r)$ *is achieved at* $s \ge 1$ *points, and that* 0 *is not a maximum point. Then for every* p *close to* 2^* *problem* (1) *has at least* $2s$ *non-radially symmetric solutions.*

The method we adopt can also be applied to study problem (1) where λ is a varying parameter and $p < 2^*$ is fixed. Thus, when λ is large enough we get theorems analogous to Theorems A, B and C. There are many papers that treat equations like (1) on $\mathbb{R}^n$, in the subcritical case. We quote for example the papers [1], [4], [7], [12], [13], [16], [20], [22], [23], [25], [26]. An extensive bibliography on this subject is contained in [12].

NOTATION. For every real function g we set $g^+ = \max\{g, 0\}$ and $g^- = \min\{g, 0\}$. We recall that $g^+, g^- \in H^1$ if $g \in H^1$, and $\nabla g^\pm = \nabla g$ a.e. on $\{x \mid \pm g \ge 0\}$.

1. The Lusternik–Schnirelman category. Examples

Let M be a closed subset of a topological space X. We recall that the *Lusternik–Schnirelman* category $\mathrm{cat}_X(M)$ of the set M in X is the least integer σ such that M can be covered by σ closed subsets $A_1, \ldots, A_\sigma$ of M such that for all i, A_i is contractible in X. This means that for every index i, there exists a continuous homotopy $H_i : [0,1] \times A_i \to X$ joining the inclusion $A_i \to X$ to a constant map. If no such integer exists, then by definition $\mathrm{cat}_X(M) = \infty$. If $M = X$ we write $\mathrm{cat}_M(M) = \mathrm{cat}(M)$.

We notice that $\mathrm{cat}(M) \ge \mathrm{cat}_X(M)$, and equality holds if there exists a continuous retraction $r : X \to M$ such that $r(x) = x$ on M. In Theorem A we are interested in the case when M is a compact subset of $\mathbb{R}^n$, and $X = M_\delta$ for some δ positive, where M_δ is the set of points whose distance from M is not greater than δ. Now we exhibit some examples in which $\mathrm{cat}_{M_\delta}(M)$ is a good approximation for $\mathrm{cat}(M)$ for small δ. We omit the simple proofs.

EXAMPLE 1.2. *In the following examples we have* $\mathrm{cat}_{M_\delta}(M) = \mathrm{cat}(M)$ *for* δ *small.*

(i) M *is the closure of a bounded open set having smooth boundary.*
(ii) M *is a smooth and compact submanifold of* $\mathbb{R}^n$.
(iii) M *is finite set. Then* $\mathrm{cat}_{M_\delta}(M) = \mathrm{cat}(M) =$ *cardinality of* M.

In the next example $\mathrm{cat}_{M_\delta}(M)$ approaches $\mathrm{cat}(M)$, even if the sets M_δ do not retract on M.

Example 1.3. *Let $x_k \to x_0$ be a convergent sequence in $\mathbb{R}^n$ such that $x_k \neq x$ for infinitely many indices k. Set $M = \{x_k \mid k \geq 1\} \cup \{x\}$. Then $\mathrm{cat}_{M_\delta}(M) < \infty$ for all δ, and $\lim_{\delta \to 0} \mathrm{cat}_{M_\delta}(M) = \infty$.*

2. The variational approach

Our first hypotheses on the map k are the following:

$$k \in C^0 \cap L^\infty(\mathbb{R}^n) \text{ and } k \text{ is positive at some point } z \in \mathbb{R}^n. \tag{2.1}$$

We notice that it is not restrictive to assume

$$\sup_{\mathbb{R}^n} k = 1. \tag{2.2}$$

For $p \in]2, 2^*[$ and for $\lambda > 0$ we set

$$\Sigma = \left\{ u \in H^1(\mathbb{R}^n) \;\middle|\; \int_{\mathbb{R}^n} (|\nabla u|^2 + \lambda |u|^2) = 1 \text{ and } \int_{\mathbb{R}^n} k(x)|u^+|^p > 0 \right\},$$

$$J_p(u) = \left(\int_{\mathbb{R}^n} k(x)|u^+|^p \, dx \right)^{-2/(p-2)}, \quad J_p : \Sigma \to \mathbb{R}.$$

Notice that σ is a nonempty smooth submanifold of the Sobolev space $H^1(\mathbb{R}^n)$, and that the functional J_p is smooth on Σ. Moreover, it is positive on Σ by (2.2) and the Sobolev embedding theorem. Now we prove that every critical point for J_p on Σ is, up to a Lagrange multiplier, a weak solution to problem (1). First we compute

$$(\nabla J|_\Sigma)(u) = \frac{2p}{p-2} J_p(u)(u - J_p(u)^{(p-2)/2}(-\Delta + \lambda)^{-1}(k(u^+)^{p-1})).$$

Therefore, a critical point for J_p on Σ is a weak solution to

$$-\Delta u + \lambda u = J_p(u)^{(p-2)/2} k(x)(u^+)^{p-1} \quad \text{in } \mathbb{R}^n.$$

Multiplying this equation by u^- we readily get $\int(|\nabla u^-|^2 + \lambda|u^-|^2) = 0$, hence $u \geq 0$ a.e. and $u = u^+$. Thus, u is a weak solution to $-\Delta u + c(x)u = J_p(u)^{(p-2)/2} k(x)^+ u^{p-1}$ for some coefficient $c(x) > 0$, which is locally bounded by the elliptic regularity theory. Therefore, standard maximum principles give $u > 0$ in $\mathbb{R}^n$, and hence u solves (1). Now we define

$$m_p(k) = \inf_{u \in \Sigma} J_p(u) = \inf_{u \in \Sigma} \left(\int_{\mathbb{R}^n} k(x)|u^+|^p \, dx \right)^{-2/(p-2)}.$$

The first step is compare the infimum $m_p(k)$ with the best Sobolev constant S:

$$S = \inf_{U \in D^1(\mathbb{R}^n)} \frac{\int |\nabla U|^2}{[\int |U|^{2^*}]^{2/2^*}}, \tag{2.3}$$

where $D^1(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $(\int_{\mathbb{R}^n} |\nabla U|^2)^{1/2}$.

Lemma 2.1. *Assume that k satisfies* (2.1) *and* (2.2). *Then*

$$m_p(k) \to S^{n/2} \quad \text{as } p \to 2^*.$$

Proof. Fix any u in Σ. Since $k \le 1$ on $\mathbb{R}^n$, by the Hölder inequality we first get

$$\begin{aligned}\Big(\int k(x)|u^+|^p\Big)^{(2^*-2)/(p-2)} &\le \Big(\int |u^+|^2\Big)^{(2^*-p)/(p-2)} \int |u^+|^{2^*} \\ &\le \lambda^{\delta(p)} \int |u^+|^{2^*},\end{aligned}$$

where $\delta(p)$ is an exponent such that $\delta(p) \to 0$ as $p \to 2^*$. From the definition of the best Sobolev constant S and from $\int |\nabla u^+|^2 \le 1$ we infer that

$$S \le \inf_{u\in\Sigma} \frac{\int |\nabla u^+|^2}{(\int |u^+|^{2^*})^{2/2^*}} \le \lambda^{2\delta(p)/2^*} m_p(k)^{2/n}.$$

This proves that $S^{n/2} \le \liminf_{p\to 2^*} m_p(k)$. Conversely, let $\varepsilon > 0$, and fix a map $\varphi \in H^1 \cap C^0(\mathbb{R}^n)$ having compact support, and such that $\varphi \ge 0$, $\int |\nabla\varphi|^2 = 1$, and $(\int |\varphi|^{2^*})^{-2/2^*} \le S + \varepsilon$. Fix a point $z \in \mathbb{R}^n$ with $k(z) > 0$. For every positive μ we set $\varphi_\mu = \mu^{-n/2^*}\varphi((x-z)/\mu)$. For μ small enough it results that $\int k(x)|\varphi|^p > 0$. Testing $m_p(k)$ with the map φ_μ, we see that for μ small,

$$\begin{aligned}m_p(k) &\le \mu^{(n-2)p/(p-2)}\Big(\int |\nabla\varphi|^2 + \lambda\mu^2 \int |\varphi|^2\Big)^{p/(p-2)} \\ &\qquad \times \Big(\int k(x)\Big|\varphi\Big(\frac{x-z}{\mu}\Big)\Big|^p\,dx\Big)^{-2/(p-2)} \\ &\le \mu^{-(n-2)(2^*-p)/(p-2)}(1+\varepsilon)^{p/(p-2)}\Big(\int k(\mu x + z)|\varphi(x)|^p\,dx\Big)^{-2/(p-2)}.\end{aligned}$$

Passing to the limit as $p \to 2^*$ and then as $\mu \to 0$ we get

$$\limsup_{p\to 2^*} m_p(k) \le (1+\varepsilon)^{n/2}\Big(k(z)\int |\varphi|^{2^*}\Big)^{-n/2^*} \le (1+\varepsilon)^{n/2} k(z)^{-n/2^*}(S+\varepsilon)^{n/2}.$$

Letting ε go to zero we get $\limsup_{p\to 2^*} m_p(k) \le k(z)^{-n/2^*} S^{n/2}$, and the conclusion readily follows, by taking the infimum over all $z \in \mathbb{R}^n$. □

The next results are based on the concentration-compactness lemma by P. L. Lions. From now on we agree that $1/0 = +\infty$. We remark that assumptions (k_1), (k_2) and (2.2) imply

$$m_p(1) \le m_p(k) \le (k_\infty^+)^{-2/(p-2)} m_p(1).$$

The left hand side inequality is trivial, from $k \le 1$. The proof of the last inequality can be found in [20], I.2, and it is based on a translation argument. In the paper by P. L. Lions the significance of the strict inequality

$$m_p(k) < (k_\infty^+)^{-2/(p-2)} m_p(1) \tag{2.4}$$

is also underlined. Inequality (2.4) holds true for example if $k_\infty \le 0$, or if k is nonconstant and $k(z) \ge k_\infty^+$ for all $z \in \mathbb{R}^n$. In Section 3 (see Corollary 3.6) we shall prove that (2.4) holds if k satisfies the weaker assumption (k_3), provided p is sufficiently close to 2^*. The next result is meaningful only if (2.4) is satisfied.

Lemma 2.2. *Let $p < 2^*$ and assume that k satisfies the assumptions* (k_1), (k_2) *and* (2.2). *Then the functional J_p satisfies the Palais–Smale condition on the set*

$$\{u \in \Sigma \mid J_p(u) < (k_\infty^+)^{-2/(p-2)} m_p(1)\}.$$

Proof. The proof is essentially contained in [20], Section I.2. Nevertheless we present it here in order to make the paper self-contained. Let $(u_h)_h$ be a sequence in Σ such that

$$(\nabla J|_\Sigma)(u_h) \to 0, \tag{2.5}$$

$$J_p(u_h) \to c < (k_\infty^+)^{-2/(p-2)} m_p(1) \tag{2.6}$$

as $h \to \infty$. First, notice that $c > 0$. We apply the concentration-compactness lemma [20] to the sequence of measures $\varrho_h = |\nabla u_h|^2 + \lambda |u_h|^2$. Standard arguments show that vanishing and dichotomy cannot occur. Thus, the sequence $(\varrho_h)_h$ is tight, that is, there exist a subsequence $(\varrho_h)_h$ and a sequence of points $(z_h)_h$ in $\mathbb{R}^n$ such that

$$\forall \varepsilon > 0 \ \exists R > 0\colon \quad \int_{B(z_h,R)} (|\nabla u_h|^2 + \lambda |u_h|^2) \ge 1 - \varepsilon. \tag{2.7}$$

Now we prove that the sequence $(z_h)_h$ is bounded. Suppose by contradiction that $|z_h| \to \infty$, and set $\widehat{u}_h = u_h(\cdot + z_h)$. Since $\widehat{u}_h \in \Sigma$, we can assume that $\widehat{u}_h \to V$ weakly in H^1 for some function V satisfying $\int (|\nabla V|^2 + \lambda |V|^2) \le 1$ by semicontinuity. From (2.7) and from the Rellich theorem we get $\widehat{u}_h \to V$ in L^p and hence also $\widehat{u}_h^+ \to V^+$ in L^p. Thus, from the assumptions on k and from $|z_h| \to \infty$ we infer $\int k(x)|u_h^+|^p \to k_\infty \int |V^+|^p$. This implies first that $k_\infty > 0$, since $c > 0$. Moreover, it proves that $\int |V^+|^p = k_\infty^{-1} c^{-(p-2)/2} > 0$. Therefore, testing $m_p(1)$ with the map V we get

$$m_p(1) \le \left(\int |V^+|^p \right)^{-2/(p-2)} = k_\infty^{-2/(p-2)} c < m_p(1),$$

a contradiction. This proves that the sequence $(z_h)_h$ is bounded. But then we can take $z_h = 0$ for all h in (2.6), that is, we have proved that

$$\forall \varepsilon > 0\ \exists R > 0\colon \quad \int_{B(0,R)} (|\nabla u_h|^2 + \lambda |u_h|^2) \geq 1 - \varepsilon.$$

Arguing as before, we find that there exists $u \in \Sigma$ such that (for a subsequence) $u_h \to u$ weakly in H^1 and strongly in L^p. In particular, from (2.6) it follows that $c = J_p(u)$, and from (2.5) it follows that u is a weak solution to

$$-\Delta u + \lambda u = c^{(p-2)/2} k(x) (u^+)^{p-1}. \tag{2.8}$$

Using u as test function in (2.8) we get $\int (|\nabla u|^2 + \lambda |u|^2) = 1$. Since $u_h \to u$ weakly in H^1, and since $\int (|\nabla u_h|^2 + \lambda |u_h|^2) = 1$ for every h, this is sufficient to conclude that $u_h \to u$ strongly in H^1. □

3. The concentration behaviour

Let U be a positive and radially symmetric function which minimizes the L^{2^*} norm on the sphere $\{u \in D^1(\mathbb{R}^n) \mid \int |\nabla u|^2 = 1\}$ (see [21]). We also recall that U is positive and smooth, and it is strictly decreasing as a function of the radius. For $\mu > 0$ and $z \in \mathbb{R}^n$ we set

$$U_{\mu,z}(x) = \mu^{-n/2^*} U\left(\frac{x - z}{\mu}\right) \quad \text{and} \quad U_\mu(x) = U_{\mu,0}(x) = \mu^{-n/2^*} U(x/\mu).$$

A simple computation shows that for every μ,

$$\int |\nabla U_{\mu,z}|^2 = 1, \quad \int |U_{\mu,z}|^{2^*} = S^{-2^*/2},$$
$$|\nabla U_{\mu,z}|^2 \to \delta_z, \quad |U_{\mu,z}|^{2^*} \to S^{-2^*/2} \delta_z \quad \text{as } \mu \to 0$$

weakly in the sense of measures, where $\delta_z =$ Dirac mass at $z \in \mathbb{R}^n$.

In the following, a_p will denote any function of $p \in\,]2, 2^*[$ such that we have $a_p - m_p(k) \to 0^+$ as $p \to 2^*$. In particular, $a_p \to S^{n/2}$ by Lemma 2.1.

PROPOSITION 3.1. *Assume that k satisfies the assumptions* (k_1), (k_2) *and* (2.2). *Then, for every sequence $u_p \in \Sigma$ with $J_p(u_p) \leq a_p$, we have $u_p \to 0$ in $L^2(\mathbb{R}^n)$ and $u_p - u_p^+ \to 0$ in $H^1(\mathbb{R}^n)$. Moreover, there exist a subsequence $p_h \to 2^*$, a sequence $(z_h)_h$ of points with $k(z_h) \to 1$, and a sequence $(\mu_h)_h$ of positive numbers with $\mu_h \to 0$, such that*

$$\begin{aligned} &|u_{p_h}^+|^{p_h} - |U_{\mu_h,z_h}|^{2^*} \to 0 && \text{in } L^1(\mathbb{R}^n), \\ &\nabla(u_{p_h} - U_{\mu_h,z_h}) \to 0 && \text{in } L^2(\mathbb{R}^n)^n, \\ &u_{p_h} - U_{\mu_h,z_h} \to 0 && \text{in } L^{2^*}(\mathbb{R}^n) \text{ as } h \to \infty. \end{aligned}$$

Proof. Let u_p be as in the statement. The proof will be divided into several steps.

Step 1: *$u_p \to 0$ in L^2, $u_p - u_p^+ \to 0$ in H^1. Compactness up to translations and dilations*. Using the arguments in the proof of Lemma 2.1 and the Sobolev theorem we get the following chain of inequalities:

$$\begin{aligned}\Big(\int k(x)|u_p^+|^p\Big)^{(2^*-2)/(p-2)} &\le \Big(\int |u_p^+|^p\Big)^{(2^*-2)/(p-2)}\\ &\le \Big(\int |u_p^+|^2\Big)^{(2^*-p)/(p-2)}\int |u_p^+|^{2^*}\\ &\le \lambda^{-\delta(p)}\int |u_p^+|^{2^*} \le \lambda^{-\delta(p)}S^{-2^*/2}\Big(\int |\nabla u_p^+|^2\Big)^{2^*/2}\\ &\le \lambda^{-\delta(p)}\Big(\int |\nabla u|^2\Big)^{2^*/2}S^{-2^*/2} \le \lambda^{-\delta(p)}S^{-2^*/2},\end{aligned}$$

where $\delta(p) \to 0$ as $p \to 2^*$. Therefore, Lemma 2.1 and $J_p(u_p) \le a_p$ give

$$S^{-2^*/2} = \lim_{p\to 2^*}\int k(x)|u_p|^p = \lim_{p\to 2^*}\int |u_p|^p = \lim_{p\to 2^*}\int |u_p|^{2^*}, \tag{3.1}$$

and also

$$\lim_{p\to 2^*}\int |\nabla u_p|^2 = \lim_{p\to 2^*}\int |\nabla u_p^+|^2 = 1.$$

First we observe that this last equality implies $u_p \to 0$ in L^2 and also $u_p^+ \to 0$ in L^2, since $u_p \in \Sigma$ for every p. We also infer that $u_p - u_p^+ \to 0$ in H^1. Hence, from now on we can assume, without restriction, that $u_p = u_p^+$. From (3.1) we see that u_p approaches the best Sobolev constant S. An application of a result by P. L. Lions [21], Theorem I.1 and Corollary I.1, proves that the sequence u_p is relatively compact in D^1 up to translations and changes of scale. This means that for a sequence $p_h \to 2^*$, there exist sequences $(\mu_h)_h$ of positive numbers, and $(z_h)_h$ of points, such that the rescaled sequence $\widehat{u}_h(x) = \mu_h^{n/2^*}u_{p_h}(\mu_h x + z_h)$ satisfies: $\nabla\widehat{u}_h \to \nabla U$ in L^2, $\widehat{u}_h \to U$ in L^{2^*} and almost everywhere. In particular, we also get $\nabla(u_{p_h} - U_{\mu_h,z_h}) \to 0$ in L^2 and $u_{p_h} - U_{\mu_h,z_h} \to 0$ in L^{2^*}.

Step 2: *$\mu_h \to 0$.* Notice that $\lim_h \int_{B(0,1)}|\widehat{u}_h|^2 = \int_{B(0,1)}U^2 > 0$ by the Rellich theorem. Therefore, Step 2 follows from

$$o(1) = \int_{\mathbb{R}^n}|u_{p_h}|^2 \ge \int_{B(z_h,\mu_h)}|u_{p_h}|^2 = \mu_h^2\int_{B(0,1)}|\widehat{u}_h|^2.$$

Step 3: *$\eta_h := (\mu_h)^{n(1-p_h/2^*)} \to 1$.* Set $f_h = \eta_h|\widehat{u}_h|^{p_h}$. Notice that $f_h \ge 0$ and $\int f_h = \int |u_p|^p \to S^{-2^*/2}$ by (3.1). Then by Fatou's lemma we infer that (for a subsequence) the pointwise limit of f_h is a.e. finite. Therefore, $\eta_h \to \eta < \infty$ and $f_h \to \eta U^{2^*}$ a.e. Now consider the sequence $g_h = \eta_h|\widehat{u}_h|^{p_h} - |\widehat{u}_h|^{2^*}$. We have

$g_h \in L^1$, $g_h \to (\eta - 1)U^{2^*}$ a.e., and $\int g_h \to 0$ by (3.1). But this immediately gives $\eta = 1$, and Step 3 is concluded.

Step 4: $|\widehat{u}_h|^{p_h} \to U^{2^*}$ *in* L^1. This easily follows from the Lebesgue and Fatou theorems, since $|\widehat{u}_h|^{p_h}, U^{2^*} \in L^1$, $|\widehat{u}_h|^{p_h} \to U^{2^*}$ almost everywhere, and $\int |\widehat{u}_h|^{p_h} \to \int U^{2^*}$ by (3.1) and Step 3. Notice that Steps 3 and 4 imply in particular that $|u_{p_h}|^{p_h} - |U_{\mu_h,z_h}|^{2^*} \to 0$ in L^1.

Step 5: *completion of the proof.* Using the first equality in (3.1) and Step 3 we get

$$S^{-2^*/2} = \int k(\mu_h x + z_h)|\widehat{u}_h(x)|^{p_h}\,dx + o(1). \tag{3.2}$$

First we assume that the sequence z_h is bounded. In this case, we can pass to a subsequence to have $z_h \to z$ for some point z. Thus the continuity of k, Step 2, Step 4 and (3.2) lead to $S^{-2^*/2} = k(z)S^{-2^*/2}$, and hence $k(z) = 1$. In case $|z_h| \to \infty$, one has only to replace $k(z)$ by k_∞ and to repeat the same argument. □

Remark 3.2. Suppose that, in addition, k satisfies (k_3). Then the set M is compact, and therefore there exists a point $z \in M$ such that $z_h \to z$ as $h \to \infty$.

Proposition 3.1 can be improved in the radially symmetric case, as is shown in the next result. In that case, the present blow-up analysis has some corollaries which will be stated in Section 5. In particular, it turns out that in the radially symmetric case the maps u_p cannot "concentrate at infinity", essentially because they are uniformly bounded in $L^2(\mathbb{R}^n)$.

Proposition 3.3. *Let k and u_p be as in Proposition* 3.1. *Suppose that for every p, the map u_p is radially symmetric. Then the conclusion of Proposition* 3.1 *holds with $z_h = 0$ for every h.*

Proof. By Proposition 3.1, the sequence $\widehat{u}_h(x) = \mu_h^{n/2^*} u_{p_h}(\mu_h x + z_h)$ converges in $D^1(\mathbb{R}^n)$ to the map U, for some sequences $\mu_h \to 0$, $(z_h)_h$ in $\mathbb{R}^n$. We just have to prove that $z_h/\mu_h \to 0$ as $h \to \infty$, since in this case the sequence $\widehat{u}_h(x)$ can be replaced with $\widehat{u}_h(x) = \mu_h^{n/2^*} u_{p_h}(\mu_h x) = \widehat{u}_h(x - z_h/\mu_h)$. Assume by contradiction that for some subsequence we have $|z_h|/\mu_h \geq 2\delta > 0$. Since $u_{p_h} - U_{\mu_h,z_h} \to 0$ in $L^{2^*}(\mathbb{R}^n)$, and since u_{p_h} is radially symmetric, we get

$$\begin{aligned}\int_{B(0,\delta)} U^{2^*} &= \int_{B(z_h,\delta\mu_h)} |u_{p_h}|^{2^*} + o(1) = \int_{B(-z_h,\delta\mu_h)} |u_{p_h}|^{2^*} + o(1)\\ &= \int_{B(-2z_h/\mu_h,\delta)} U^{2^*} + o(1).\end{aligned}$$

This immediately leads to a contradiction, since $U = U(|x|)$ is smooth and strictly decreasing, and therefore from $|-2z_h/\mu_h| \geq 4\delta$ it follows that $\min_{|x|\leq\delta} U^{2^*} > \max_{|x|\geq 2\delta} U^{2^*}$. □

Now, let $p \in]2, 2^*[$, and consider the minimization problem

$$m_p(1) = \inf_{V \in \Sigma} \bigg(\int |V|^p \bigg)^{-2/(p-2)}. \tag{3.3}$$

It is well known that (3.3) has a positive solution (see for example [20]), which is unique up to translations by a result of Kwong [15]. We denote by V_p the radially symmetric solution of (3.3). An application of Proposition 3.3 with $k \equiv 1$ gives the next result.

Corollary 3.4. *As $p \to 2^*$, we have*

(i) $V_p \to 0$ *in* $L^2(\mathbb{R}^n)$;
(ii) $|\nabla V_p|^2 \to \delta_0$ *weakly in the sense of measures;*
(iii) $|V_p|^p \to S^{-2^*/2}\delta_0$ *weakly in the sense of measures.*

As in the paper by Benci and Cerami [5], we define two continuous maps $\beta : \Sigma \to \mathbb{R}^n$ and $\Phi_p : \mathbb{R}^n \to \Sigma$. For $p < 2^*$ we set

$$\Phi_p(z) := V_p(\cdot - z) \quad \text{for } z \in \mathbb{R}^n.$$

Let $R > 0$ be large enough, so that in particular M is contained in the ball $B_R = \{x \mid |x| \le R\}$. Fix a smooth and bounded map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that φ has compact support, and $\varphi(x) = x$ if $|x| \le R$. We define a "barycentre" function

$$\beta(u) = \int_{\mathbb{R}^n} \varphi(x)(|\nabla u|^2 + \lambda u^2)\, dx \quad \text{for } u \in \Sigma.$$

We also set

$$J_p^{a_p} = \{u \in \Sigma \mid J_p(u) \le a_p\}.$$

Corollary 3.5. *Assume that k satisfies* (k$_1$), (k$_2$), (k$_3$) *and* (2.2). *Then, as $p \to 2^*$,*

(i) $\beta(\Phi_p(z)) = z + o(1)$ *uniformly for $z \in B_R$;*
(ii) $\Phi_p(z) \in \Sigma$ *and* $J_p(\Phi_p(z)) = m_p(k) + o(1)$ *uniformly for $z \in M$;*
(iii) $\sup\{d(\beta(u), M) \mid u \in J_p^{a_p}\} \to 0$.

Proof. Assertions (i) and (ii) are easy consequences of Corollary 3.4 (use also the fact that M is compact). Now we prove (iii). For a sequence $p \to 2^*$, let $u_p \in J_p^{a_p}$. Then, by Proposition 3.1 and Remark 3.2, we find that for a subsequence p_h, and for sequences $\mu_h \to 0$ and $z_h \to z$ with $z \in M$, we have $u_{p_h} \to 0$ in L^2 and $\nabla(u_{p_h} - U_{\mu_h,z_h}) \to 0$ in L^2. Then by the continuity of β we get $\beta(u_{p_h}) = \int \varphi(x)|\nabla U_{\mu_h,z_h}| + o(1) = z$, since $|\nabla U_{\mu,z}|^2 \to \delta_z$ uniformly for z on bounded sets, and since $\varphi(z) = z$ on M. $\square$

COROLLARY 3.6. *Assume that k satisfies* (k_1), (k_2), (k_3) *and* (2.2). *Then for every $t \in]k_\infty^+, 1]$ there exists $p_t < 2^*$ such that for $p \in [p_t, 2^*[$ we have*

(i) $\Phi_p(z) \in \Sigma$ *if* $k(z) \geq t$;
(ii) $\max\{J_p(\Phi_p(z)) \mid z \in \mathbb{R}^n,\ k(z) \geq t\} < (k_\infty^+)^{-2/(p-2)} m_p(1)$.

In particular, $m_p(k) < (k_\infty^+)^{-2/(p-2)} m_p(1)$ for every p close to 2^*.

PROOF. (i) follows from Corollary 3.4(iii), since the set $\{z \in \mathbb{R}^n \mid k(z) \geq t\}$ is compact. To prove (ii) it is sufficient to assume $k_\infty > 0$. Fix a $t \in]k_\infty, 1]$, and write $t = \vartheta k_\infty$ for $\vartheta > 1$. Using Corollary 3.4(iii) and Lemma 2.1, we deduce that for p close to 2^*,

$$J_p(\Phi_p(z))^{-(p-2)/2} = k(z) S^{-2^*/2} + o(1) \geq \vartheta k_\infty m_p(1)^{-2^*/n} + o(1)$$

uniformly on $\{z \in \mathbb{R}^n \mid k(z) \geq t\}$. Thus,

$$\begin{aligned} k_\infty^{2/(p-2)} m_p(1)^{-1} \max_{k(z) \geq t} J_p(\Phi_p(z)) &\leq \vartheta^{-2/(p-2)} m_p(1)^{(2^*-2)/(p-2)} + o(1) \\ &= \vartheta^{-2/(p-2)} + o(1), \end{aligned}$$

and the conclusion follows. □

4. Proofs

We follow the method developed by Benci and Cerami in [5].

PROOF OF THEOREM A. The map k satisfies (k_1), (k_2), (k_3) and the non-restrictive condition (2.2). Assume that the functional J_p has a finite number of critical points, and fix a $\delta > 0$. For every p sufficiently close to 2^*, we fix a number $a_p < (k_\infty^+)^{-2/(p-2)} m_p(1)$ such that a_p is not a critical value for J_p, and such that

$$\Phi_p(z) \in \Sigma \quad \text{and} \quad J_p(\Phi_p(z)) < a_p \quad \forall z \in M, \tag{4.1}$$

$$a_p - m_p(k) \to 0 \quad \text{as } p \to 2^* \tag{4.2}$$

(use Corollary 3.5(ii) and Corollary 3.6). Next, fix a radius R such that $M \subseteq B_R$. Define the maps β and Φ_p as in Section 3. By Corollary 3.5(i), (iii) for p close to 2^* (p will depend on δ),

$$|\beta(\Phi_p(z)) - z| < \delta \quad \forall z \in B_R, \tag{4.3}$$

$$\beta(J_p^{a_p}) \subseteq M_\delta. \tag{4.4}$$

We claim that

(4.5) *the composite map $\beta \circ \Phi_p$ is homotopic to the inclusion $M \to M_\delta$.*

In fact, it suffices to consider the homotopy $\alpha(t,x)=x+t(\beta(\Phi_p(x))-x)$, since by (4.1) and (4.3) we have $d(\alpha(t,x),M)\le|\beta(\Phi_p(x))-x|\le\delta$ for every $x\in M$ and $t\in M$ and $t\in[0,1]$, that is, α maps $[0,1]\times M$ into M_δ.

Notice that by Lemma 2.2 the functional J_p satisfies the Palais–Smale condition in $J_p^{-1}([m_p(k),a_p])$. Hence, by standard Lusternik–Schnirelman theory, to conclude the proof it suffices to show that

$$\operatorname{cat}(J_p^{a_p})\ge\operatorname{cat}_{M_\delta}(M). \tag{4.6}$$

In the following, we denote by Φ_p the restriction of the map Φ_p to M. Hence, $\Phi_p:M\to J_p^{a_p}$ by (4.1). Suppose that $A_1,\dots,A_\sigma$ is a closed covering of $J_p^{a_p}$ such that for every i, there exists a homotopy $H_i:[0,1]\times A_i\to J_p^{a_p}$ with $H_i(0,u)=u$ for $u\in A_i$ and $H_i(1,\cdot)=\text{constant}$ for $i=1,\dots,\sigma$. Set $C_i=\Phi_p^{-1}(A_i)$. Then C_i is closed in M for all i, and the union of the sets C_i covers M. In order to prove (4.6) it suffices to show that the sets C_i are contractible in M_δ. This is readily done by using the homotopy $h_i(t,x)=\beta(H_i(t,\Phi_p(x)))$, $h_i:[0,1]\times C_i\to M_\delta$ (use (4.4) and (4.5)). This completes the proof of Theorem A. □

Proof of Theorem B. The map k satisfies (k_1), (k_2), (k_3^*) and the non-restrictive condition (2.2). Assume again that the functional J_p has a finite number of critical points, and fix $\delta>0$ such that $\operatorname{cat}_{M_\delta}(M)>1$. Take p close to 2^*, and define a_p as before. Fix the value $t\in\,]k_\infty,\max_{\mathbb{R}^n}k[$ such that M is contractible in $C:=\{z\in\mathbb{R}^n\mid k(z)\ge t\}$ (assumption (k_3^*)). Fix a radius R such that $C\subseteq B_R$, and define the maps β and Φ_p as in Section 3. For p close to 2^* we get the validity of (4.3) and (4.4), and moreover $\Phi_p(C)\subseteq\Sigma$ by Corollary 3.6. In addition, if p is close to 2^*, then we can choose $b_p>a_p$ such that b_p is not a critical level for J_p, and such that

$$b_p>\max_{x\in C}J_p(\Phi_p(x)), \tag{4.7}$$

$$b_p<k_\infty^{-2/(p-2)}m_p(1). \tag{4.8}$$

Notice that this is possible if p is large enough, by Corollary 3.6(ii). Notice also that by Lemma 2.2 and (4.8) the functional J_p satisfies the Palais–Smale condition on $J_p^{b_p}$. Assume that J_p has no critical points with energy in $[a_p,b_p]$. In this case we can use a deformation lemma and (4.1) to construct a map $\alpha:J_p^{b_p}\to J_p^{a_p}$ such that $\alpha(\Phi_p(z))=\Phi_p(z)$ for $z\in M$ (notice that $\Phi_p(M)$ is a compact subset of Σ). Let $h:[0,1]\times M\to C$ be a continuous homotopy joining the inclusion $M\to C$ to a constant map, and then define the map

$$H(s,x)=\beta(\alpha(\Phi_p(h(s,x)))),\quad H:[0,1]\times M\to M_\delta$$

(use (4.7) and (4.4)), which is a homotopy between $\beta\circ\Phi_p$ and a constant map. Since, as before, the map $\beta\circ\Phi_p$ is homotopic to the inclusion $M\to M_\delta$, this proves that M is contractible in M_δ, contrary to the assumption $\operatorname{cat}_{M_\delta}(M)>1$. □

5. The radially symmetric case

We conclude this paper with some remarks in case k is a radially symmetric function satisfying (k_1), (k_2) and the nonrestrictive condition $\max k = 1$. Set

$$\Sigma^{\mathrm{s}} = \{u \in \Sigma \mid u \text{ is radially symmetric}\}, \quad m_p^s(k) = \inf_{u\in\Sigma^{\mathrm{s}}} J_p(u),$$

so that $m_p^{\mathrm{s}}(k) \geq m_p(k) > 0$. Since k is radially symmetric, it turns out that every critical point of the functional J_p on Σ^{s} is, up to a Lagrange multiplier, a solution to (1). We recall that by a result of Strauss [26] (see also [10] and [7]), for every $p \in [2, 2^*[$ the restriction of the embedding $H^1(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ to the subspace of radially symmetric functions is compact. Therefore, standard arguments show that the infimum $m_p^{\mathrm{s}}(k)$ is achieved on Σ^{s}, and hence problem (1) has always a radially symmetric solution (see also [26]). The aim of this section is to give some estimates for $m_p^{\mathrm{s}}(k)$ as $p \to 2^*$, in order to prove the existence of non-radially symmetric solutions.

Lemma 5.1.

$$S^{n/2} \leq \liminf_{p\to 2^*} m_p^{\mathrm{s}}(k) \leq \limsup_{p\to 2^*} m_p^{\mathrm{s}}(k) \leq (k(0)^+)^{-n/2^*} S^{n/2}.$$

Proof. The left-hand side inequality follows from $m_p^{\mathrm{s}}(k) \geq m_p(k)$ and from Lemma 2.1. Assume $k(0) > 0$, fix an $\varepsilon > 0$, and choose a nonnegative and radially symmetric function $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\int |\nabla\varphi|^2 = 1$ and $(\int |\varphi|^{2^*})^{-2/2^*} \leq S + \varepsilon$. Arguing as in the second part of the proof of Lemma 2.1 we see that for every $\mu > 0$ small enough,

$$m_p^{\mathrm{s}}(k) \leq \mu^{-(n-2)(2^*-p)/(p-2)}(1+\varepsilon)^{p/(p-2)} \left(\int k(\mu x)|\varphi(x)|^p\,dx\right)^{-2/(p-2)}.$$

Passing to the limit as $p \to 2^*$ and then as $\mu \to 0$ we get

$$\limsup_{p\to 2^*} m_p^{\mathrm{s}}(k) \leq (1+\varepsilon)^{n/2}\left(k(0)\int |\varphi|^{2^*}\right)^{-n/2} \leq (1+\varepsilon)^{n/2} k(0)^{-n/2^*}(S+\varepsilon)^{n/2}.$$

Letting ε go to zero we get $\limsup_{p\to 2^*} m_p^{\mathrm{s}}(k) \leq k(0)^{-n/2^*} S^{n/2}$, and the conclusion follows. $\square$

Proposition 5.2. $\liminf_{p\to 2^*} m_p^{\mathrm{s}}(k) > S^{n/2}$ *if and only if* $k(0) < 1$.

Proof. If $k(0) = 1$ then by Lemma 5.1, $m_p^{\mathrm{s}}(k) \to S^{n/2}$. Conversely, if it is possible to find a sequence $(u_p)_p$ in Σ^{s} such that $J_p(u_p) - m_p(k) \to 0$ as $p \to 2^*$, then by Proposition 3.1 a subsequence u_{p_h} concentrates along a sequence of points z_h, with $k(z_h) \to 1$. On the other hand, u_p is radially symmetric for every p, and hence by Proposition 3.3 we can take $z_h = 0$ for all h, which implies in particular $k(0) = 1$.

Corollary 5.3. *Assume that k satisfies also* (k_3) *and* $k(0) < 1$. *Then, for p close to* 2^*, *the least energy solution to* (1) *is not radially symmetric.*

Proof. By Lemma 2.2 and Corollary 3.6 the infimum $m_p(k)$ is achieved on Σ. Since $m_p(k) \to S^{n/2}$ as $p \to 2^*$, for p close to 2^* we have $m_p(k) < m_p^{\mathrm{s}}(k)$, and the conclusion follows. □

Proof of Theorem C. Let k be as in Theorem C, and assume also that $\max k = 1$. Under these assumptions, the set M is the union of s spheres, and hence $\mathrm{cat}_{M_\delta}(M) \geq 2s$ for δ small (see Section 2). Theorem C follows from the proof of Theorem A. We just have to notice that $a_p < m_p^{\mathrm{s}}(k)$ for p close to 2^*, since $a_p \to S^{n/2} < \liminf_{p \to 2^*} m_p^{\mathrm{s}}(k)$ by Lemma 2.1 and Proposition 5.2. □

It would be of interest to give more information on the behaviour of $m_p^{\mathrm{s}}(k)$ as $p \to 2^*$. Since a deeper analysis of this subject goes far beyond the aim of the present paper, we do not enter into details.

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