Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 6, 1995, 295–308

CRITICAL POINTS WHEN THERE IS NO SADDLE POINT GEOMETRY

Martin Schechter

Dedicated to Louis Nirenberg on the occasion of his 70th birthday

We show how mountain pass methods can be used even when there is no mountain pass or saddle point geometry. The functional can grow on linking subsets. Applications are given.

1. Introduction

The original saddle point theorem of Rabinowitz [R] can be described as follows. Let M be a closed subspace of a Hilbert space H such that $M \neq H$ and $N = M^{\perp} \neq H$. Let G be a C^1 functional on H such that

$$G(v) \leq \alpha, \quad v \in N, \tag{1.1}$$

$$G(w) \geq \alpha, \quad w \in \partial B_\delta \cap M, \tag{1.2}$$

$$G(sw_0 + v) \leq m_R, \quad s \geq 0,\ v \in N,\ \|sw_0 + v\| = R, \tag{1.3}$$

for some $w_0 \in \partial B_1 \cap M$, where $0 < \delta < R$ and $B_r = \{u \in H : \|u\| \leq r\}$. A slight generalization of Rabinowitz's theorem is

Theorem 1.1. *In addition to* (1.1)–(1.3) *assume*

$$\dim N < \infty, \tag{1.4}$$

$$m_R \leq \alpha. \tag{1.5}$$

1991 *Mathematics Subject Classification*. Primary 35J65, 58E05, 49B27, 47H15.
Research supported in part by an NSF grant.

Then there are a sequence $\{u_k\} \subset H$ *and a* $c \geq \alpha$ *such that*

$$G(u_k) \to c, \quad G'(u_k) \to 0. \tag{1.6}$$

Although (1.6) does not guarantee a solution of

$$G'(u) = 0 \tag{1.7}$$

it does so in many applications. Theorem 1.1 was generalized by Silva [Si] and the author [Sc1–4] to

THEOREM 1.2. *If one replaces* (1.5) *in Theorem* 1.1 *by*

$$\sup_{R>\delta} m_R \leq m < \infty \tag{1.8}$$

then there is a sequence satisfying (1.6) *with* $\alpha \leq c \leq m$.

In the present paper we derive theorems which imply

THEOREM 1.3. *Assume*

$$\textit{either } \dim N < \infty \textit{ or } \dim M < \infty, \tag{1.9}$$

(1.10) $\psi(t)$ *is a nondecreasing positive function with* $m_R - \alpha < \int_0^R \psi(t)\,dt$.

Then for each $\delta > 0$ *there is a* $u \in H$ *such that*

$$\alpha - \delta < G(u) < m_R + \delta, \quad \|G'(u)\| < \psi(\|u\| + \delta). \tag{1.11}$$

COROLLARY 1.4. *If* $\beta \geq 0$ *and*

$$m_R/R^{\beta+1} \to 0 \quad \textit{as } R \to \infty \tag{1.12}$$

then there is a $\{u_k\} \subset H$ *such that*

$$\|G'(u_k)\|/(\|u_k\| + 1)^\beta \to 0, \quad G(u_k) \to c, \quad \alpha \leq c \leq \infty. \tag{1.13}$$

We consider applications to semilinear elliptic boundary value problems. Let Ω be a smooth, bounded domain in $\mathbb{R}^n$, and let $A \geq \lambda_0 > 0$ be a selfadjoint operator on $L^2(\Omega)$ such that $C_0^\infty(\Omega) \subset D := D(A^{1/2}) \subset H^m(\Omega)$, $m > 0$, with eigenvalues $\lambda_0 < \lambda_1 < \ldots$ and eigenfunctions that are in $L^\infty(\Omega)$. Let $f(x,t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$ such that

$$|f(x,t)| \leq C(|t| + 1). \tag{1.14}$$

Let

$$F(x,t) = \int_0^t f(x,s)\,ds. \tag{1.15}$$

We assume

(I) For some eigenvalue λ_ℓ, $\ell \geq 1$, we have

$$2F(x,t) \leq \lambda_\ell t^2, \quad |t| < \delta, \tag{1.16}$$

$$2F(x,t) \geq \lambda_{\ell-1} t^2, \quad t \in \mathbb{R}, \tag{1.17}$$

$$2F(x,t) \geq \lambda_\ell t^2 - V(x)^2 h(t) - W(x), \quad t \in \mathbb{R}, \tag{1.18}$$

where $\delta > 0, W \in L^1(\Omega)$, $h(t)$ is a locally bounded function satisfying

$$h(t)/t^2 \to 0 \quad \text{as } t^2 \to \infty \tag{1.19}$$

and $V(x) \in L^2(\Omega)$ maps D into $L^2(\Omega)$.

(II) $f(x,t)/t \to \alpha_\pm(x)$ a.e. as $t \to \pm\infty$.

(III) If

$$Au = \alpha_+ u^+ - \alpha_- u^-, \tag{1.20}$$

where $u^\pm = \max\{\pm u, 0\}$, then $u \equiv 0$.

We have

Theorem 1.5. *Under the above hypotheses the equation*

$$Au = f(x,u) \tag{1.21}$$

has at least one nontrivial solution.

We can change the directions of the inequalities (1.16)–(1.18). In fact, we can replace hypothesis (I) with

(I′) For some eigenvalue $\lambda_\ell, \ell \geq 0$, we have

$$2F(x,t) \geq \lambda_\ell t^2, \quad |t| < \delta, \tag{1.22}$$

$$2F(x,t) \leq \lambda_{\ell+1} t^2, \quad t \in \mathbb{R}, \tag{1.23}$$

$$2F(x,t) \leq \lambda_\ell t^2 + V(x)^2 h(t) + W(x), \quad t \in \mathbb{R}, \tag{1.24}$$

with V, W, h as before.

Theorem 1.6. *Under hypotheses* (I′), (II) *and* (III), (1.21) *has at least one nontrivial solution.*

There are other geometries covered by our theorems. For instance, we have

Theorem 1.7. *Assume that* (1.4) *holds and that*

$$G(v) \leq m_R, \quad v \in N \cap \partial B_R, R > 0, \tag{1.25}$$

$$G(w) \geq \alpha, \quad w \in M. \tag{1.26}$$

If $\beta \geq 0$ *and* (1.12) *holds, then there is a sequence satisfying* (1.13).

THEOREM 1.8. *Assume that*

$$G(0) \le \alpha \le G(u), \quad u \in \partial B_\delta,\ \delta > 0, \tag{1.27}$$

and there is a $\varphi_0 \in \partial B_1$ *such that*

$$G(R\varphi_0) \le m_R, \quad R > \delta.$$

If $\beta \ge 0$ *and* (1.12) *holds, then there is a sequence satisfying* (1.13).

Applications of these theorems will be given in forthcoming publications. Our abstract theorems are given in the next section. Proofs of the theorems of this section are given in Section 3.

2. The abstract theorems

We recall the definition of linking given in [ST]. Let E be a Banach space and let Φ be the set of all continuous maps $\Gamma(t)$ from $E \times [0,1]$ to E such that

(a) $\Gamma(0) = I$,
(b) there is an $x_0 \in E$ such that $\Gamma(1)x = x_0$ for each $x \in E$,
(c) $\Gamma(t)x \to x_0$ as $t \to 1$ uniformly on bounded subsets of E,
(d) for each $t \in [0,1), \Gamma(t)$ is a homeomorphism of E onto itself,
(e) $\Gamma(t)^{-1}$ is continuous from $E \times [0,1)$ to E.

DEFINITION. A subset A of E *links* a subset B of E if $A \cap B = \emptyset$ and for each $\Gamma \in \Phi$ there is a $t \in [0,1]$ such that $\Gamma(t)A \cap B \neq \emptyset$.

Let A, B be subsets of E such that A links B, and let G be a C^1 functional on E. Define

$$a = \inf_{\Gamma\in\Phi} \sup_{0\le s\le 1, u\in A} G(\Gamma(s)u), \quad a_0 = \sup_A G, \quad b_0 = \inf_B G. \tag{2.1}$$

Since A links B we have

$$b_0 \le a. \tag{2.2}$$

Assume that

$$d(A,B) > 0. \tag{2.3}$$

Let

$$B' := \{v \in B : G(v) < a_0\}. \tag{2.4}$$

Note that

$$B' = \emptyset \quad \text{iff} \quad a_0 \le b_0. \tag{2.5}$$

Let $\psi(t)$ be a positive nondecreasing function on $[0,\infty)$ such that

$$a_0 - b_0 < \int_0^R \psi(t)\,dt \tag{2.6}$$

for some finite $R \le d' := d(B', A)$ (we take $d' = \infty$ if $B' = \emptyset$). Our first result is

THEOREM 2.1. *Assume in addition that*

$$-\infty < b_0, a < \infty. \tag{2.7}$$

Then for every $\delta > 0$ there is a $u \in E$ such that

$$b_0 - \delta \le G(u) \le a + \delta, \quad \|G'(u)\| < \psi(d(u, A)). \tag{2.8}$$

PROOF. Assume first that $a = a_0$. If (2.8) were not true, then there would be a $\delta > 0$ such that

$$\psi(d(u, A)) \le \|G'(u)\| \tag{2.9}$$

holds for all u in the set

$$Q = \{u \in E : b_0 - 3\delta \le G(u) \le a + 3\delta\}. \tag{2.10}$$

By reducing δ if necessary we can find $\theta < 1$ and $T < R$ such that

$$a - b_0 + \delta < \theta \int_0^T \psi(t)\,dt, \quad \delta < \theta\psi(0)T. \tag{2.11}$$

Let

$$\begin{aligned} &Q_0 = \{u \in Q : b_0 - 2\delta \le G(u) \le a + 2\delta\},\\ &Q_1 = \{u \in Q : b_0 - \delta \le G(u) \le a + \delta\},\\ &Q_2 = E \setminus Q_0, \quad \eta(u) = d(u, Q_2)/[d(u, Q_1) + d(u, Q_2)]. \end{aligned}$$

There is a locally Lipschitz continuous map $Y(u)$ of $\widehat{E} = \{u \in E : G'(u) \ne 0\}$ to itself such that

$$\|Y(u)\| \le 1, \quad \theta\|G'(u)\| \le (G'(u), Y(u)), \quad u \in \widehat{E}. \tag{2.12}$$

Let $\sigma(t)$ be the flow generated by $\eta(u)Y(u)$. Then

$$\sigma(t)v - v = \int_0^t \eta(\sigma(\tau)v)Y(\sigma(\tau)v)\,d\tau.$$

Consequently,

$$d(v, A) - t \le d(\sigma(t)v, A) \le d(v, A) + t, \quad t > 0. \tag{2.13}$$

We also have

$$\begin{aligned} dG(\sigma(t)v)/dt &= (G'(\sigma), \sigma') = \eta(\sigma)(G'(\sigma), Y(\sigma)) \ge \theta\eta(\sigma)\|G'(\sigma)\|\\ &\ge \theta\eta(\sigma)\psi(d(\sigma, A)) \ge \theta\eta(\sigma)\psi(d(v, A) - t) \end{aligned} \tag{2.14}$$

in view of (2.9), (2.12) and (2.13). Now suppose $v \in B$ is such that there is a $t_1 \in [0, T]$ for which $\sigma(t_1)v \notin Q_1$. Then $G(\sigma(t_1)v) > a + \delta$. Consequently,

$$G(\sigma(T)v) > a + \delta. \tag{2.15}$$

On the other hand, if $\sigma(t)v \in Q_1$ for all $t \in [0, T]$, then

$$G(\sigma(T)v) \geq G(v) + \theta \int_0^T \psi(d(\sigma(t)v, A))\, dt. \tag{2.16}$$

If $v \in B'$, this gives

$$\begin{aligned} G(\sigma(T)v) &\geq b_0 + \theta \int_0^T \psi(d(v, A) - t)\, dt \geq b_0 + \theta \int_{d-T}^d \psi(\tau)\, d\tau \\ &\geq b_0 + \theta \int_0^T \psi(\tau)\, d\tau > a + \delta \end{aligned}$$

by (2.11), where $d = d(v, A) \geq d' \geq R > T$. Thus (2.15) holds in this case as well. If $v \in B \setminus B'$, then

$$G(\sigma(T)v) \geq a + \theta \int_0^T \psi(d(\sigma(t)v, A))\, dt \geq a + \theta\psi(0)T > a + \delta$$

by (2.11) and (2.16). Hence (2.15) holds for all $v \in B$.

We shall show that A links $B_1 := \sigma(T)B$. If so, this will contradict the definition of a. For there is a $\Gamma \in \Phi$ such that

$$G(\Gamma(s)u) < a + (\delta/2), \quad 0 \leq s \leq 1,\ u \in A. \tag{2.17}$$

But if A links B_1, then there is a $t_1 \in [0, 1]$ such that $\Gamma(t_1)A \cap B_1 \neq \emptyset$. This means that there is a $u_1 \in A$ such that $\Gamma(t_1)u_1 \in B_1$. In view of (2.15) this would imply $G(\Gamma(t_1)u_1) > a + \delta$, contradicting (2.17).

Thus it remains to show that A links B_1. To this end, note that $\sigma(t)v \notin A$ for $v \in B$ and $t \in [0, T]$. For $v \in B'$ this follows from (2.13) and the fact that $T < R \leq d'$. If $v \in B \setminus B'$ we have, by (2.14),

$$G(\sigma(t)v) \geq a + \theta \int_0^t \eta(\sigma(\tau)v)\psi(d(\sigma(\tau)v, A))\, d\tau.$$

Thus

$$G(\sigma(t)v) > a, \quad t > 0, \tag{2.18}$$

unless $\eta(v) = 0$. But this would mean that $v \in \overline{Q}_2$, and consequently that $G(v) \geq a + 2\delta$. Hence (2.18) holds for all $v \in B \setminus B'$. Therefore $\sigma(t)v$ cannot intersect A for all $v \in B$ and $t \in [0, T]$. Let Γ be any map in Φ. Define

$$\Gamma_1(t) = \begin{cases} \sigma(2tT)^{-1}, & 0 \leq t \leq 1/2, \\ \sigma(T)^{-1}\Gamma(2t - 1), & 1/2 < t \leq 1. \end{cases}$$

Then $\Gamma_1 \in \Phi$. Since A links B, there is a $t_1 \in [0,1]$ such that $\Gamma_1(t_1)A \cap B \neq \emptyset$. If $0 \leq t_1 \leq 1/2$, this would mean $\sigma(2t_1T)^{-1}A \cap B \neq \emptyset$, or, equivalently, that $A \cap \sigma(2t_1T)B \neq \emptyset$, contradicting the fact that $\sigma(t)B$ does not intersect A for $t \in [0,T]$. Thus we must have $1/2 < t_1 \leq 1$. This says that $\sigma(T)^{-1}\Gamma(2t_1 - 1)A \cap B \neq \emptyset$, or, equivalently, $\Gamma(2t_1 - 1)A \cap \sigma(T)B \neq \emptyset$. Thus A links $B_1 = \sigma(T)B$, and the proof is complete for the case $a = a_0$. If $a \neq a_0$, then it follows from Corollary 2.8 of [Sc5] that there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to a, \quad G'(u_k) \to 0.$$

Thus for each $\delta > 0$ we can find a u_k such that $a - \delta \leq G(u_k) \leq a + \delta$, $\|G'(u_k)\| < \psi(0) \leq \psi(d(u_k, A))$ which gives (2.8) for this case as well. □

Theorem 2.2. *Assume* (2.7), *and let*

$$A'' := \{u \in A : G(u) > b_0\}. \tag{2.19}$$

Assume that (2.6) *holds for some* $R \leq d'' := d(A'', B)$. *Then for each* $\delta > 0$ *there is a* $u \in E$ *such that*

$$b_0 - \delta \leq G(u) \leq a + \delta, \quad \|G'(u)\| < \psi(d(u,B)). \tag{2.20}$$

Proof. As before, we may assume $a = a_0$. If the conclusion of the theorem were false, there would be a $\delta > 0$ such that

$$\psi(d(u,B)) \leq \|G'(u)\|, \quad u \in Q, \tag{2.21}$$

where Q is given by (2.10). Let θ, T, Q_0, Q_1, Q_2, η and Y be determined as in the proof of Theorem 2.1. Let $\sigma(t)$ be the flow generated by $-\eta(u)Y(u)$. Then we have

$$d(u,B) - t \leq d(\sigma(t)u, B) \leq d(u,B) + t \tag{2.22}$$

and

$$\begin{aligned} dG(\sigma(t)u)/dt &= -\eta(\sigma)(G'(\sigma), Y(\sigma)) \leq -\theta\eta(\sigma)\|G'(\sigma)\| \\ &\leq -\theta\eta(\sigma)\psi(d(\sigma,B)). \end{aligned} \tag{2.23}$$

Hence

$$G(\sigma(t)u) \leq G(u) - \theta \int_0^t \eta(\sigma(\tau)u)\psi(d(\sigma(\tau)u, B))\, d\tau. \tag{2.24}$$

Now suppose $u \in A$ is such that there is a $t_1 \in [0,T]$ for which $\sigma(t_1)u \notin Q_1$. Then

$$G(\sigma(T)u) \leq G(\sigma(t_1)u) < b_0 - \delta. \tag{2.25}$$

On the other hand, if $\sigma(t)u \in Q_1$ for $0 \le t \le T$, then

$$G(\sigma(T)u) \le G(u) - \theta \int_0^T \psi(d(\sigma(t)u, B))\, dt. \tag{2.26}$$

If $u \in A''$, this implies

$$\begin{aligned} G(\sigma(T)u) &\le a - \theta \int_0^T \psi(d(u,B) - t)\, dt \\ &\le a - \theta \int_{d-T}^{d} \psi(\tau)\, d\tau \le a - \theta \int_0^T \psi(\tau)\, d\tau < b_0 - \delta, \end{aligned}$$

where $d = d(u,B) \ge d'' \ge R > T$. Thus (2.25) holds. If $u \in A \setminus A''$, then

$$G(\sigma(T)u) \le b_0 - \theta\psi(0)T < b_0 - \delta.$$

Hence (2.25) holds for all $u \in A$.

Let $A_1 = \sigma(T)A, a_{10} = \sup_{A_1} G$ and

$$a_1 = \inf_{\Gamma\in\Phi} \sup_{0\le s\le 1, u\in A_1} G(\Gamma(s)u). \tag{2.27}$$

I claim that

(a) $a_{10} < b_0 \le a_1 \le a$,
(b) A_1 links B.

If (a) and (b) are true, then it follows from Theorem 2.1 of [ST] that there is a sequence $\{u_k\} \subset D$ such that

$$G(u_k) \to a_1, \quad G'(u_k) \to 0.$$

This will contradict (2.21), establishing the theorem. First let us prove (b). We show that $\sigma(t)A$ does not intersect B for $0 \le t \le T$. For $u \in A''$ this follows from (2.22) and the fact that $T < R \le d''$. If $u \in A \setminus A''$, we see from (2.24) that

$$G(\sigma(t)u) < b_0, \quad t \ge 0, \tag{2.28}$$

unless $\eta(u) = 0$, i.e., unless $u \in \overline{Q}_2$. But then $G(u) \le b_0 - 2\delta$. Hence (2.27) holds for all $u \in A \setminus A''$. This will prevent $\sigma(t)u$ from intersecting B at any time. This now implies (b). Let Γ be any map in Φ. Let

$$\Gamma_1(s) = \begin{cases} \sigma(2sT), & 0 \le s \le 1/2, \\ \Gamma(2s-1)\sigma(T), & 1/2 < s \le 1. \end{cases}$$

Then $\Gamma_1 \in \Phi$. Since A links B, there is an $s_1 \in [0,1]$ such that $\Gamma_1(s_1)A \cap B \ne \emptyset$. Since $\sigma(2sT)A \cap B = \emptyset$ for $0 \le s \le 1/2$, we must have $s_1 > 1/2$. Hence $\Gamma(2s_1 - 1)\sigma(T)A \cap B \ne \emptyset$, showing that $A_1 = \sigma(T)A$ links B.

The first inequality in (a) follows from (2.25) while the second follows from (b) and (2.27). To prove the last, let Γ be any map in Φ. Let

$$\widetilde{\Gamma}(s) = \begin{cases} \sigma(2sT)^{-1}, & 0 \le s \le 1/2, \\ \Gamma(2s-1)\sigma(T)^{-1}, & 1/2 < s \le 1. \end{cases}$$

Then $\widetilde{\Gamma} \in \Phi$ and

$$G(\widetilde{\Gamma}(s)\sigma(T)u) = \begin{cases} G(\sigma(T-2sT)u) \le G(u), & 0 \le s \le 1/2, \\ G(\Gamma(2s-1)u), & 1/2 < s \le 1. \end{cases}$$

Therefore

$$\sup_{0\le s\le 1, u\in A_1} G(\widetilde{\Gamma}(s)u) \le \sup_{0\le s\le 1, u\in A} G(\Gamma(s)u).$$

Thus $a_1 \le a$, and the proofs of (a) and the theorem are complete. □

Corollary 2.3. *Let $\{A_k, B_k\}$ be a sequence of pairs of sets satisfying the hypotheses of Theorem 2.1 such that $d'_k = d(A_k, B'_k) \to \infty$ as $k \to \infty$ and for some $\beta \ge 0$,*

$$(a_{k0} - b_{k0})/(d'_k)^{\beta+1} \to 0 \quad \text{as } k \to \infty. \tag{2.29}$$

Then there is a sequence $\{u_k\} \subset E$ such that

$$b_{k0} - (1/k) \le G(u_k) \le a_k + (1/k), \quad \|G'(u_k)\|/(d(u_k, A_k) + 1)^\beta. \tag{2.30}$$

Similarly, if $d''_k = d(A''_k, B_k) \to \infty$ and

$$(a_{k0} - b_{k0})/(d''_k)^{\beta+1} \to 0 \tag{2.31}$$

then there is a sequence $\{u_k\} \subset E$ such that

$$b_{k0} - (1/k) \le G(u_k) \le a_k + (1/k), \quad \|G'(u_k)\|/(d(u_k, B_k) + 1)^\beta \to 0. \tag{2.32}$$

Proof. For each k take

$$\psi_k(t) = (\beta+1)(a_{k0} - b_{k0})(t+1)^\beta / R_k^{\beta+1}$$

with R_k equal to d'_k or d''_k, as the case may be. Then

$$\int_0^{R_k} \psi_k(t)\,dt = (a_{k0} - b_{k0})[(R_k+1)^{\beta+1} - 1]/R_k^{\beta+1} > a_{k0} - b_{k0}.$$

By Theorems 2.1 and 2.2 there is a u_k such that

$$b_{k0} - (1/k) \le G(u_k) \le a_k + (1/k)$$

and either $\|G'(u_k)\| < \psi_k(d(u_k, A_k))$ or $\|G'(u_k)\| < \psi(d(u_k, B_k))$, as the case may be. We now merely note that $\psi_k(t_k)/(t_k+1)^\beta \to 0$ as $k \to \infty$ for any sequence $t_k \ge 0$. □

3. The applications

In this section we shall prove Theorems 1.5 and 1.6. We let N_ℓ denote the subspace of $L^2(\Omega)$ spanned by the eigenfunctions of A corresponding to the eigenvalues $\lambda_0, \dots, \lambda_\ell$. Let $M_\ell = N_\ell^\perp \cap D$. Thus $D = M_\ell \oplus N_\ell$. Let

$$a(u,v) = (Au,v), \quad \|u\|_D^2 = a(u,u), \quad u,v \in D, \tag{3.1}$$

and

$$G(u) = \|u\|_D^2 - 2\int_\Omega F(x,u)\,dx, \quad u \in D. \tag{3.2}$$

It is readily verified under hypothesis (1.14) that G is a C^1 functional on D and

$$(G'(u),v) = 2a(u,v) - 2(f(u),v), \quad u,v \in D, \tag{3.3}$$

where we write $f(u)$ for $f(x,u)$. It therefore follows that u is a solution of (1.21) iff

$$G'(u) = 0. \tag{3.4}$$

Now (1.17) implies

$$G(v) \le \|v\|_D^2 - \lambda_{\ell-1}\|v\|^2 \le 0, \quad v \in N_{\ell-1}. \tag{3.5}$$

We also have, by (1.18),

$$G(v) \le \|v\|_D^2 - \lambda_\ell\|v\|^2 + \int V(x)^2 h(v)\,dx + B, \tag{3.6}$$

where $B = \int_\Omega W(x)\,dx$.

Let $\varepsilon > 0$ be given. By (1.19) there is a K such that $|h(t)|/t^2 < \varepsilon$ for $|t| > K$. Thus

$$\begin{aligned}\int V(x)^2 h(v)\,dx &\le \int_{|v|<K} + \int_{|v|>K} \le C_1 \int V(x)^2\,dx + \varepsilon \int V(x)^2 v(x)^2\,dx \\ &\le C_1\|V\|^2 + \varepsilon C_2 \|v\|_D^2,\end{aligned}$$

where

$$\|Vv\|^2 \le C_2\|v\|_D^2, \quad v \in D. \tag{3.7}$$

Thus by (3.6),

$$G(v) \le C_1\|V\|^2 + \varepsilon C_2 R^2 + B, \quad v \in N_\ell,\ \|v\|_D = R.$$

Thus $m_R = \sup\{G(v) : v \in N_\ell,\ \|v\|_D = R\}$ satisfies $\limsup_{R\to\infty} m_R/R^2 \le \varepsilon C_2$. Therefore

$$m_R/R^2 \to 0 \quad \text{as } R \to \infty. \tag{3.8}$$

For $w \in M_{\ell-1}$, write $w = w' + y$, where $w' \in M_\ell$ and $y \in E(\lambda_\ell)$, the eigenspace of λ_ℓ. Since $E(\lambda_\ell)$ is finite-dimensional and contained in $L^\infty(\Omega)$, there is a $\varrho > 0$ such that

$$\|y\|_D \le \varrho \quad \text{implies} \quad |y(x)| \le \delta/2, \tag{3.9}$$

where $\delta > 0$ is given in (1.16). Thus if $\|w\|_D \le \varrho$ and $|w(x)| \ge \delta$ then

$$\delta \le |w(x)| \le |w'(x)| + |y(x)| \le |w'(x)| + \delta/2. \tag{3.10}$$

Hence

$$|w(x)| \le 2|w'(x)|. \tag{3.11}$$

Thus by (1.16) and (1.14),

$$G(w) \ge \|w'\|_D^2 - \lambda_\ell \|w'\|^2 - C \int_{|w|>\delta} (w^2 + |w|)\, dx. \tag{3.12}$$

By the Sobolev imbedding theorem, there is a $p > 2$ such that

$$\|w\|_p \le C_p \|w\|_D, \quad w \in D. \tag{3.13}$$

Hence

$$\begin{aligned} \int_{|w|>\delta} (w^2 + |w|)\, dx &\le (\delta^{2-p} + \delta^{1-p}) \int_{|w|>\delta} |w|^p\, dx \\ &\le C \int_{2|w'|>\delta} |w'|^p\, dx \le C\|w'\|_p^p \le C'\|w'\|_D^p \end{aligned}$$

by (3.11) and (3.13). Hence by (3.12),

$$\begin{aligned} G(w) &\ge (1 - (\lambda_\ell/\lambda_{\ell+1}) - C'\|w'\|_D^{p-2})\|w'\|_D^2 \\ &\ge \varepsilon \|w'\|_D^2, \quad w \in M_{\ell-1},\ \|w\|_D \le \varrho, \end{aligned} \tag{3.14}$$

for ϱ sufficiently small. Now suppose

$$\inf\{G(w) : w \in M_{\ell-1},\ \|w\|_D = \varrho\} = 0. \tag{3.15}$$

Then there is a sequence $\{w_k\} \subset M_{\ell-1}$ such that $\|w_k\|_D = \varrho$ and $G(w_k) \to 0$. Write $w_k = w_k' + y_k$, where $w_k' \in M_\ell$ and $y_k \in E(\lambda_\ell)$. By (3.14) we see that $\|w_k'\|_D \to 0$. Hence $\|y_k\|_D \to \varrho$. Thus for a renamed subsequence, $y_k \to y$ in D and $\|y\|_D = \varrho$. By (3.9), $|y(x)| \le \delta/2$. Hence by (1.16),

$$2F(x, y(x)) \le \lambda_\ell y(x)^2, \quad x \in \Omega. \tag{3.16}$$

Also

$$\|y\|_D^2 - 2\int_\Omega F(x, y)\, dx = G(y) = 0.$$

In other words,

$$\int_\Omega \{2F(x, y) - \lambda_\ell y^2\}\, dx = 0.$$

Since the integrand is never positive, we have

$$2F(x,y(x)) \equiv \lambda_\ell y(x)^2, \quad x \in \Omega. \tag{3.17}$$

For $\zeta(x) \in C_0^\infty(\Omega)$ and t small we have

$$2F(x, y+t\zeta) \le \lambda_\ell (y+t\zeta)^2.$$

Hence for $t > 0$,

$$2t^{-1}[F(x,y+t\zeta) - F(x,y)] \le \lambda_\ell[(y+t\zeta)^2 - y^2]/t.$$

Taking the limit as $t \to 0$, we obtain

$$f(x,y)\zeta \le \lambda_\ell y\zeta, \quad \zeta \in C_0^\infty(\Omega).$$

This implies $f(x,y) \equiv \lambda_\ell y$. Since $Ay = \lambda_0 y$, we see that y is a solution of (1.21). Since $\|y\|_D = \varrho$, $y \ne 0$. Thus if (3.15) holds, we have a nontrivial solution of (1.21). Thus we may assume that the left hand side of (3.15) is positive. But in this case the hypotheses of Corollary 1.4 are satisfied with $\beta = 1$. Hence there is a sequence $\{u_k\} \subset D$ such that

$$G'(u_k)/(\|u_k\|_D + 1) \to 0, \quad G(u_k) \ge \varepsilon. \tag{3.18}$$

I claim that

$$R_k = \|u_k\|_D \le C. \tag{3.19}$$

For if $R_k \to \infty$, then we take $\widetilde{u}_k = u_k/R_k$. Then $\|\widetilde{u}_k\|_D = 1, \widetilde{u}_k \to \widetilde{u}$ weakly in D, strongly in $L^2(\Omega)$ and a.e. in Ω for a renamed subsequence. Hence by (3.18),

$$(G'(u_k), u_k)/(2R_k^2) = 1 - (f(u_k), u_k)/R_k^2 \to 1 - \alpha(\widetilde{u}) = 0,$$

where

$$\alpha(u) = \int_\Omega \{\alpha_+(x)(u^+)^2 + \alpha_-(x)(u^-)^2\}\, dx.$$

This shows that $\widetilde{u} \ne 0$. Moreover, we have

$$(G'(u_k), v)/(2R_k) = a(\widetilde{u}_k, v) - (f(u_k)/R_k, v) \to a(\widetilde{u}, v) - \alpha(\widetilde{u}, v) = 0, \quad v \in D,$$

where

$$\alpha(u,v) = \int_\Omega \{\alpha_+ u^+ - \alpha_- u^-\} v\, dx.$$

Hence $\widetilde{u}$ is a solution of (1.20). By hypothesis (III) this implies that $\widetilde{u} = 0$, contradicting our previous conclusion. Hence (3.19) holds.

This implies the existence of a renamed subsequence such that $u_k \to u$ weakly in D, strongly in $L^2(\Omega)$ and a.e. in Ω. By (3.18),

$$(G'(u_k), v) = 2a(u_k, v) - 2(f(u_k), v) \to 2a(u,v) - 2(f(u), v) = 0, \quad v \in D.$$

Hence u is a solution of (1.21). We have also

$$\|u_k\|^2 = (G'(u_k), u_k)/2 + (f(u_k), u_k) \to (f(u), u) = \|u\|_D^2.$$

Thus $u_k \to u$ strongly in D, and $0 < \varepsilon \le G(u_k) \to G(u)$. This shows that $u \ne 0$, and the proof of Theorem 1.5 is complete. □

Proof of Theorem 1.6. By following the methods of the previous proof, we show that

$$G(v) \le -\varepsilon\|v'\|_D^2, \quad v \in N_\ell,\ \|v\|_D^2 \le \varrho,$$

for $\varrho > 0$ sufficiently small, where $v = v' + y$, $v' \in N_{\ell-1}$, $y \in E(\lambda_\ell)$. Also

$$G(w) \ge 0, \quad w \in M_\ell,$$

and

$$G(w) \ge -m_R, \quad w \in M_{\ell-1},\ \|w\|_D = R,$$

where m_R satisfies (3.8). Reasoning as before we show that either (1.21) has a nontrivial solution in $E(\lambda_\ell)$ or

$$G(v) \le -\varepsilon_1 < 0, \quad v \in N_\ell,\ \|v\|_D = \varrho.$$

Now we can apply Corollary 1.4 to $-G$ to obtain the desired conclusion. □

Proof of Theorem 1.3. We take $A = \{v \in N : \|v\| \le R\} \cup \{sw_0 + v : s \ge 0,\ v \in N,\ \|sw_0 + v\| = R\}$, $B = \partial B_\delta \cap M$. Then A and B link each other (Proposition 1.2 of [ST]), and $a_0 \le m_R$, $\alpha \le b_0$. The hypotheses of Theorem 2.2 are satisfied, and we note that $d(u, B) \le \|u\| + \delta$. □

Proof of Corollary 1.4. We take $\psi_R(t) = (\beta+1)m_R(1+t)^\beta/R^{\beta+1}$. Then

$$\int_0^R \psi_R(t)\,dt = m_R[(1+R)^{\beta+1} - 1]/R^{\beta+1} > m_R.$$

Hence there is a u_R such that $\alpha - (1/R) \le G(u_R) \le m_R + (1/R)$ and

$$\|G'(u_R)\|/(\|u_R\| + 1) < (\beta+1)m_R/R^{\beta+1} \to 0.$$

This gives the desired conclusion. □

Proof of Theorem 1.7. Let $A_k = N \cap \partial B_k, B = M$ and apply Corollary 2.3. □

Proof of Theorem 1.8. Let $A_k = \{0, k\varphi_0\}, B = \partial B_\delta$ and apply Corollary 2.3. □

REFERENCES

[BN] H. BREZIS AND L. NIRENBERG, *Remarks on finding critical points*, Comm. Pure Appl. Math. **44** (1991), 939–964.

[R] P. H. RABINOWITZ, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conf. Ser. in Math., vol. 65, Amer. Math. Soc., 1986.

[Sc1] M. SCHECHTER, *New saddle point theorems*, Proceedings of an International Symposium on Generalized Functions and Their Applications, Varanasi, India, December, 1991, pp. 23–26.

[Sc2] ———, *A generalization of the saddle point method with applications*, Ann. Polon. Math. **57** (1992), 269–281.

[Sc3] ———, *Critical points over splitting subspaces*, Nonlinearity **6** (1993), 417–727.

[Sc4] ———, *Splitting subspaces and critical points*, Appl. Anal. **49** (1993), 33–48.

[Sc5] ———, *The intrinsic mountain pass* (to appear).

[ST] M. SCHECHTER AND K. TINTAREV, *Pairs of critical points produced by linking subsets with applications to semilinear elliptic problems*, Bull. Soc. Math. Belg. **44** (1992), 249–261.

[Si] E. A. DE B. E. SILVA, *Linking theorems and applications to semilinear elliptic problems at resonance*, Nonlinear Anal. **16** (1991), 455–477.

Manuscript received January 3, 1996

MARTIN SCHECHTER
University of California at Irvine
Irvine, CA 92717, USA