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A TOPOLOGICAL SHOOTING METHOD AND THE EXISTENCE OF KINKS OF THE EXTENDED FISHER–KOLMOGOROV EQUATION

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To Louis Nirenberg on his 70th birthday

1. Introduction

In three recent papers [PT1-3] we have developed a topological shooting method to establish the existence of monotone *kinks* as well as stationary *periodic* and *chaotic* solutions of the fourth order equation

$$\text{(EFK)} \qquad \frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3, \quad \gamma > 0.$$

For $\gamma = 0$ this equation reduces to the famous Fisher–Kolmogorov equation. For $\gamma > 0$ this fourth order diffusion equation has often been referred to as the *Extended Fisher–Kolmogorov equation* [DS] and has served as a model equation for the study of bi-stable systems arising in a variety of situations in physics [CER, CH, DS, HLS], such as second order phase transitions (Lifschitz points [Z]). The term "bi-stable" refers here to the fact that the uniform states $u = \pm 1$ are stable as solutions of the related equation

$$\frac{du}{dt} = u - u^3.$$

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The purpose of the present paper is to give a concise description of the method, uncluttered by technical details for which we refer to [PT1-3], and to apply it to prove the existence of a countably infinite number of kinks, or heteroclinic orbits, connecting the stable states $u = \pm 1$, when $\gamma > 1/8$. Thus, we shall study the problem

$$\text{(I)} \begin{cases} \gamma u^{\text{iv}} = u'' + u - u^3 \quad \text{in } \mathbb{R}, & \text{(1.1a)} \\ (u, u', u'', u''') \to (\pm 1, 0, 0, 0) \quad \text{as } x \to \pm\infty. & \text{(1.1b)} \end{cases}$$

To narrow down the number of solutions we shall restrict ourselves to *odd* solutions, and so set

$$u(0) = 0 \quad \text{and} \quad u''(0) = 0. \tag{1.1c}$$

Thus, to obtain solutions of Problem (I), we can focus on solutions of (1.1a) on $\mathbb{R}^+$ which satisfy (1.1c) at the origin and (1.1b) at infinity. To analyse this problem by means of a shooting method we relinquish the condition at infinity in favour of two additional conditions at the origin:

$$u'(0) = \alpha \quad \text{and} \quad u'''(0) = \beta. \tag{1.1d}$$

The four initial conditions (1.1c) and (1.1d) are not independent, because (1.1a) has a first integral. If we multiply (1.1a) by $2u'$ and integrate, we obtain

$$\mathcal{E}(u) := 2\gamma u' u''' - \gamma (u'')^2 - (u')^2 + \frac{1}{2}(1 - u^2)^2 = \frac{\mu}{2}, \tag{1.2}$$

where μ is a constant. Since we seek a solution u which satisfies (1.1b) we need to set

$$\mu = 0. \tag{1.3}$$

At the origin we can evaluate $\mathcal{E}(u)$ from the initial conditions (1.1c) and (1.1d). Using (1.3), we find that

$$\alpha \neq 0 \quad \text{and} \quad \beta = \beta(\alpha) := \frac{1}{2\alpha\gamma}\left(\alpha^2 - \frac{1}{2}\right). \tag{1.4}$$

Because $-u$ is also an odd solution it suffices to consider only $\alpha > 0$.

Thus, the construction of kinks entails the analysis of the initial value problem

$$\text{(II)} \begin{cases} \gamma u^{\text{iv}} = u'' + u - u^3 \quad \text{in } \mathbb{R}^+, & \text{(1.5a)} \\ u(0) = 0, \quad u'(0) = \alpha, \quad u''(0) = 0, \quad u'''(0) = \beta(\alpha). & \text{(1.5b)} \end{cases}$$

Plainly, for every $\alpha \neq 0$, this problem has a unique local solution $u = u(x, \alpha)$, and we need to find values of α such that (i) $u(x, \alpha)$ exists for all $x > 0$ and (ii) $(u, u', u'', u''') \to (1, 0, 0, 0)$ as $x \to \infty$.

In [PT1] we began our analysis of Problem (II) by establishing the existence of *kinks* for values of γ in the range $0 < \gamma \leq 1/8$. For this parameter regime we proved that Problem (II) has a unique *monotone* solution which satisfies (1.1b) at $+\infty$. Extending this solution as an odd function we obtain a kink, which is a natural generalization of the well known kink of the Fisher–Kolmogorov equation

$$u(x) = \tanh(x/\sqrt{2}),$$

which is also monotone.

The value $\gamma = 1/8$ is critical because at this value the nature of the linearization of equation (1.1a) at the point $(1, 0, 0, 0)$ changes. That is, for $\gamma \le 1/8$ all four eigenvalues are real, two are negative and two are positive, and for $\gamma > 1/8$ all the eigenvalues are complex, two having a positive real part and two a negative real part. This changes the dynamics dramatically, and the method used in [PT1] could not be applied directly for these larger values of γ. In [PTV] it was subsequently shown by variational methods that kinks do exist for all $\gamma > 0$. Each of these kinks has exactly one zero.

In [PT2] we developed a new topological argument, based on a careful study of the dependence of the first critical value of the graph of $u(x, \alpha)$ on α, to prove that for any $\gamma > 1/8$, equation (1.1a) has at least two *periodic solutions*, u_1 and u_2, such that

$$\max_{x\in\mathbb{R}} |u_1(x)| < 1 \quad \text{and} \quad \max_{x\in\mathbb{R}} |u_2(x)| > 1.$$

Denoting these periodic solutions by $u_i(x, \gamma)$, it was found that

$$u_1(x, \gamma) \to \overline{u}(x) \quad \text{and} \quad u_2(x, \gamma) \to \overline{u}(x) \quad \text{pointwise as } \gamma \downarrow 1/8.$$

Here $\overline{u}$ is the unique monotone kink at $\gamma = 1/8$. Thus these two periodic solutions bifurcate from the kink $\overline{u}$ as γ passes through 1/8. For $\gamma \le 1/8$ the EFK equation was shown to have no periodic solutions with amplitude larger than 1, as is well known to be the case for the FK equation.

In [PT3] this argument was further extended to prove the existence of an infinite, *chaotic* family of bounded solutions with zero energy ($\mu = 0$). Each of these solutions has an infinite sequence (ξ_k) of positive local maxima such that $\xi_k \to \infty$ as $k \to \infty$, and is labelled according to the location of $u(\xi_k)$ in relation to the level $u = 1$: above or below. This was possible thanks to the fact that

(i) the critical points on the graph of u are all isolated, and
(ii) the maxima do not lie on the line $u = +1$.

Thus, corresponding to the sequence (ξ_k) we can define a sequence $\sigma = (\sigma_k)$ of zeros and ones by setting

$$\sigma_k = \begin{cases} 0 & \text{if } u(\xi_k) < 1, \\ 1 & \text{if } u(\xi_k) > 1, \end{cases}$$

and for every such sequence σ we can find a corresponding solution u_σ.

In the present paper we return to the analysis of kinks, and shall further extend our method to enable us to establish the existence of many different kinks, and especially kinks with a prescribed number of zeros, in addition to those found earlier [PTV] which had the property that $u(x) \geq 0$ for $x \geq 0$ and, since they were odd, $u(x) \leq 0$ for $x \leq 0$.

Theorem A. *For any $\gamma > 1/8$ and any integer $n \geq 0$ there exists an odd kink with precisely $2n+1$ zeros on $\mathbb{R}$. If $u(x)$ is such a kink, then:*

(a) *$|u(x)| < \sqrt{2}$ for all $x \in \mathbb{R}$.*
(b) *If n is even, then $u'(0) > 0$, and if n is odd, $u'(0) < 0$.*
(c) *Let $m = n/2$ if n is even and $m = (n+1)/2$ if n is odd. Then the sequences (η_k) of local minima and (ξ_k) of local maxima on $\mathbb{R}^+$ have the properties:*

$$\begin{cases} -1 < u(\eta_1) < u(\xi_1) < 1 & \text{if } n \text{ is even},\\ -1 < u(\eta_1) < 0 & \text{if } n \text{ is odd}, \end{cases}$$

$$u(\eta_m) < 0 < u(\xi_m) < 1, \quad u(\xi_k) > 1, \quad u(\eta_k) \geq 1/\sqrt{3} \quad \text{for } k \geq m+1.$$

If $m \geq 3$ we have

$$-1 < u(\eta_k) < 0 < u(\xi_k) < 1 \quad \text{for } 2 \leq k \leq m-1.$$

Remark. If n is even, the sequence of zeros of u' begins with a *maximum* ξ_1 and continues with a minimum η_1, and if u is odd, it begins with a *minimum* η_1 and continues with a maximum, which, to be consistent with the previous case, we denote by ξ_2.

The kinks established in Theorem A all have a particular structure, in which we can distinguish three regions: an inner region $(-M, M)$ in which the kink is approximately periodic, with amplitude smaller than 1, and two outer regions, $(-\infty, -M)$ and (M, ∞), with tails which join the solution in the inner region up with one of the stable uniform states $u = \pm 1$. We conjecture that there are many other types of kinks as well.

An important ingredient in the method we use to prove this theorem is a series of properties of the critical points and values of solution graphs. They will be formulated and in part derived in Section 2. Then, in Section 3, we give an outline of the inductive method developed in [PT2,3] of constructing an infinite sequence of nested closed intervals $I_k \subset \mathbb{R}^+$ of initial slopes α, and the manner it is applied to prove Theorem A. This method uses ideas from studies in [HT1,2] of periodic and chaotic solutions of the Falkner–Skan equation, and chaotic solutions of the Lorenz equations. In Section 4 we present the inductive step, and in Section 5 we then use it to construct a kink with a single zero (at the origin). In Section 6 we extend the method to construct kinks with any given (odd) number of zeros. Finally, in Sections 7 and 8, we prove two global properties of solution graphs, which lie at the heart of the construction of the intervals I_k.

2. Critical points of solution graphs

As indicated in the introduction, the critical points of the solution graphs $u(x,\alpha)$, that is, the zeros of $u'(x,\alpha)$, play a pivotal role in the construction of our heteroclinic orbits, or kinks. In this section we summarize the most important properties of these points. Some were derived in [PT1-3] and others will be proved here.

We begin with a preliminary lemma which implies that all critical points are isolated.

LEMMA 2.1 [PT2]. *Suppose that u is a nonconstant solution of* (1.1a) *and that $u'(x_0) = 0$ at some $x_0 \in \mathbb{R}$.*

(a) *If $u''(x_0) = 0$ then $u(x_0) = \pm 1$ and $u'''(x_0) \neq 0$.*
(b) *If $u(x_0) = \pm 1$ then $u''(x_0) = 0$ and $u'''(x_0) \neq 0$.*

Thus we can number the critical points of the graph of $u(x,\alpha)$. We denote the positive local maxima by ξ_k and the minima by η_k with $k = 1, 2, \ldots$ At inflection points these points coincide. Specifically, we define

$$\xi_1 = \sup\{x > 0 : u' > 0 \text{ on } [0, x)\}. \tag{2.1a}$$

When $u''(\xi_1) < 0$ we set

$$\eta_1 = \sup\{x > \xi_1 : u' < 0 \text{ on } (\xi_1, x)\}. \tag{2.1b}$$

When $u''(\xi_1) = 0$, and so $u(\xi_1) = 1$ by Lemma 2.1, we set

$$\eta_1 = \xi_1. \tag{2.1c}$$

This defines the first terms in the sequences (ξ_k) and (η_k). We can now continue formally to larger values of k. For $k \geq 2$ we define

$$\xi_k = \begin{cases} \sup\{x > \eta_{k-1} : u' > 0 \text{ on } (\eta_{k-1}, x)\} & \\ & \text{if } u' > 0 \text{ in } (\eta_{k-1}, \eta_{k-1} + \delta_1), \\ \eta_{k-1} & \text{otherwise}, \end{cases} \tag{2.2a}$$

where δ_1 is some small positive number. Similarly, we set

$$\eta_k = \begin{cases} \sup\{x > \xi_k : u' < 0 \text{ on } (\xi_k, x)\} & \text{if } u' < 0 \text{ in } (\xi_k, \xi_k + \delta_2), \\ \xi_k & \text{otherwise}, \end{cases} \tag{2.2b}$$

in which δ_2 is some small positive number. It is readily seen that

$$\xi_k \le \eta_k \le \xi_{k+1}, \quad k \ge 1. \tag{2.3}$$

Remark. In (2.3) one of the inequalities must be strict. To see this, suppose that $\xi_k = \eta_k$. Then, because the zeros of u' are isolated by Lemma 2.1, it follows from (2.2b) that $u' > 0$ in a right neighbourhood of ξ_k, so that u has an inflection point at ξ_k, where $u''' > 0$. Hence, by (2.2a), $\eta_k < \xi_{k+1}$.

On the other hand, if $\eta_k = \xi_{k+1}$, then by Lemma 2.1 and (2.2a), $u' < 0$ in a right neighbourhood of ξ_k and u has an inflection point at η_k, where $u''' < 0$. Therefore, by (2.2b), $\xi_k < \eta_k$. In particular, this implies that

$$\xi_k < \xi_{k+1} \quad \text{and} \quad \eta_k < \eta_{k+1} \quad \text{for every } k \ge 1. \tag{2.4}$$

In the following two lemmas we present a few properties of the critical points, which were established in earlier studies on periodic and chaotic solutions of the EFK equation. In particular, we emphasize that, as α changes, critical points are preserved and cannot disappear by coalescing with one another.

Lemma 2.2 [PT2,3]. (a) *Suppose that $\gamma > 1/8$. Then the sequences (ξ_k) and (η_k) are well defined for every $\alpha > 0$.*
(b) *$\xi_k \in C(\mathbb{R}^+)$ and $\eta_k \in C(\mathbb{R}^+)$.*

Remark. Recall that if $0 < \gamma \le 1/8$ then there exists a unique value α_0 for which the corresponding solution $u(x, \alpha_0)$ tends monotonically to 1 (the kink), so that $\xi_1(\alpha_0) = \infty$ and the sequence (ξ_k) is not well defined.

In the following lemma we estimate the critical values $u(\xi_1)$ and $u(\eta_1)$ for small and large values of α in relation to the critical levels $u = \pm 1$.

Lemma 2.3 [PT3]. (a) *Let $\gamma > 1/8$. Then*

$$u(\xi_1(\alpha), \alpha) > \sqrt{2} \quad \textit{if } \alpha > 4,$$
$$u(\xi_1(\alpha), \alpha) < 1 \quad \textit{and} \quad u(\eta_1(\alpha), \alpha) < -\sqrt{2} \quad \textit{if } \alpha < 1/(160\gamma).$$

(b) *Let $\gamma > 0$. Then*

$$u(\eta_1(\alpha), \alpha) < -\sqrt{2} \quad \textit{if } \alpha > 4.$$

An important ingredient in the construction of the sequence of nested intervals relies on an interesting property of the behaviour of $u(\xi_k)$ and $u(\eta_k)$ when these values cross the level $u = 1$. This crucial property is the subject of the next lemma.

LEMMA 2.4. *Suppose that for some* $k \geq 1$,

$$u(\xi_k) = 1, \quad u''(\xi_k) = 0, \quad u'''(\xi_k) > 0 \quad \text{at } \alpha = \alpha^*, \tag{2.5}$$

and for some $\delta > 0$,

$$u(\xi_k(\alpha), \alpha) > 1 \quad \text{for } \alpha^* < \alpha < \alpha^* + \delta. \tag{2.6}$$

Then there exists an $\varepsilon > 0$ *such that*

$$u(\xi_k(\alpha), \alpha) > u(\eta_k(\alpha), \alpha) > 1 \quad \text{for } \alpha^* < \alpha < \alpha^* + \varepsilon. \tag{2.7}$$

PROOF. Suppose, to the contrary, that there exists a sequence $(\alpha_i) \subset (a^*, a^* + \delta)$ such that $\alpha_i \searrow a^*$ and $u(\eta_k(\alpha_i), \alpha_i) \leq 1$. Then, since $u(\xi_k(\alpha_i), \alpha_i) > 1$ for large i, it follows that for each large i there exists a point $x_i \in (\xi_k, \eta_k]$ such that

$$u(x_i, \alpha_i) = 1 \quad \text{and} \quad u'(x_i, \alpha_i) \leq 0. \tag{2.8}$$

By continuity,

$$x_i \to \xi_k(\alpha^*) \quad \text{as } i \to \infty. \tag{2.9}$$

Suppose that there exists a subsequence of (x_i), which we still denote by (x_i), along which $u' = 0$. Then, because $u(\xi_k(\alpha_i), \alpha_i) > 1$, it follows that $u'''(x_i) < 0$. Hence, by continuity,

$$\lim_{i \to \infty} u'''(x_i) \leq 0.$$

However, $u'''(x_i) \to u'''(\xi_k(a^*), a^*)$ as $i \to \infty$, and $u'''(\xi_k(a^*), a^*) > 0$. Therefore we arrive at a contradiction. Thus we may strengthen (2.8) to

$$u(x_i, \alpha_i) = 1 \quad \text{and} \quad u'(x_i, \alpha_i) < 0. \tag{2.8a}$$

At the points x_i, the energy identity states that

$$(2\gamma u''' - u')u' = \gamma(u'')^2 \geq 0$$

and hence, because $u' < 0$ along the sequence,

$$2\gamma u''' \leq u' < 0. \tag{2.10}$$

If we now let $i \to \infty$, we obtain the same contradiction as before.

3. Outline of the method

As we have seen, for $\gamma > 1/8$ kinks cannot be monotone, and this complicates the use of the shooting method to prove their existence. However, by adapting ideas from [PT3], it turns out to be still possible to modify the shooting method to establish the existence of kinks, thanks to the following observation.

Lemma 3.1. *Let $\gamma > 0$ and let $u(x)$ be a solution of equation* (1.1a) *on $\mathbb{R}^+$ such that for some constants $M > 1$ and $a > 0$,*

$$1/\sqrt{3} \le u(x) \le M \quad \text{for } x > a. \tag{3.1}$$

Then $(u, u', u'', u''') \to (1, 0, 0, 0)$ as $x \to \infty$.

This lemma reduces the problem of finding a kink to finding constants $M > 1$ and $a > 0$ and an initial slope α such that if we shoot the solution into the interval $[1/\sqrt{3}, M]$ at $a > 0$, then it remains there for all $x \ge a$.

The proof of Lemma 3.1 relies on the observation that if u is a solution of equation (1.1a) then the function

$$H(x) = \mathcal{H}(u(x)), \tag{3.2}$$

where

$$\mathcal{H}(u) = \frac{1}{4}(u^2 - 1)^2 + \frac{\gamma}{2}(u'')^2, \tag{3.3}$$

is a *convex* function if $u(x) \ge 1/\sqrt{3}$. For the details of the proof we refer to Section 7.

The first step in proving the existence of a kink is to identify a closed interval $I_1 \subset \mathbb{R}^+$ of values of α which guarantee that

$$u(\eta_1(\alpha), \alpha) \ge 1/\sqrt{3} \quad \text{for } \alpha \in I_1. \tag{3.4}$$

The following lemma then implies that $u(\xi_1(\alpha), \alpha) < \sqrt{2}$.

Lemma 3.2 [PT3]. *Let $\gamma > 0$ and let u be an odd solution of equation* (1.1a) *such that $\mathcal{E}(u) = 0$. If there exists a point $a \in \mathbb{R}^+$ such that*

$$|u(a)| \ge \sqrt{2}, \tag{3.5}$$

then

$$|u| > \sqrt{2} \quad \textit{whenever} \quad u' = 0 \ \textit{on}\ (a, \omega),$$

where $[a, \omega)$ is the maximal interval in $[a, \infty)$ on which u exists.

As a result of Lemma 3.2, we observe that if $u(\xi_1) \ge \sqrt{2}$, then $u(\eta_1) < -\sqrt{2}$, which contradicts (3.4).

We then construct a closed subinterval $I_2 \subset I_1$ such that

$$1/\sqrt{3} \le u(\eta_2(\alpha), \alpha) \le u(\xi_2(\alpha), \alpha) < \sqrt{2} \quad \text{for } \alpha \in I_2, \tag{3.6}$$

where the upper bound follows again from Lemma 3.2. Continuing this process, we show that we can inductively obtain a nested sequence (I_k) of closed intervals such that

$$1/\sqrt{3} \le u(\eta_j(\alpha), \alpha) \le u(\xi_j(\alpha), \alpha) < \sqrt{2}, \quad 1 \le j \le k, \quad \text{for } \alpha \in I_k. \tag{3.7}$$

Let $I = \bigcap_{k=1}^{\infty} I_k$. Because the intervals I_k are all closed, their intersection I is nonempty. For any $\alpha \in I$, we have

$$1/\sqrt{3} \le u(\eta_k(\alpha), \alpha) \le u(\xi_k(\alpha), \alpha) < \sqrt{2} \quad \text{for all } k \ge 1. \tag{3.8}$$

This means in particular that

$$1/\sqrt{3} \le u(x, \alpha) \le \sqrt{2} \quad \text{for all } x \in (\eta_1, \infty) \tag{3.9}$$

and so, by Lemma 3.1, $(u, u', u'', u''') \to (1, 0, 0, 0)$ as $x \to \infty$ whenever $\alpha \in I$, and we have established the existence of a kink with one zero (at the origin).

We complete this section with a key property of solutions of equation (1.1a) which is used to execute this programme.

LEMMA 3.3. *Let $\gamma > 0$ and let u be a solution of equation* (1.1a) *on $\mathbb{R}$ such that for some $a \in \mathbb{R}$,*

$$0 \le u(a) \le 1/\sqrt{3}, \quad u'(a) = 0, \quad u''(a) > 0 \quad \textit{and} \quad u'''(a) \ge 0. \tag{3.10}$$

Then

$$|u| > \sqrt{2} \quad \textit{whenever} \quad u' = 0 \textit{ on } (a, \omega). \tag{3.11}$$

Here $[a, \omega)$ is the maximal interval in $[a, \infty)$ on which u exists.

This lemma implies that if u is an *odd* solution of (1.1a) which satisfies

$$0 \le u(a) \le 1/\sqrt{3}, \quad u'(a) = 0, \quad u''(a) > 0 \quad \text{for some } a > 0, \tag{3.12}$$

then

$$u'''(a) > 0. \tag{3.13}$$

To see this, we note that if instead, $u'''(a) \le 0$, then the function $v(x) = u(a - y)$ satisfies (3.10) at $y = 0$, but v has a critical point at $y = 2a$ where $v = 1/\sqrt{3}$. This contradicts (3.11).

We give the proof of Lemma 3.3 in Section 8.

4. The induction process

In this section we define the crucial induction step, in which we construct a sequence of nested closed intervals which share a certain property, which we call *Property* R:

PROPERTY R. We say that an interval $[a, b] \subset \mathbb{R}^+$ has *Property* R for some $i \ge 1$ if

(a) $u(\xi_i(\alpha), \alpha) > 1$ for $\alpha \in (a, b]$,
(b) $\xi_i(a) = \eta_i(a)$ and $u(\xi_i(a), a) = u(\eta_i(a), a) = 1$,
(c) $u(\eta_i(\alpha), \alpha) > 1/\sqrt{3}$ for $\alpha \in [a, b)$,

(d) $u(\eta_i(b), b) = 1/\sqrt{3}$,
(e) $u(\xi_{i+j}(b), b) > 1$ and $u(\eta_{i+j}(b), b) < -1$ for all $j \geq 1$.

The induction process is based on the following lemma.

Lemma 4.1. *If $[a, b] \subset \mathbb{R}^+$ has Property R for $i = k$, where $k \geq 1$, then there exists an interval $[c, d] \subset (a, b)$ which has Property R for $i = k + 1$.*

Proof. We prove the lemma by constructing a subinterval $[c, d]$ which has Property R. We proceed in a series of steps.

Step 1. Define

$$\alpha_1 = \inf\{\alpha_0 < b : u(\eta_k(\alpha), \alpha) < 1 \text{ for } \alpha_0 < \alpha \leq b\}. \tag{4.1}$$

Because $u(\eta_k(b), b) = 1/\sqrt{3}$, we know that α_1 is well defined. In addition, from Lemma 2.4 we know that $u(\eta_k(\alpha), \alpha) > 1$ for $a < \alpha < a + \delta$ when δ is sufficiently small. Therefore, $\alpha_1 > a$. Because $u(\xi_k(\alpha_1), \alpha_1) > 1$ it follows that

$$\eta_k = \xi_{k+1} \quad \text{and} \quad u(\eta_k) = u(\xi_{k+1}) = 1 \quad \text{at } \alpha = \alpha_1. \tag{4.2}$$

Step 2. Define

$$\alpha_2 = \inf\{\alpha_0 < b : u(\xi_{k+1}(\alpha), \alpha) > 1 \text{ for } \alpha_0 < \alpha \leq b\}. \tag{4.3}$$

Because $u(\xi_{k+1}(b), b) > 1$ by Property R(e), it follows that α_2 is well defined. Using Lemma 2.4, but now for η_k, we find that $u(\xi_{k+1}(\alpha), \alpha) < 1$ for $\alpha \in (\alpha_1, \alpha_1 + \delta)$, where δ is again a sufficiently small number, and hence $\alpha_2 > \alpha_1$. Remembering from (4.1) that $u(\eta_k(\alpha_2), \alpha_2) < 1$, we conclude that

$$\xi_{k+1} = \eta_{k+1} \quad \text{and} \quad u(\xi_{k+1}) = u(\eta_{k+1}) = 1 \quad \text{at } \alpha = \alpha_2. \tag{4.4}$$

Step 3. Define

$$\alpha_3 = \sup\{\alpha_0 > \alpha_2 : u(\eta_{k+1}(\alpha), \alpha) > 1/\sqrt{3} \text{ for } \alpha_2 \leq \alpha < \alpha_0\}. \tag{4.5}$$

From Property R(e) we know that $u(\eta_{k+1}(b), b) < -1$. Thus continuity implies that α_3 is well defined and satisfies

$$u(\eta_{k+1}(\alpha_3), \alpha_3) = 1/\sqrt{3}. \tag{4.6}$$

We assert that the interval $[c, d] = [\alpha_2, \alpha_3]$ has Property R for $i = k + 1$. Indeed, it follows from (4.3) that R(a) is satisfied and from (4.4) that R(b) is satisfied. From (4.5) we deduce that R(c) holds and (4.6) states that R(d) holds. Finally, Lemma 3.3 ensures the validity of R(e).

This completes the proof of Lemma 4.1.

5. Existence of a kink with one zero

To start the induction process we need an initial interval $[a, b] \subset \mathbb{R}^+$ which has Property R. In this section we construct such an interval.

As a first step we define

$$\alpha_1 := \inf\{\alpha^* > 0 : u(\xi_1(\alpha), \alpha) > 1 \text{ for } \alpha^* < \alpha < \infty\}. \tag{5.1}$$

Lemma 5.1. *The point α_1 is well defined. We have*

(a) $1/16 < \alpha_1 < 4$ *and* $u(\xi_1(\alpha_1), \alpha_1) = 1$.
(b) $\xi_1(\alpha_1) = \eta_1(\alpha_1)$, $u'''(\xi_1(\alpha_1), \alpha_1) > 0$ *and* $u(\eta_1(\alpha_1), \alpha_1) = 1$.

Proof. It follows from Lemma 2.3(a) that $u(\xi_1(\alpha), \alpha) - 1$ has different signs at $\alpha = 1/16$ and at $\alpha = 4$. By Lemma 2.2, ξ_1 depends continuously on α and so does $u(x, \alpha)$. Thus, $u(\xi_1(\alpha), \alpha)$ depends continuously on α and hence there exists an $\alpha \in (1/16, 4)$ with $u(\xi_1(\alpha), \alpha) - 1 = 0$. The largest such value will be α_1.

As $u' > 0$ on $(0, \xi_1)$ we conclude from the definition of η_1 that $\xi_1(\alpha_1) = \eta_1(\alpha_1)$, and so $u(\xi_1(\alpha_1), \alpha_1) = u(\eta_1(\alpha_1), \alpha_1) = 1$. Because $u' = 0$ and $u'' = 0$ at ξ_1, Lemma 2.1 implies that $u'''(\xi_1) > 0$.

Next, we define

$$\beta_1 := \sup\{\alpha^* > \alpha_1 : u(\eta_1(\alpha), \alpha) > 1/\sqrt{3} \text{ for } \alpha_1 < \alpha < \alpha^*\}. \tag{5.2}$$

Lemma 5.2. *The point β_1 is well defined. We have*

(a) $\alpha_1 < \beta_1 < 4$, $u(\eta_1(\beta_1), \beta_1) = 1/\sqrt{3}$ *and* $u'''(\eta_1(\beta_1), \beta_1) > 0$.
(b) $u(\xi_k(\beta_1), \beta_1) > \sqrt{2}$ *and* $u(\eta_k(\beta_1), \beta_1) < -\sqrt{2}$ *for* $k \geq 2$.

Proof. It follows from Lemma 2.3(b) that $u(\eta_1(\alpha), \alpha) < -\sqrt{2}$ if $\alpha > 4$, and from Lemma 5.1 that $u(\eta_1(\alpha_1), \alpha_1) = 1$. Hence, by the continuity of $u(\eta_1(\alpha), \alpha)$ as a function of α, we conclude that β_1 is well defined, that $\alpha_1 < \beta_1 < 4$ and that $u(\eta_1(\beta_1), \beta_1) = 1/\sqrt{3}$.

Next, we show that $u'''(\eta_1(\beta_1), \beta_1) > 0$. Suppose, to the contrary, that $u'''(\eta_1(\beta_1), \beta_1) \leq 0$. If we now set $y = \eta_1(\beta_1) - x$ and $v(y) = u(x)$, then v satisfies the conditions of Lemma 3.3 with $a = 0$. This means that the critical values of $v(y)$ for $y > 0$ lie either above $\sqrt{2}$ or below $-\sqrt{2}$. However, since u is an odd solution, we know that $v(2\eta_1) = -1/\sqrt{3}$, which contradicts this assertion. This completes the proof of (a).

To prove (b) we use Lemma 3.3 again. (a) states that all the conditions of Lemma 3.3 are satisfied at $a = \eta_1$, and so all the critical values have absolute value larger than $\sqrt{2}$, which proves (b).

Lemma 5.3. *The interval* $I_1 = [\alpha_1, \beta_1]$ *has Property* R *for* $i = 1$.

Proof. It follows from the definition (5.1) of α_1 that R(a) is satisfied, and Lemma 5.1 states that R(b) is satisfied. The definition (5.2) of β_1 ensures that R(c) holds. Finally, we conclude from Lemma 5.2 that R(d) is true, and that we may apply Lemma 3.3 to show that R(e) holds.

We are now ready to establish the existence of kinks with one zero.

Theorem 5.4. *Let* $\gamma > 0$. *Then there exists an odd solution* $u(x)$ *of Problem* (I) *such that*

$$u(x) > 0 \quad (< 0) \quad \text{for } x > 0 \quad (< 0).$$

The sequences (ξ_k) *of local maxima and* (η_k) *of local minima on* $\mathbb{R}^+$ *are such that*

$$1/\sqrt{3} < u(\eta_k) < u(\xi_k) < \sqrt{2} \quad \text{for all } k = 1, 2, \ldots$$

Proof. As we have seen in Lemma 5.3, the interval I_1 constructed above has Property R for $i = 1$, and so, by Lemma 4.1, we can construct by induction a nested sequence (I_k) of closed intervals. Their intersection $I = \bigcap_{k=1}^{\infty} I_k$ is thus a nonempty set.

Let $\alpha_0 \in I$, let $u_0 = u(\cdot, \alpha_0)$, and let $[0, \omega_0)$ be its maximal interval of existence. Since for every $k \geq 1$, the interval I_k has Property R for $i = k$, it follows that

$$1/\sqrt{3} \leq u_0(x) \leq \sqrt{2} \quad \text{for } \eta_1(\alpha_0) \leq x < \omega_0.$$

This implies by standard theory that $\omega_0 = \infty$ and therefore

$$1/\sqrt{3} \leq u_0(x) \leq \sqrt{2} \quad \text{for } \eta_1(\alpha_0) \leq x < \infty.$$

Thus u_0 satisfies the conditions of Lemma 3.1 with $M = \sqrt{2}$, and we may conclude that $(u_0, u_0', u_0'', u_0''') \to (1, 0, 0, 0)$ as $x \to \infty$.

6. Existence of kinks with more than one zero

In this section we construct a family of odd kinks which have an odd number of zeros on $\mathbb{R}$: one zero at the origin, n zeros on $\mathbb{R}^+$ and n zeros on $\mathbb{R}^-$. The graph of each of these kinks can be divided into three regions: an inner region $(-M, M)$ in which the kink is approximately periodic, and two outer regions, $(-\infty, -M)$ and (M, ∞), in which the kink has tails which join up with the uniform stable states $u = \pm 1$ at $x = \pm\infty$.

These kinks are constructed by continuation of a periodic solution $\overline{u}$ of equation (1.1a) with the property

$$0 < \overline{u}(\xi_k) < 1 \quad \text{and} \quad -1 < \overline{u}(\eta_k) < 0 \quad \text{for } k = 1, 2, \ldots \tag{6.1}$$

In [PT2] we found that such a solution exists for any $\gamma > 1/8$.

We shall distinguish two cases:

(i) $u'(0) > 0$. In this case n is *even* and we write $n = 2m$ with $m = 1, 2, \ldots$

(ii) $u'(0) < 0$. In this case n is *odd* and we write $n = 2m-1$ with $m = 1, 2, \ldots$

Case (i): $u'(0) > 0$. To demonstrate the main ideas, we first construct a kink with 5 zeros ($n = 2$). Set $\overline{\alpha} = \overline{u}'(0) > 0$. By (6.1), we have $0 < u(\xi_1(\overline{\alpha}), \overline{\alpha}) < 1$ and so we can define

$$\alpha_1 = \sup\{\alpha^* > \overline{\alpha} : u(\xi_1) < 1 \text{ on } [\overline{\alpha}, \alpha^*)\}. \tag{6.2}$$

Lemma 6.1. *We have*

(a) $\overline{\alpha} < \alpha_1 < 4$ *and* $u(\xi_1(\alpha_1), \alpha_1) = 1$.

(b) $\xi_1(\alpha_1) = \eta_1(\alpha_1)$, $u'''(\xi_1(\alpha_1), \alpha_1) > 0$ *and* $u(\eta_1(\alpha_1), \alpha_1) = 1$.

Proof. Part (a) follows from the continuity of ξ_1 on $\mathbb{R}^+$ (Lemma 2.2) and Lemma 2.3(a). Part (b) follows from Lemma 2.1 and the definition of η_1.

Because (6.1) also implies that $-1 < u(\eta_1(\overline{\alpha}), \overline{\alpha}) < 0$, we can define

$$\alpha_2 = \sup\{\alpha^* > \overline{\alpha} : u(\eta_1) < 0 \text{ on } [\overline{\alpha}, \alpha^*)\}. \tag{6.3}$$

Lemma 6.2. *We have*

(a) $\overline{\alpha} < \alpha_2 < \alpha_1$ *and* $u(\eta_1(\alpha_2), \alpha_2) = 0$.

(b) $u(\xi_2(\alpha_2), \alpha_2) > 1$ *and* $u(\eta_2(\alpha_2), \alpha_2) < -1$.

Proof. Since $u(\eta_1(\overline{\alpha}), \overline{\alpha}) < 0$ and $u(\eta_1(\alpha_1), \alpha_1) = 1 > 0$ by Lemma 6.1, (a) follows from the continuity of $u(\eta_1(\alpha), \alpha)$. Part (b) follows from Lemma 3.3.

Next, we define

$$\alpha_3 = \inf\{\alpha^* < \alpha_2 : u(\xi_2) > 1 \text{ on } (\alpha^*, \alpha_2]\}. \tag{6.4}$$

Lemma 6.3. *We have*

(a) $\overline{\alpha} < \alpha_3 < \alpha_2$ *and* $u(\xi_2(\alpha_3), \alpha_3) = 1$.

(b) $\xi_2(\alpha_3) = \eta_2(\alpha_3)$, $u'''(\xi_2(\alpha_3), \alpha_3) > 0$ *and* $u(\eta_2(\alpha_3), \alpha_3) = 1$.

Proof. We recall from (6.1) and Lemma 6.2 that $u(\xi_2(\overline{\alpha}), \overline{\alpha}) < 1$ and $u(\xi_2(\alpha_2), \alpha_2) > 1$. Hence, by continuity, (a) follows.

Observe that $u(\eta_1) < 0 < u(\xi_1) < 1 = u(\xi_2)$ at α_3. Hence, by Lemma 2.1, $u'''(\xi_2) > 0$, and (b) follows.

Because

$$u(\eta_2(\alpha_3), \alpha_3) = 1 \quad \text{and} \quad u(\eta_2(\alpha_2), \alpha_2) < -1 \tag{6.5}$$

by Lemma 6.2, we can define

$$\alpha_4 = \sup\{\alpha^* > \alpha_3 : u(\eta_2) > 1/\sqrt{3} \text{ on } [\alpha_3, \alpha^*)\}. \tag{6.6}$$

Lemma 6.4. *We have*

(a) $\alpha_3 < \alpha_4 < \alpha_2$ *and* $u(\eta_2(\alpha_4), \alpha_4) = 1/\sqrt{3}$.
(b) $u(\xi_{2+j}(\alpha_4), \alpha_4) > 1$ *and* $u(\eta_{2+j}(\alpha_4), \alpha_4) < -1$ *for* $j = 1, 2, \ldots$

Proof. Part (a) follows from (6.5) and the continuity of $u(\eta_2(\alpha), \alpha)$, and (b) follows from Lemma 3.3.

Lemma 6.5. *The interval* $I_2 = [\alpha_3, \alpha_4]$ *has Property* R *for* $i = 2$.

Proof. Because $I_2 \subset (\alpha_3, \alpha_2)$, it follows that $u(\xi_2) > 1$ on $(\alpha_3, \alpha_4]$ with $u(\xi_2) = 1$ at α_3, and so R(a) is satisfied.

R(b) is satisfied by Lemma 6.3(b), and R(c) and R(d) are satisfied in view of the definition of α_4. Finally, R(e) is satisfied by Lemma 6.4(b).

Summarizing, we have constructed an interval I_2 which has Property R such that

$$\text{(6.7a)}\quad u(\eta_1) < 0 < u(\xi_1) < 1 < u(\xi_2) \quad \text{and} \quad u(\eta_2) \geq 1/\sqrt{3} \quad \text{when } \alpha \in (\alpha_3, \alpha_4],$$
$$\text{(6.7b)}\quad u(\eta_1) < 0 < u(\xi_1) < 1 = u(\xi_2) \quad \text{and} \quad u(\eta_2) > 1/\sqrt{3} \quad \text{when } \alpha = \alpha_3.$$

We can now use the Induction Lemma to construct a nested sequence $(I_k)_{k=2}^{\infty}$ of closed subintervals such that if $\alpha_0 \in I = \bigcap_{k=2}^{\infty} I_k$, then $u_0 = u(\cdot, \alpha_0)$ exists for all $x \in \mathbb{R}$ and

$$1/\sqrt{3} \leq u_0(x) < \sqrt{2} \quad \text{for } \eta_2 \leq x < \infty. \tag{6.8}$$

This implies by Lemma 3.1, and the fact that u_0 is odd, that

$$(u_0, u_0', u_0'', u_0''') \to (\pm 1, 0, 0, 0) \quad \text{as } x \to \pm\infty. \tag{6.9}$$

Lemma 6.6. *For* $\alpha_0 \in I = \bigcap_{k=2}^{\infty} I_k$, *the solution* $u(x, \alpha_0)$ *is a kink with precisely* 5 *zeros.*

Proof. It follows from (6.9) that u_0 is a kink. Thus, it remains to count its zeros. By (6.7) it has one positive zero in the interval (ξ_1, η_2) and one in (η_1, ξ_2). By (6.8), these are the only zeros on $\mathbb{R}^+$. Thus, $n = 2$ and the total number of zeros of u_0 on $\mathbb{R}$ equals $2n + 1 = 5$.

We next show how this algorithm can be extended to construct an interval I_n $(n > 2)$ such that if $\alpha_0 \in I = \bigcap_{k=n}^{\infty} I_k$, then the solution $u(x, \alpha_0)$ has precisely $2n + 1$ zeros on $\mathbb{R}$.

Let $n > 2$ be an *even* integer; for convenience we shall write $n = 2m$ $(m > 1)$. Define

$$a_m = \sup\{\alpha > \overline{\alpha} : u(\xi_m) < 1 \text{ on } [\overline{\alpha}, \alpha)\}.$$

Because of Lemmas 2.3 and 3.2, it follows that $u(\xi_i) > \sqrt{2}$ at $\alpha = 4$, for $i \geq 1$. Hence $a_m \in (\overline{\alpha}, 4)$.

For the critical values of $u(x,\alpha)$ on $(0,\xi_m(a_m))$ when $\alpha \in [\overline{\alpha}, a_m)$, we can make the following statements.

LEMMA 6.7. *For any* $\alpha \in [\overline{\alpha}, a_m)$ *we have*

(a) $u(\xi_i) < 1$ *for* $i = 1,\dots,m$,
(b) $u(\xi_i) > 0$ *for* $i = 1,\dots,m-1$,
(c) $u(\eta_i) < 0$ *for* $i = 1,\dots,m-1$,
(d) $u(\eta_i) > -1$ *for* $i = 1,\dots,m-1$.

PROOF. (a) Suppose that

$$a_1 = \sup\{\alpha > \overline{\alpha} : u(\xi_1) < 1 \text{ on } [\overline{\alpha},\alpha)\} \in (\overline{\alpha}, a_m).$$

Then at a_1 we have $u(\xi_1) = 1$, $u''(\xi_1) = 0$ and $u'''(\xi_1) > 0$, and hence $u(\xi_2) > 1$. Thus

$$a_2 = \sup\{\alpha > \overline{\alpha} : u(\xi_2) < 1 \text{ on } [\overline{\alpha},\alpha)\} \in (\overline{\alpha}, a_1).$$

Since $u(\eta_1) \le u(\xi_1) < 1$ at a_2, it follows that $u'''(\xi_2) > 0$ and hence $u(\xi_3) > 1$ at a_2. Continuing inductively we conclude that $u(\xi_m) > 1$ at a value of α in the interval $(\overline{\alpha}, a_m)$. This contradicts the definition of a_m.

(b) Suppose that for some $i \in \{1,\dots,m-1\}$ we have

$$\widehat{\alpha} = \sup\{\alpha > \overline{\alpha} : u(\xi_i) > 0 \text{ on } [\overline{\alpha},\alpha)\} \in (\overline{\alpha}, a_m).$$

Then Lemma 3.3 implies that $u(\xi_{i+1}) > 1$ at $\widehat{\alpha}$, which contradicts (a).

(c) Suppose that for some $i \in \{1,\dots,m-1\}$ we have

$$\widehat{\alpha} = \sup\{\alpha > \overline{\alpha} : u(\eta_i) < 0 \text{ on } [\overline{\alpha},\alpha)\} \in (\overline{\alpha}, a_m).$$

Then Lemma 3.3 implies that $u(\xi_{i+1}) > 1$ at $\widehat{\alpha} \in (\overline{\alpha}, a_m)$, which contradicts (a).

(d) Suppose that for some $i \in \{1,\dots,m-1\}$ we have

$$\widehat{\alpha} = \sup\{\alpha > \overline{\alpha} : u(\eta_i) > -1 \text{ on } [\overline{\alpha},\alpha)\} \in (\overline{\alpha}, a_m).$$

Then, since, by (a), $u(\xi_i) > 0$, it follows that $u'''(\eta_i) < 0$ and hence $\xi_{i+1} = \eta_i$. Thus $u(\xi_{i+1}) = -1$, which contradicts (b). This completes the proof of Lemma 6.7.

We now distinguish two cases:

(I) $u(\xi_m(\alpha),\alpha) > 0$ for all $\alpha \in [\overline{\alpha}, a_m)$.
(II) $b_m = \sup\{\alpha > \overline{\alpha} : u(\xi_m) > 0 \text{ on } [\overline{\alpha},\alpha)\} < a_m$.

CASE (I). Since, as in Lemma 6.1, $u(\eta_m) = 1$ at a_m, we have

$$c_m = \sup\{\alpha > \overline{\alpha} : u(\eta_m) < 0 \text{ on } [\overline{\alpha},\alpha)\} \in (\overline{\alpha}, a_m).$$

At c_m we have $u(\eta_m) = 0$ and therefore, by Lemma 3.3, $u(\xi_{m+1}) > 1$. Since $u(\xi_{m+1}) \in (0,1)$ at $\overline{\alpha}$, it follows from continuity that

$$d_m = \inf\{\alpha < c_m : u(\xi_{m+1}) > 1 \text{ on } (\alpha, c_m]\} \in (\overline{\alpha}, c_m).$$

At d_m we have $u(\eta_m) \le u(\xi_m) < 1$ and $u(\xi_{m+1}) = 1$. This implies that $u'''(\xi_{m+1}) > 0$ and $u(\eta_{m+1}) = 1$. At c_m we have $u(\eta_{m+1}) < -1$ by Lemma 3.3. Thus

$$e_m = \sup\{\alpha > d_m : u(\eta_{m+1}) > 1/\sqrt{3} \text{ on } [d_m, \alpha)\} \in (d_m, c_m).$$

It follows by Lemma 6.7 and by inspection that

Lemma 6.8. *The interval $I_{m+1} = [d_m, e_m]$ has Property* R *for $i = m+1$. If $\alpha \in I_{m+1}$, then*

$$-1 < u(\eta_i) < 0 < u(\xi_i) < 1 \quad \textit{for } i = 1, \ldots, m-1, \tag{6.10a}$$

$$0 < u(\xi_m) < 1, \tag{6.10b}$$

$$u(\xi_{m+1}) \ge 1 \quad \textit{and} \quad u(\eta_{m+1}) > 1/\sqrt{3}. \tag{6.10c}$$

In fact, $u(\xi_{m+1}) > 1$ in $(d_m, e_m]$.

Case (II). We divide this case into two subcases:

(IIa) $u(\eta_m(\alpha), \alpha) < 0$ for all $\alpha \in [\overline{\alpha}, b_m]$.

(IIb) $b_m^* = \sup\{\alpha > \overline{\alpha} : u(\eta_m) < 0 \text{ on } [\overline{\alpha}, \alpha)\} \in (\overline{\alpha}, b_m)$.

Remark. This subdivision is made in order to ensure that there is a zero between ξ_m and η_m. In this way we keep control of the number of zeros of the solution.

Case (IIa). From Lemma 3.3 we know that $u(\xi_{m+1}) > 1$ at b_m and so, because $u(\xi_{m+1}) < 1$ at $\overline{\alpha}$, we conclude that

$$c_m^* = \inf\{\alpha < b_m : u(\xi_{m+1}) > 1 \text{ on } (\alpha, b_m]\} \in (\overline{\alpha}, b_m).$$

Since by assumption, $u(\eta_m) < 0$ at c_m^*, it follows that $u(\eta_{m+1}) = 1$ at c_m^*. On the other hand, by Lemma 3.3, $u(\eta_{m+1}) < -1$ at b_m and so

$$d_m^* = \sup\{\alpha > c_m^* : u(\eta_{m+1}) > 1/\sqrt{3} \text{ on } [d_m, \alpha)\} \in (c_m^*, b_m).$$

It follows by Lemma 6.7 and by inspection that

Lemma 6.9A. *The interval $I_{m+1} = [c_m^*, d_m^*]$ has Property* R *for $i = m+1$. If $\alpha \in I_{m+1}$, then* (6.10) *holds.*

Case (IIb). We know from Lemma 3.3 that $u(\xi_{m+1}) > 1$ at b_m^*. Remembering that $u(\xi_{m+1}) < 1$ at $\overline{\alpha}$, we conclude that

$$c_m^* = \inf\{\alpha < b_m^* : u(\xi_{m+1}) > 1 \text{ on } (\alpha, b_m^*]\} \in (\overline{\alpha}, b_m^*).$$

From the definition of b_m^* we know that $u(\eta_m) < 0$ at c_m^* and so $u(\eta_{m+1}) = 1$ at c_m^*. Since by Lemma 3.3, $u(\eta_{m+1}) < -1$ at b_m^* we conclude that

$$d_m^* = \sup\{\alpha > c_m^* : u(\eta_{m+1}) > 1/\sqrt{3} \text{ on } [c_m^*, \alpha)\} \in (c_m^*, b_m^*).$$

Again it follows by Lemma 6.7 and by inspection that

Lemma 6.9B. *The interval* $I_{m+1} = [c_m^*, d_m^*]$ *has Property* R *for* $i = m+1$. *If* $\alpha \in I_{m+1}$, *then* (6.10) *holds.*

Case (ii): $u'(0) < 0$. For this case the analysis is very similar to the one for the previous case. Rather than repeating all the details of Case (i), we sketch the construction of a kink with 7 zeros ($n = 3$). The few differences and the many similarities will be apparent from this example.

We set $\overline{\alpha} = \overline{u}'(0) < 0$, where $\overline{u}$ is a periodic solution of equation (1.1a) with the properties (6.1).

Since $u'(0) < 0$, the first zero of u' is now a local *minimum* or an inflection point, and we write

$$\eta_1 = \sup\{x > 0 : u' < 0 \text{ on } [0, x)\}.$$

Proceeding in the manner of Section 2, we denote the next critical point by ξ_2 and continue with η_2, $\xi_3, \ldots$ When we set $v = -u$ in Lemma 5.2 of [PT3] we see that $u(\eta_1(\alpha), \alpha) > -1$ for $-1/(16\gamma) < \alpha < 0$, and hence we can choose the periodic solution $\overline{u}$ with initial slope $\overline{\alpha}$, so that $u(\eta_1(\alpha), \alpha) > -1$ for $\overline{\alpha} \le \alpha < 0$.

From Lemma 5.6 of [PT3] we similarly know that there exists an $\widehat{\alpha} < 0$ such that $u(\xi_2(\alpha), \alpha) > \sqrt{2}$ when $\alpha \in (\widehat{\alpha}, 0)$. Hence, we can define

$$\alpha_1 = \sup\{\alpha^* > \overline{\alpha} : u(\xi_2(\alpha), \alpha) < 1 \text{ for } \overline{\alpha} < \alpha < \alpha^*\},$$

and we see, because ξ_2 is continuous and $u(\eta_1(\alpha_1), \alpha_1) < 0$, that

$$\begin{gathered} \overline{\alpha} < \alpha_1 < 0 \quad \text{and} \quad u(\xi_2(\alpha_1), \alpha_1) = 1; \\ \xi_2(\alpha_1) = \eta_2(\alpha_1), \quad u'''(\xi_2(\alpha_1), \alpha_1) > 0, \quad u(\eta_2(\alpha_1), \alpha_1) = 1. \end{gathered}$$

We first suppose that

$$u(\xi_2(\alpha), \alpha) > 0 \quad \text{for } \overline{\alpha} < \alpha < \alpha_1. \tag{6.11}$$

Because $u(\eta_2(\overline{\alpha}), \overline{\alpha}) < 0$ and $u(\eta_2(\alpha_1), \alpha_1) = 1$, we can define

$$\alpha_2 = \sup\{\alpha^* > \overline{\alpha} : u(\eta_2(\alpha), \alpha) < 0 \text{ for } \overline{\alpha} < \alpha < \alpha^*\}.$$

As in Lemma 6.2, we find that

$$\begin{gathered} \overline{\alpha} < \alpha_2 < \alpha_1 \quad \text{and} \quad u(\eta_2(\alpha_2), \alpha_2) = 0; \\ u(\xi_3(\alpha_2), \alpha_2) > 1, \quad u(\eta_3(\alpha_2), \alpha_2) < -1. \end{gathered}$$

In turn, because $u(\xi_3(\overline{\alpha}),\overline{\alpha}) < 1$ and $u(\xi_3(\alpha_2),\alpha_2) > 1$, we can define

$$\alpha_3 = \inf\{\alpha^* < \alpha_2 : u(\xi_3(\alpha),\alpha) > 1 \text{ for } \alpha^* < \alpha \le \alpha_2\}$$

and we obtain

$$\begin{gathered}\overline{\alpha} < \alpha_3 < \alpha_2 \quad \text{and} \quad u(\xi_3(\alpha_3),\alpha_3) = 1;\\ \xi_3(\alpha_3) = \eta_3(\alpha_3), \quad u'''(\xi_3(\alpha_3),\alpha_3) > 0, \quad u(\eta_3(\alpha_3),\alpha_3) = 1.\end{gathered}$$

Finally, since $u(\eta_3(\alpha_3),\alpha_3) = 1$ and $u(\eta_3(\alpha_2),\alpha_2) < -1$, we can define

$$\alpha_4 = \sup\{\alpha^* > \alpha_3 : u(\eta_3(\alpha),\alpha) > 1/\sqrt{3} \text{ for } \alpha_3 \le \alpha < \alpha^*\}$$

and we have

$$\begin{gathered}\alpha_3 < \alpha_4 < \alpha_2 \quad \text{and} \quad u(\eta_3(\alpha_4),\alpha_4) = 1/\sqrt{3};\\ u(\xi_{3+j}(\alpha_4),\alpha_4) > 1 \quad \text{and} \quad u(\eta_{3+j}(\alpha_4),\alpha_4) < -1 \quad \text{for } j = 1,2,\ldots\end{gathered}$$

As in Case (i) we find that the interval $I_3 = [\alpha_3,\alpha_4]$ has Property R for $i = 3$, and that for $\alpha \in I_3$,

$$-1 < u(\eta_1) < 0 < u(\xi_2) < 1 < u(\xi_3) \quad \text{and} \quad u(\eta_2) < 0 \quad \text{when } \alpha \in (\alpha_3,\alpha_4], \tag{6.12a}$$

$$-1 < u(\eta_1) < 0 < u(\xi_2) < 1 = u(\xi_3) \quad \text{and} \quad u(\eta_2) < 0 \quad \text{when } \alpha = \alpha_3. \tag{6.12b}$$

Next, suppose that (6.11) does not hold and so

$$\widehat{\alpha}_1 = \sup\{\alpha^* > \overline{\alpha} : u(\xi_2(\alpha),\alpha) > 0 \text{ for } \overline{\alpha} < \alpha < \alpha^*\} \in (\overline{\alpha},\alpha_1).$$

Then

$$\begin{gathered}\overline{\alpha} < \widehat{\alpha}_1 < 0 \quad \text{and} \quad u(\xi_2(\widehat{\alpha}_1),\widehat{\alpha}_1) = 0;\\ u(\eta_2(\widehat{\alpha}_1),\widehat{\alpha}_1) < -1, \quad u(\xi_3(\widehat{\alpha}_1),\widehat{\alpha}_1) > 1, \quad u(\eta_3(\widehat{\alpha}_1),\widehat{\alpha}_1) < -1.\end{gathered}$$

Suppose that

$$u(\eta_2(\alpha),\alpha) < 0 \quad \text{for } \overline{\alpha} < \alpha < \widehat{\alpha}_1. \tag{6.13}$$

Then we proceed as in the previous case to construct I_3.

Finally, suppose that (6.13) is false and that

$$\widetilde{\alpha}_2 = \sup\{\alpha^* > \overline{\alpha} : u(\eta_2(\alpha),\alpha) < 0 \text{ for } \overline{\alpha} < \alpha < \alpha^*\} \in (\overline{\alpha},\widehat{\alpha}_1).$$

Then we have $u(\xi_3(\widetilde{\alpha}_2),\widetilde{\alpha}_2) > 1$ and $u(\eta_3(\widetilde{\alpha}_2),\widetilde{\alpha}_2) < -1$ and we can proceed from this point to complete the construction of I_3.

In both cases the inequalities (6.12) are valid.

Using the Iteration Lemma we find that there exists an $\alpha_0 \in I_3$ such that $u(x, \alpha_0)$ is a kink, which, by (6.12), has precisely $n = 3$ positive zeros and hence 7 zeros on $\mathbb{R}$.

7. Proof of Lemma 3.1

We recall

LEMMA 3.1. *Let* $\gamma > 0$ *and let* $u(x)$ *be a solution of* (1.1a) *on* (a, ∞) *such that*

$$1/\sqrt{3} \le u(x) \le M \quad \textit{for } x > a \tag{7.1}$$

for some constant $M > 1$. *Then* $\lim_{x\to\infty}(u, u', u'', u''') = (1, 0, 0, 0)$.

In the proof the functional

$$\mathcal{H}(u) := \frac{1}{4}(u^2 - 1)^2 + \frac{\gamma}{2}(u'')^2$$

plays a central role. The following properties of $\mathcal{H}$ are readily verified: if u is a smooth function, then the function $H(x) = \mathcal{H}(u(x))$ yields upon differentiation

$$H'(x) = (u^3 - u)u' + \gamma u'' u''' \tag{7.2}$$

and, if u is a solution of (1.1a), then

$$H''(x) = (3u^2 - 1)(u')^2 + (u'')^2 + \gamma(u''')^2. \tag{7.3}$$

In the following lemma we show that $H(x)$ and $H'(x)$ tend to a limit as $x \to \infty$.

LEMMA 7.1. *Suppose that* $u(x)$ *is a solution of* (1.1a) *on* (a, ∞) *for some* $a \in \mathbb{R}$, *and that* (7.1) *holds for some* $M > 1$. *Then*

$$H'(x) \to 0 \quad \textit{and} \quad H(x) \to l \quad \textit{as } x \to \infty, \tag{7.4}$$

where l *is a nonnegative number.*

PROOF. In view of (7.1), $H'' \ge 0$ on (a, ∞) and so $H'(x)$ tends to a limit as $x \to \infty$:

$$\lim_{x\to\infty} H'(x) = L.$$

Since $H(x) \ge 0$ it is clear that $L \ge 0$. Suppose that $L > 0$. Then $H(x) \to \infty$ as $x \to \infty$ and so $|u''(x)| \to \infty$ as $x \to \infty$. This means that $u(x)$ becomes unbounded as $x \to \infty$, which contradicts (7.1). Thus $L = 0$.

Remembering that $H'' \ge 0$, we may conclude that

$$H'(x) \le 0 \quad \text{for } x > a. \tag{7.5}$$

Hence, since $H \ge 0$, it follows that $\lim_{x\to\infty} H(x) =: l \ge 0$ exists.

Lemma 7.2. *Suppose that* $u(x)$ *is a solution of* (1.1a) *on* (a,∞) *such that* (7.1) *holds. Then* $\lim_{x\to\infty} H(x) = 0$.

An immediate corollary of Lemma 7.2 is that $u(x)$ tends to 1.

Corollary 7.3. *Suppose that* $u(x)$ *is a solution of* (1.1a) *on* (a,∞) *such that* (7.1) *holds. Then* $\lim_{x\to\infty}(u,u',u'',u''') = (1,0,0,0)$.

Proof. Lemma 7.2 and the definition of H imply that

$$\lim_{x\to\infty}(u,u'') = (1,0). \tag{7.6}$$

Next, we claim that

$$\lim_{x\to\infty} u'(x) = 0. \tag{7.7}$$

If this is not so, then there exists a constant $\delta \neq 0$ and an unbounded increasing sequence $(x_k) \subset \mathbb{R}^+$ such that

$$\lim_{k\to\infty} u'(x_k) = \delta. \tag{7.8}$$

Suppose that $\delta > 0$ (the details of the case $\delta < 0$ are similar). We consider the sequence of intervals $[x_k, x_k+1]$. From (7.6) and (7.8) we conclude that $u' \geq \delta/2$ on $[x_k, x_k+1]$ for k large enough. Hence, for such k,

$$u(x_k+1) - u(x_k) = \int_{x_k}^{x_k+1} u'(t)\,dt \geq \delta/2. \tag{7.9}$$

It follows from (7.6) that $u(x_k+1) - u(x_k) \to 0$ as $k\to\infty$. This contradicts (7.9) and thus proves (7.7).

Next, we combine (7.6) with (1.1a) and observe that $\lim_{x\to\infty} u^{\mathrm{iv}}(x) = 0$. We may now repeat the argument we used to prove (7.7), with u, u', u'' replaced by u'', u''', u^{iv}, to show that $\lim_{x\to\infty} u'''(x) = 0$. We omit the details.

Proof of Lemma 7.2. Suppose to the contrary that $l > 0$. Then, because of (7.5), $H(x) \geq l$ for $x > a$.

We first show that u' must vanish along a sequence (x_n) which tends to infinity. Suppose instead that u is ultimately monotone. Then, since $|u(x)|$ is uniformly bounded, $\lim_{x\to\infty} u(x) =: m$ exists. Since $H(x) \to l$ as $x\to\infty$ this means that

$$\lim_{x\to\infty}(u''(x))^2 \text{ exists} = \frac{2}{\gamma}\left\{l - \frac{1}{4}(m^2-1)^2\right\}. \tag{7.10}$$

Because $|u(x)|$ is uniformly bounded, the limit in (7.10) can only be zero, and so $m = \sqrt{1 \pm 2\sqrt{l}} \neq 1$. Therefore, taking the limit in the equation (7.1) we find that

$$\gamma u^{\mathrm{iv}}(x) \to m - m^3 \neq 0 \quad \text{as } x\to\infty$$

and so $|u(x)| \to \infty$ as $x\to\infty$, which contradicts (7.1).

Thus, there exists an infinite sequence (x_n) of zeros of u' with $x_n \to \infty$ as $n \to \infty$. We choose the sequence so that $u' \neq 0$ between two consecutive points x_n and x_{n+1}. Observe that along the sequence, the energy functional $\mathcal{E}(u)$ becomes

$$\frac{\gamma}{2}(u'')^2 = \frac{1}{4}(u^2-1)^2 \quad \text{at } x_n,$$

and so

$$H(x_n) = \frac{1}{2}\{u^2(x_n)-1\}^2, \tag{7.11a}$$

as well as

$$H(x_n) = \gamma\{u''(x_n)\}^2. \tag{7.11b}$$

Therefore, by Lemma 7.1,

$$\gamma\{u''(x_n)\}^2 \to l \quad \text{as } n \to \infty. \tag{7.12}$$

It follows that the critical points are eventually all isolated local maxima or minima. Along the maxima we have

$$u(x_n) \to \sqrt{1+\sqrt{2l}} \quad \text{as } n \to \infty. \tag{7.13}$$

To force a contradiction we turn to the integral

$$J_n := \int_{x_n}^{x_{n+1}} H''(x)\,dx = H'(x_{n+1}) - H'(x_n). \tag{7.14}$$

From Lemma 7.1 we conclude that

$$J_n \to 0 \quad \text{as } n \to \infty. \tag{7.15}$$

Let (x_k) be a sequence of maxima and write

$$u(x_k) = 1+\delta_k, \quad u''(x_k) = -\mu_k, \quad \delta_k, \mu_k > 0.$$

Then

$$\delta_k \to \sqrt{1+\sqrt{2l}} - 1 \quad \text{and} \quad \mu_k \to \sqrt{l/\gamma}.$$

In addition, we set $u'''(x_k) = \varepsilon_k$. Then $H'(x_k) = -\gamma\mu_k\varepsilon_k$. Remembering (7.5) and Lemma 7.1, we conclude that $\varepsilon_k \geq 0$ and $\varepsilon_k \to 0$ as $k \to \infty$.

In a neighbourhood of x_k we have $u > 1$ and $u'' < 0$, and as long as these inequalities remain true, we have by (1.1a), after two integrations over (x_k, x),

$$u''(x_k+y) < -\mu_k + \varepsilon_k y < -\frac{1}{2}\mu_k \tag{7.16}$$

for k large enough.

To determine the interval over which $u > 1$ we use equation (1.1a) again, but now we estimate u^{iv} from below. We have

$$\gamma u^{\text{iv}} > u'' - \sigma, \quad \sigma = -(1+\delta) + (1+\delta)^3 > 0,$$

where we have omitted the subscript k from δ and μ. Upon integration over $(x_k, x_k + y)$ we obtain, as long as $u > 1$,

$$\begin{aligned}
\gamma u'''(x_k + y) &> u'(x_k + y) - \sigma y, \\
\gamma u''(x_k + y) &> -(\gamma\mu + \delta) - \frac{1}{2}\sigma y^2, \\
\gamma u'(x_k + y) &> -(\gamma\mu + \delta)y - \frac{1}{3!}\sigma y^3, \\
\gamma u(x_k + y) &> \gamma(1 + \delta) - \frac{1}{2}(\gamma\mu + \delta)y^2 - \frac{1}{4!}\sigma y^4.
\end{aligned}$$

Thus, there exists an interval $(x_k, x_k + \eta_k)$ on which $u > 1$, and $\eta_k \to \eta^* > 0$ as $k \to \infty$. It follows from (7.16) that for k large enough,

$$J_k \geq \int_{x_k}^{x_k+\eta_k} \{u''(x)\}^2\,dx > \frac{1}{4}\mu_k^2\eta_k.$$

Hence

$$\liminf_{k\to\infty} J_k \geq \frac{l}{4\gamma}\eta^* > 0,$$

which contradicts (7.15).

Thus $l = 0$, the proof of Lemma 7.2 is complete and so is the proof of Lemma 3.1.

8. Proof of Lemma 3.3

We recall

Lemma 3.3. *Let $\gamma > 0$ and let u be a solution of equation* (1.1a) *on $\mathbb{R}$ such that for some $a \in \mathbb{R}$,*

$$0 \leq u(a) \leq 1/\sqrt{3}, \quad u'(a) = 0, \quad u''(a) > 0 \quad \textit{and} \quad u'''(a) \geq 0.$$

Then

$$|u| > \sqrt{2} \quad \textit{whenever} \quad u' = 0 \textit{ on } (a, \omega).$$

Here $[a, \omega)$ is the maximal interval in $[a, \infty)$ on which u exists.

Proof. Without loss of generality we may set $a = 0$. It will be convenient to scale the variable x, and set $x^* = \gamma^{-1/4}x$ and $u^*(x^*) = u(x)$. Then the function u^* satisfies the equation

$$u^{\mathrm{iv}} = \sigma u'' + u - u^3, \quad \sigma = 1/\sqrt{\gamma}, \tag{8.1}$$

where we have dropped the asterisks. The energy identity then becomes

$$2u'u''' - (u'')^2 - \sigma(u')^2 + \frac{1}{2}(1 - u^2)^2 = 0. \tag{8.2}$$

Thus, if $u(0) = \theta \in [0, 1)$, then

$$u''(0) = \frac{1}{\sqrt{2}}(1 - \theta^2). \tag{8.3}$$

It is clear from (1.1) that as long as $u < 1$, then

$$u''' > 0, \quad u'' > u''(0), \quad u' > 0$$

and so

$$x_0 = \sup\{x > 0 : u < 1 \text{ on } [0, x)\} < \infty.$$

At $x = x_0$ we have

$$u''' > 0, \quad u'' > u''(0), \quad u' > \kappa, \tag{8.4}$$

where $\kappa = 2^{-1/4}(1-\theta)\sqrt{1+\theta}$. The first two inequalities are evident. To prove the third inequality, we suppose that $u'(x_0) \le \kappa$. Then, because $u'' > 0$ on $[0, x_0]$, it follows that $u' < \kappa$ on $[0, x_0)$ and so we find, upon integration over $(0, x_0)$, $1 - \theta < \kappa x_0$ or $x_0 > (1-\theta)/\kappa$. Since $u''' > 0$ on $(0, x_0)$, this implies that

$$u'(x_0) = \int_0^{x_0} u''(s)\, ds > u''(0)x_0 = \frac{1}{\sqrt{2}}(1-\theta^2)\frac{1-\theta}{\kappa} = \kappa,$$

a contradiction.

For later reference we note that if $\theta = 1/\sqrt{3}$, then $\kappa > 0.446$.

Next we continue the solution above $u = 1$ and define

$$x_1 = \sup\{x > x_0 : u'' > 0 \text{ on } [x_0, x)\}.$$

We shall show that $u(x)$ rises above $\sqrt{2}$ on the interval (x_0, x_1).

Suppose to the contrary that

$$u(x) \le \sqrt{2} \quad \text{for } x_0 \le x \le x_1. \tag{8.5}$$

When we use this bound in equation (8.1) and integrate repeatedly over (x_0, x), with $x \in (x_0, x_1)$, and write $y = x - x_0$, we obtain, in view of the conditions (8.4) at $x = x_0$,

$$u^{\mathrm{iv}}(x) > -\sqrt{2}, \quad u'''(x) > -\sqrt{2}y, \quad u''(x) > \frac{1-\theta^2}{\sqrt{2}} - \frac{1}{\sqrt{2}}y^2,$$

and so

$$x_1 - x_0 > \sqrt{1-\theta^2}. \tag{8.6}$$

Two further integrations yield

$$u'(x) > \kappa + \frac{1-\theta^2}{\sqrt{2}}y - \frac{1}{3\sqrt{2}}y^3,$$

$$u(x) > 1 + \kappa\, y + \frac{1-\theta^2}{2\sqrt{2}}y^2 - \frac{1}{12\sqrt{2}}y^4. \tag{8.7}$$

By (8.6), the lower bound (8.7) for $u(x)$ holds for $x = x_0 + y^*$, where $y^* = \sqrt{1-\theta^2}$. Thus, we find that

$$u(x_0 + y^*) > 1 + \kappa\sqrt{1-\theta^2} + \frac{5}{12\sqrt{2}}(1-\theta^2)^2 =: \phi(\theta), \tag{8.8}$$

where

$$\phi(\theta) = 1 + 2^{-1/4}(1+\theta)(1-\theta)^{3/2} + \frac{5}{12\sqrt{2}}(1-\theta^2)^2. \tag{8.9}$$

For $\theta \in [0, 1/\sqrt{3}]$ an elementary computation yields $u(x_0 + y^*) > \sqrt{2}$, which contradicts the initial assumption (8.5). This proves the assertion.

The proof is completed with an application of Lemma 3.2.

Remark. The upper bound for $u(a)$ in Lemma 3.3 can be raised to the first zero $\theta_0 \in [0,1)$ of the function $\phi(\theta)$. It is found that $.63 < \theta_0 < 1$.

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