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# DEFORMATION PROPERTIES FOR CONTINUOUS FUNCTIONALS AND CRITICAL POINT THEORY

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(*Submitted by A. Granas*)

*Dedicated to the memory of Karol Borsuk*

## 1. Introduction

In the last ten years, several authors have extended the classical critical point theory (see e. g., [17–19]) to various nonsmooth settings. We refer to [3] for locally Lipschitz continuous functionals, to [6, 7, 9, 15] for certain classes of lower semi-continuous functionals and to [20] for $C^1$ perturbations of convex lower semi-continuous functionals. All these developments are independent and are not comparable among them.

In [10], a new generalized notion of $\|df(u)\|$ has been introduced, where $f : X \to \mathbb{R}$ is a function defined on a metric space $X$; this notion allows to treat continuous functionals and some classes of lower semi-continuous functionals.

On the other hand, the whole classical theory is not covered in [10] since, for instance, a deformation theorem is proved only for compact sets and only a relation between the number of critical points of the functional $f$ and the supremum of $\mathrm{cat}(K; X)$ with $K$ compact in $X$, is established.

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The main purpose of this paper is to prove a natural extension of the Deformation Theorem and of the Noncritical Interval Theorem (Theorems (2.14), (2.15)) to continuous functionals. Consequently, critical point theory for continuous functionals can be developed following the classical lines of [17–19]. On the contrary, in [10] an essential tool was constituted by the Variational Principle of Ekeland.

If the functional $f$ is only lower semi-continuous, a general critical point theory seems not to be possible, according to the simple example mentioned at the beginning of section 4. However, it is always possible to define a continuous function $\mathcal{G}_f$ and, in some particular cases, to establish a bijective correspondence between the critical points of $f$ and those of $\mathcal{G}_f$, reducing the problem to the continuous case.

We wish to point out that, if $X$ is a Finsler manifold of class $C^1$ and $f$ is of class $C^1$, we improve the results of [21]. As a particular case, if the manifold $X$ is of class $C^2$, we recover the classical theory without the construction of a pseudo-gradient vector field. Moreover, we include and unify all the mentioned results of [3, 6, 7, 9, 15, 20].

A further development of our approach is contained in [5], where also the equivariant case is treated.

In section 2, we prove the basic deformation theorems for continuous functionals. In section 3, we apply them to get some typical results of critical point theory. Finally, in section 4, we show some possible extensions to the lower semi-continuous case.

## 2. Deformation properties

In this section we recall from [10] some basic facts concerning the notion of weak slope and we prove some deformation properties for continuous functionals.

In the following $X$ will denote a metric space endowed with the metric $d$.

(2.1) DEFINITION. Let $f : X \to \mathbb{R}$ be a continuous function and $u \in X$. We denote by $|df|(u)$ the supremum of the $\sigma$'s in $[0, +\infty[$ such that there exist $\delta > 0$ and $\mathcal{H} : B(u; \delta) \times [0, \delta] \to X$ continuous with

$$d(\mathcal{H}(v, t), v) \leq t, \tag{2.2}$$

$$f(\mathcal{H}(v, t)) \leq f(v) - \sigma t. \tag{2.3}$$

The extended real number $|df|(u)$ is called the *weak slope* of $f$ at $u$. If $X$ is a Finsler manifold of class $C^1$ and $f$ is a function of class $C^1$, it turns out [10, Corollary 2.12] that $|df|(u) = \|df(u)\|$. For further comparisons between the weak slope and other notions in the literature, the reader is referred to [10].

(2.4) DEFINITION. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function. The *effective domain* of $f$ is the set

$$\mathcal{D}(f) = \{u \in X : \ f(u) < +\infty\}.$$

According to [7], we define $\mathcal{G}_f : \operatorname{epi}(f) \to \mathbb{R}$ by

$$\mathcal{G}_f(u, \xi) = \xi,$$

where $\operatorname{epi}(f) = \{(u, \xi) \in X \times \mathbb{R} : \ f(u) \le \xi\}$.

In the following, $\operatorname{epi}(f)$ will be endowed with the metric

$$d\left((u, \xi), (v, \mu)\right) = \left(d(u, v)^2 + (\xi - \mu)^2\right)^{1/2}.$$

Since $\mathcal{G}_f$ is Lipschitz continuous of constant 1, it follows that $|d\mathcal{G}_f|(u, \xi) \le 1$.

(2.5) PROPOSITION (See [10, Proposition 2.3]). *Let $f : X \to \mathbb{R}$ be a continuous function and let $(u, \xi) \in \operatorname{epi}(f)$. Then*

$$|d\mathcal{G}_f|(u, \xi) = \begin{cases} \dfrac{|df|(u)}{\sqrt{1 + |df|(u)^2}}, & \text{if } f(u) = \xi \quad \text{and} \quad |df|(u) < +\infty; \\ 1, & \text{if } f(u) < \xi \quad \text{or} \quad |df|(u) = +\infty. \end{cases}$$

By the previous proposition it is possible to define consistently the weak slope also in the lower semi-continuous case.

(2.6) DEFINITION. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function and let $u \in \mathcal{D}(f)$. We set

$$|df|(u) = \begin{cases} \dfrac{|d\mathcal{G}_f|(u, f(u))}{\sqrt{1 - |d\mathcal{G}_f|(u, f(u))^2}}, & \text{if } |d\mathcal{G}_f|(u, f(u)) < 1; \\ +\infty, & \text{if } |d\mathcal{G}_f|(u, f(u)) = 1. \end{cases}$$

(2.7) PROPOSITION (See [10, Proposition 2.6]). *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function. Then, for every sequence $(u_h)$ in $\mathcal{D}(f)$ converging to $u \in \mathcal{D}(f)$ with $(f(u_h))$ converging to $f(u)$, we have*

$$|df|(u) \le \liminf_h |df|(u_h).$$

Now we can prove the first result concerning deformations.

(2.8) THEOREM. *Let $f : X \to \mathbb{R}$ and $\sigma : X \to [0, +\infty[$ be two continuous functions such that*

$$|df|(u) \ne 0 \implies |df|(u) > \sigma(u).$$

*Then there exist two continuous maps* $\eta : X\times[0,+\infty[\to X$ *and* $\tau : X \to [0,+\infty[$ *such that*

$$\begin{aligned}&d(\eta(u,t),u) \le t,\\&f(\eta(u,t)) \le f(u),\\&t \le \tau(u) \Longrightarrow f(\eta(u,t)) \le f(u) - \sigma(u)t,\\&|df|(u) \ne 0 \Longrightarrow \tau(u) > 0.\end{aligned}$$

PROOF. By Proposition (2.7) $|df|$ is lower semi-continuous. Hence, for every $u$ with $|df|(u) \ne 0$ there exists $\delta_u > 0$ with

$$B(u;\delta_u) \subseteq \{v \in X : \ |df|(v) \ne 0\}$$

and $\mathcal{H}_u : B(u;\delta_u) \times [0,\delta_u] \to X$ continuous satisfying (2.2) and (2.3), with $\sigma$ substituted by $\sup\{\sigma(v) : v \in B(u;\delta_u)\}$. By Milnor's lemma (see for instance [16, Lemma 2.4]), the open cover $\{B(u;\delta_u/2) : \ |df|(u) \ne 0\}$ of $\{u \in X : \ |df|(u) \ne 0\}$ admits a locally finite refinement $\{V_{j,\lambda} : \ j \in \mathbb{N}, \lambda \in \Lambda_j\}$ such that

$$\lambda \ne \mu \Longrightarrow V_{j,\lambda} \cap V_{j,\mu} = \emptyset.$$

Let $\vartheta_{j,\lambda} : X \to [0,1]$ be a family of continuous functions with

$$\operatorname{supt}\vartheta_{j,\lambda} \subseteq V_{j,\lambda},$$
$$|df|(v) \ne 0 \ \Longrightarrow \sum_{j=1}^{\infty}\sum_{\lambda\in\Lambda_j}\vartheta_{j,\lambda}(v) = 1.$$

For every $(j,\lambda)$ let $V_{j,\lambda} \subseteq B(u_{j,\lambda};\delta_{u_{j,\lambda}}/2)$. Set $\delta_{j,\lambda} = \delta_{u_{j,\lambda}}$ and $\mathcal{H}_{j,\lambda} = \mathcal{H}_{u_{j,\lambda}}$. Let $\tau : X \to [0,+\infty[$ be a continuous function such that

$$\begin{aligned}&|df|(v) = 0 \Longrightarrow \tau(v) = 0,\\&|df|(v) \ne 0 \Longrightarrow 0 < \tau(v) < \tfrac{1}{2}\min\{\delta_{j,\lambda} : v \in \overline{V}_{j,\lambda}\}.\end{aligned}$$

We want to define a sequence of continuous maps

$$\eta_h : \{(v,t) \in X \times [0,+\infty[ : \ t \le \tau(v)\} \to X$$

such that

$$d(\eta_h(v,t),v) \le \Big(\sum_{j=1}^{h}\sum_{\lambda\in\Lambda_j}\vartheta_{j,\lambda}(v)\Big)t, \tag{2.9}$$

$$f(\eta_h(v,t)) \le f(v) - \sigma(v)\Big(\sum_{j=1}^{h}\sum_{\lambda\in\Lambda_j}\vartheta_{j,\lambda}(v)\Big)t. \tag{2.10}$$

First of all we set

$$\eta_1(v,t) = \begin{cases}\mathcal{H}_{1,\lambda}(v,\vartheta_{1,\lambda}(v)t), & \text{if } v \in \overline{V}_{1,\lambda};\\ v, & \text{if } v \notin \bigcup_{\lambda\in\Lambda_1} V_{1,\lambda}.\end{cases}$$

Now assume we have defined $\eta_{h-1}$ satisfying (2.9) and (2.10). For every $v \in \overline{V}_{h,\lambda}$ we have

$$d(\eta_{h-1}(v,t),v) \leq \left(\sum_{j=1}^{h-1} \sum_{\lambda\in\Lambda_j} \vartheta_{j,\lambda}(v)\right)t \leq \tau(v) < \tfrac{1}{2}\delta_{h,\lambda},$$

hence $\eta_{h-1}(v,t) \in B(u_{h,\lambda};\delta_{h,\lambda})$. Therefore the map

$$\eta_h(v,t) = \begin{cases} \mathcal{H}_{h,\lambda}(\eta_{h-1}(v,t), \vartheta_{h,\lambda}(v)t), & \text{if } v \in \overline{V}_{h,\lambda}; \\ \eta_{h-1}(v,t), & \text{if } v \notin \bigcup_{\lambda\in\Lambda_h} V_{h,\lambda} \end{cases}$$

is well defined and satisfies (2.9) and (2.10).

Since the family $\{V_{j,\lambda}\}$ is locally finite, for every $u$ with $|df|(u) \neq 0$ there exist a neighborhood $U$ of $u$ and $h_0 \in \mathbb{N}$ such that $\eta_h(v,t) = \eta_{h_0}(v,t)$ for every $v \in U$, $t \in [0,\tau(v)]$ and $h \geq h_0$. Therefore the map $\eta : X \times [0,+\infty[ \to X$ defined by

$$\eta(u,t) = \lim_h \eta_h(u, \min\{t,\tau(u)\})$$

is continuous at the points $(u,t)$ with $|df|(u) \neq 0$. By (2.9) and (2.10) it follows

$$\begin{aligned} &d(\eta(u,t),u) \leq t, \\ &f(\eta(u,t)) \leq f(u), \\ &t \leq \tau(u) \Longrightarrow f(\eta(u,t)) \leq f(u) - \sigma(u)t, \end{aligned}$$

so that $\eta$ is continuous also at the points $(u,t)$ with $|df|(u) = 0$. □

(2.11) THEOREM. *Let $X$ be a complete metric space, $f : X \to \mathbb{R}$ a continuous function, $C$ a closed subset of $X$ and $\delta, \sigma > 0$ such that*

$$d(u,C) \leq \delta \Longrightarrow |df|(u) > \sigma.$$

*Then there exists a continuous map $\eta : X \times [0,\delta] \to X$ such that*

$$\begin{aligned} &d(\eta(u,t),u) \leq t, \\ &f(\eta(u,t)) \leq f(u), \\ &d(u,C) \geq \delta \Longrightarrow \eta(u,t) = u, \\ &u \in C \Longrightarrow f(\eta(u,t)) \leq f(u) - \sigma t. \end{aligned}$$

PROOF. By the lower semi-continuity of $|df|$ there exists a continuous function $\tilde{\sigma} : X \to [0,+\infty[$ such that

$$\begin{aligned} &d(u,C) \leq \delta \Longrightarrow \tilde{\sigma}(u) = \sigma, \\ &|df|(u) \neq 0 \Longrightarrow \tilde{\sigma}(u) < |df|(u). \end{aligned}$$

Let $\eta_1 : X \times [0,+\infty[ \to X$ and $\tau_1 : X \to [0,+\infty[$ be two continuous maps obtained applying the previous theorem to $\tilde{\sigma}$. Let us set recursively

$$\eta_h(u,t) = \begin{cases} \eta_{h-1}(u,t), & \text{if } 0 \leq t \leq \tau_{h-1}(u); \\ \eta_1(\eta_{h-1}(u,\tau_{h-1}(u)), t - \tau_{h-1}(u)), & \text{if } t \geq \tau_{h-1}(u), \end{cases}$$

$$\tau_h(u) = \tau_{h-1}(u) + \tau_1(\eta_{h-1}(u, \tau_{h-1}(u))).$$

Let us show that for every $(u,t)$ with $d(u,C) + t \le \delta$ we have

$$\lim_h \tau_h(u) > t.$$

By contradiction, let us suppose that $\tau_h(u) \le t$ for every $h$. Since

$$d(\eta_h(u,\tau_h(u)), \eta_{h-1}(u, \tau_{h-1}(u))) \le \tau_h(u) - \tau_{h-1}(u),$$
$$d(\eta_h(u,\tau_h(u)), C) \le d(\eta_h(u,\tau_h(u)), u) + d(u,C) \le \tau_h(u) + d(u,C),$$

then $(\eta_h(u,\tau_h(u)))$ is a Cauchy sequence in $\{v : \ d(v,C) \le \delta\}$. If

$$\lim_h \eta_h(u, \tau_h(u)) = v,$$

it follows

$$\tau_1(v) = \lim_h \tau_1(\eta_h(u,\tau_h(u))) = \lim_h (\tau_{h+1}(u) - \tau_h(u)) = 0,$$

hence a contradiction.

Therefore we can define a continuous map

$$\eta : \{(u,t) : d(u,C) + t \le \delta\} \to X$$

by

$$\eta(u,t) = \lim_h \eta_h(u,t).$$

It is readily seen that

$$d(\eta(u,t), u) \le t,$$
$$f(\eta(u,t)) \le f(u) - \sigma t.$$

If we set

$$\eta(u,t) = \eta(u, (\delta - d(u,C))^+)$$

whenever $d(u,C) + t \ge \delta$, the map $\eta$ has the required properties. □

(2.12) Definition. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function. A point $u \in \mathcal{D}(f)$ is said to be *critical* if $|df|(u) = 0$.

(2.13) Definition. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function and let $c \in \mathbb{R}$. We say that $f$ satisfies the *Palais-Smale condition at level $c$* ($(PS)_c$ in short), if every sequence $(u_h)$ in $\mathcal{D}(f)$ with $|df|(u_h) \to 0$ and $f(u_h) \to c$ contains a subsequence $(u_{h_k})$ converging in $X$.

In the following, for every $c \in \mathbb{R}$ we set

$$K_c = \{u \in X : \ |df|(u) = 0, \ f(u) = c\},$$
$$f^c = \{u \in X : \ f(u) \le c\}.$$

Now we can prove the two main results of this section.

(2.14) THEOREM (Deformation Theorem). *Let $f : X \to \mathbb{R}$ be a continuous function defined on a complete metric space $X$, and let $c \in \mathbb{R}$. Assume that $f$ satisfies the Palais-Smale condition at level $c$.*

*Then, given $\bar{\varepsilon} > 0$, $\mathcal{O}$ a neighborhood of $K_c$ (if $K_c = \emptyset$, we allow $\mathcal{O} = \emptyset$) and $\lambda > 0$, there exist $\varepsilon > 0$ and $\eta : X \times [0,1] \to X$ continuous with:*

(a) $d(\eta(u,t), u) \leq \lambda t$;

(b) $f(\eta(u,t)) \leq f(u)$;

(c) $f(u) \notin ]c - \bar{\varepsilon}, c + \bar{\varepsilon}[ \implies \eta(u,t) = u$;

(d) $\eta(f^{c+\varepsilon} \setminus \mathcal{O}, 1) \subseteq f^{c-\varepsilon}$.

PROOF. Fix $\bar{\varepsilon} > 0$, a neighborhood $\mathcal{O}$ of $K_c$ and $\lambda > 0$. First of all, let us suppose that $f$ is Lipschitz continuous of constant 1.

From the Palais-Smale condition at level $c$ and Proposition (2.7), we deduce that $K_c$ is compact. Let $r > 0$ be such that $B(K_c; 2r) \subseteq \mathcal{O}$. Let $\delta, \sigma > 0$ be such that $2\delta \leq \bar{\varepsilon}$, $\delta \leq r$ and

$$c - 2\delta \leq f(u) \leq c + 2\delta, \quad u \notin B(K_c; r) \implies |df|(u) > \sigma.$$

Set

$$C = \{u \in X : \ c - \delta \leq f(u) \leq c + \delta, \quad u \notin B(K_c; 2r)\}.$$

Since $f$ is Lipschitz continuous of constant 1, we have

$$d(u, C) \leq \delta \implies |df|(u) > \sigma.$$

Let $\eta' : X \times [0, \delta] \to X$ be a continuous map as in Theorem (2.11). We can assume, without loss of generality, $\lambda \leq \delta$ and define $\eta : X \times [0,1] \to X$ by $\eta(u,t) = \eta'(u, \lambda t)$.

Properties $(a)$ and $(b)$ are obvious. Since $f$ is Lipschitz with constant 1, $f(u) \notin ]c - \bar{\varepsilon}, c + \bar{\varepsilon}[$ implies $d(u, C) \geq \delta$, hence $\eta(u,t) = u$. Finally, set $\varepsilon = \min\{\frac{\sigma\lambda}{2}, \delta\}$. If $u \in f^{c+\varepsilon} \setminus \mathcal{O}$ and $f(u) \geq c - \varepsilon$, it follows $u \in C$, hence

$$f(\eta(u,1)) = f(\eta'(u,\lambda)) \leq f(u) - \sigma\lambda \leq c + \varepsilon - \sigma\lambda \leq c - \varepsilon.$$

If $u \in f^{c+\varepsilon} \setminus \mathcal{O}$ and $f(u) \leq c - \varepsilon$, we deduce from $(b)$ that $f(\eta(u,1)) \leq c - \varepsilon$.

Now let us consider the general case. Being closed in $X \times \mathbb{R}$, $\mathrm{epi}(f)$ is complete. Let us denote by $\widetilde{K}_c$ the set of critical points of $\mathcal{G}_f$ at level $c$. By Proposition (2.5), the function $\mathcal{G}_f$ satisfies the Palais-Smale at level $c$. Moreover, $(\mathcal{O} \times \mathbb{R}) \cap \mathrm{epi}(f)$ is a neighborhood of $\widetilde{K}_c$ and $\mathcal{G}_f$ is Lipschitz continuous of constant 1.

By the previous step, we can find $\varepsilon > 0$ and a continuous map

$$\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2) : \mathrm{epi}(f) \times [0,1] \to \mathrm{epi}(f)$$

such that

$$\begin{aligned}
&d(\tilde{\eta}((u,\xi),t),(u,\xi)) \le \lambda t, \\
&\tilde{\eta}_2((u,\xi),t) \le \xi, \\
&\xi \notin ]c-\overline{\varepsilon}, c+\overline{\varepsilon}[ \Longrightarrow \tilde{\eta}((u,\xi),t) = (u,\xi), \\
&\xi \le c+\varepsilon,\ u \notin \mathcal{O} \Longrightarrow \tilde{\eta}_2((u,\xi),1)) \le c-\varepsilon.
\end{aligned}$$

Let us define $\eta : X \times [0,1] \to X$ by $\eta(u,t) = \tilde{\eta}_1((u,f(u)),t)$. Since $\tilde{\eta}$ takes its values in epi$(f)$, we have

$$f(\tilde{\eta}_1((u,f(u)),t)) \le \tilde{\eta}_2((u,f(u)),t).$$

Then $(a)$, $(b)$, $(c)$ and $(d)$ easily follow. □

(2.15) THEOREM (Noncritical Interval Theorem). *Let $X$ be a complete metric space, $f : X \to \mathbb{R}$ be a continuous function, $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$ $(a < b)$. Assume that $f$ has no critical points $u$ with $a \le f(u) \le b$ and that $(PS)_c$ holds whenever $c \in [a,b]$, $c \le f(u)$ for some $u \in X$. Then there exists $\eta : X \times [0,1] \to X$ continuous with:*

$(a)$ $\eta(u,0) = u$;

$(b)$ $f(\eta(u,t)) \le f(u)$;

$(c)$ $f(u) \le b \implies f(\eta(u,1)) \le a$;

$(d)$ $f(u) \le a \implies \eta(u,t) = u$.

PROOF. First of all assume $b \le f(u)$ for some $u \in X$. Consider the case in which $f$ is Lipschitz continuous of constant 1. By the Palais-Smale condition, we can find $\delta, \sigma > 0$ such that

$$a - \delta \le f(u) \le b + \delta \implies |df|(u) > \sigma.$$

If we set

$$C = \{u \in X : a \le f(u) \le b\},$$

it follows

$$d(u,C) \le \delta \implies |df|(u) > \sigma.$$

Let $\eta' : X \times [0,\delta] \to X$ be a continuous map as in Theorem (2.11). Define recursively $\eta_h : X \times [0,1] \to X$ by

$$\begin{aligned}
\eta_1(u,t) &= \eta'(u,\delta t), \\
\eta_h(u,t) &= \eta_1(\eta_{h-1}(u,t),t),
\end{aligned}$$

and consider $n$ such that $n\sigma\delta \ge b-a$. If $\vartheta : \mathbb{R} \to [0,1]$ is the function such that $\vartheta(s) = 0$ for $s \le 0$, $\vartheta(s) = s$ for $0 \le s \le 1$, and $\vartheta(s) = 1$ for $s \ge 1$, then the map

$$\eta(u,t) = \eta_n\left(u, t\,\vartheta\left(\frac{f(u)-a}{b-a}\right)\right)$$

has the required properties.

Now consider the case in which $f$ is only continuous. The function $\mathcal{G}_f$ satisfies the assumptions of the theorem and is Lipschitz continuous of constant 1. Let

$$\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2) : \mathrm{epi}(f) \times [0,1] \to \mathrm{epi}(f)$$

be a continuous map given by the previous step. Then the map

$$\eta(u,t) = \tilde{\eta}_1((u, f(u)), t)$$

has the required properties.

Finally, assume $b > f(u)$ for all $u \in X$. Of course, it is sufficient to treat the case $b = \sup f$. Let $(b_h)$ be a strictly increasing sequence with

$$b_1 = a \quad \text{and} \quad \lim_h b_h = b.$$

For every $h \geq 1$ let $\eta'_h : X \times [0,1] \to X$ be a continuous map satisfying the thesis with $a$ and $b$ substituted by $b_h$ and $b_{h+1}$. Let us define $\eta_h : X \times [0,1] \to X$ by

$$\begin{aligned} &\eta_1(u,t) = \eta'_1(u,t), \\ &\eta_h(u,t) = \eta_{h-1}(\eta'_h(u,t), t). \end{aligned}$$

Arguing by induction, it is easy to see that

$$\begin{aligned} &\eta_h(u,0) = u, \\ &f(\eta_h(u,t)) \leq f(u), \\ &f(u) \leq b_{h+1} \Longrightarrow f(\eta_h(u,1)) \leq a, \\ &f(u) \leq b_h \Longrightarrow \eta_h(u,t) = \eta_{h-1}(u,t), \\ &f(u) \leq a \Longrightarrow \eta_h(u,t) = u. \end{aligned}$$

Therefore the map

$$\eta(u,t) = \lim_h \eta_h(u,t)$$

has the required properties. □

Now we want to give "symmetric" versions of the previous results, when $X$ is a Banach space endowed with the usual $\mathbf{Z}_2$-action. Since 0 is a fixed point of the action, we have to treat the origin as a critical point, even if we do not know whether $|df|(0) = 0$ for every even continuous function $f : X \to \mathbb{R}$.

(2.16) THEOREM. *Let $X$ be a Banach space, $f : X \to \mathbb{R}$ be an even continuous function and $c \in \mathbb{R}$. Assume that $f$ satisfies the Palais-Smale condition at level $c$.*

*Then, given $\bar{\varepsilon} > 0$, $\mathcal{O}$ a neighborhood of $K_c \cup \{0\}$ and $\lambda > 0$, there exist $\varepsilon > 0$ and $\eta : X \times [0,1] \to X$ continuous having properties (a) to (d) of Theorem (2.14) and*

*(e) $\eta(\cdot, t) : X \to X$ is odd for each $t \in [0,1]$.*

Proof. Since $f$ is even, $|df|(-u) = |df|(u)$ for all $u \in X$ (in particular, $K_c$ is a symmetric set for each $c \in \mathbb{R}$).

If $u \neq 0$ and $|df|(u) > \sigma > 0$, let $\delta > 0$ and $\mathcal{H} : B(u;\delta) \times [0,\delta] \to X$ be a continuous map satisfying (2.2) and (2.3). Of course, we can suppose $\delta < \|u\|$. Then

$$\widetilde{\mathcal{H}} : (B(u;\delta) \cup B(-u;\delta)) \times [0,\delta] \to X$$

defined by

$$\widetilde{\mathcal{H}}(v,t) = \begin{cases} \mathcal{H}(v,t), & \text{if } v \in B(u;\delta); \\ -\mathcal{H}(-v,t), & \text{if } v \in B(-u;\delta), \end{cases}$$

is continuous, odd with respect to the first variable and satisfies (2.2) and (2.3).

Then, all the constructions of Theorems (2.8), (2.11) and (2.14) can be repeated in a symmetric way, yielding the result. □

(2.17) Theorem. *Let $X$ be a Banach space, $f : X \to \mathbb{R}$ be an even continuous function, $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\} (a < b)$. Assume that $f$ has no critical points $u$ with $a \le f(u) \le b$, $f(0) \notin [a,b]$ and that $(PS)_c$ holds whenever $c \in [a,b]$, $c \le f(u)$ for some $u \in X$.*

*Then there exists $\eta : X \times [0,1] \to X$ continuous having properties (a) to (d) of Theorem (2.15) and*

*(e) $\eta(\,\cdot\,,t) : X \to X$ is odd for each $t \in [0,1]$.*

Proof. The argument is similar to that of the previous theorem. □

## 3. Applications to critical point theory

In this section we apply the Deformation Theorem and the Noncritical Interval Theorem, to extend some classical results of critical point theory to continuous functionals.

Throughout this section $X$ will denote a metric space endowed with the metric $d$.

(3.1) Definition. Let $A$ be a closed non-empty subset of $X$. We denote by $\operatorname{cat}(A;X)$ the least integer $n$ such that $A$ can be covered by $n$ open subsets of $X$, each of which is contractible in $X$. If no such integer $n$ exists, we put $\operatorname{cat}(A;X) = \infty$. We set also $\operatorname{cat}(\emptyset;X) = 0$ and $\operatorname{cat}(X) = \operatorname{cat}(X;X)$.

For our purposes, we prefer the definition of the Lusternik-Schnirelman category given by means of *open* covers, as in [11, 12]. Of course, if $X$ is an $ANR$, the above definition agrees with the usual one ([17, 18]), involving *closed* covers.

Let us recall the basic properties of the category.

(3.2) PROPOSITION. *Let $A$ and $B$ be two closed subsets of $X$. Then the following facts hold*:

(*a*) $A \subseteq B \Longrightarrow \mathrm{cat}(A;X) \leq \mathrm{cat}(B;X)$;

(*b*) $\mathrm{cat}(A \cup B;X) \leq \mathrm{cat}(A;X) + \mathrm{cat}(B;X)$;

(*c*) *if $\mathcal{H} : X \times [0,1] \to X$ is a deformation with $\mathcal{H}(A \times \{1\}) \subseteq B$, we have*

$$\mathrm{cat}(A;X) \leq \mathrm{cat}(B;X);$$

(*d*) *there exists an open subset $U$ of $X$ such that $A \subseteq U$ and*

$$\mathrm{cat}(\overline{U};X) = \mathrm{cat}(A;X).$$

If $f : X \to \mathbb{R}$ is a continuous function and $1 \leq m \leq \mathrm{cat}(X)$, we set

$$\mathcal{A}_m = \{A \subseteq X : \ A \text{ is closed } \text{ and } \mathrm{cat}(A;X) \geq m\},$$
$$c_m = \inf_{A \in \mathcal{A}_m} \sup_A f.$$

The first result concerning the Lusternik-Schnirelman category is the following one.

(3.3) THEOREM. *Let $X$ be a complete metric space and $f : X \to \mathbb{R}$ a continuous function. Assume that $(PS)_c$ holds whenever $c \leq f(u)$ for some $u \in X$. If $1 \leq m \leq n \leq \mathrm{cat}(X)$, the following facts hold*:

(*a*) $c_m \leq c_n \leq \sup\{f(u) : \ |df|(u) = 0\}$ (*with the convention* $\sup \emptyset = -\infty$);

(*b*) *if $-\infty < c_m \leq f(u)$ for some $u \in X$, then $K_{c_m} \neq \emptyset$*;

(*c*) *if $-\infty < c_m = c_n \leq f(u)$ for some $u \in X$, then*

$$\mathrm{cat}(K_{c_m};X) \geq n - m + 1.$$

PROOF. Since we have proved in our setting the Deformation Theorem and the Noncritical Interval Theorem, it is sufficient to adapt the classical arguments of [17, 18] to our situation. □

The next results on the Lusternik-Schnirelman category require some regularity of the metric space $X$.

(3.4) DEFINITION. The metric space $X$ is said to be *weakly locally contractible*, if every $x \in X$ admits a neighborhood $U$ contractible in $X$.

It is readily seen that $X$ is weakly locally contractible if and only if

$$\mathrm{cat}(\{x\};X) = 1, \qquad \text{for all } x \in X.$$

If $X$ is weakly locally contractible and $K$ is a compact subset of $X$, it is easy to show that $\mathrm{cat}(K;X) < \infty$.

(3.5) THEOREM. *Let $X$ be a weakly locally contractible complete metric space, $f : X \to \mathbb{R}$ be a continuous function and $a, b \in \mathbb{R}$ $(a < b)$. Assume that $(PS)_c$ holds for all $c \in [a, b]$.*

*Then* $\operatorname{cat}(f^a; X) < \infty$ *implies* $\operatorname{cat}(f^b; X) < \infty$.

PROOF. It is sufficient to show that the set

$$\{c \in [a, b] : \operatorname{cat}(f^c; X) < \infty\}$$

is open and closed in $[a, b]$. This fact can be proved in a standard way by means of the Deformation Theorem. □

(3.6) THEOREM. *Let $X$ be a weakly locally contractible complete metric space and $f : X \to \mathbb{R}$ be a continuous function. Assume that $f$ is bounded from below and that $(PS)_c$ holds whenever $c \leq f(u)$ for some $u \in X$.*

*Then the following facts hold:*

(a) *$f$ has at least* $\operatorname{cat}(X)$ *critical points;*

(b) *if* $\operatorname{cat}(X) = \infty$, $\sup_X f$ *is not achieved, and*

$$\lim_m c_m = \sup_X f;$$

(c) *if* $\operatorname{cat}(X) = \infty$ *and $(PS)_c$ holds for all $c \in \mathbb{R}$, we have*

$$\sup_X f = +\infty.$$

PROOF. It follows from Theorems (3.3) and (3.5) by a standard technique. □

Now we want to prove a result of saddle point type. It is an extension of the results of [1, 19] in the regular case and of [4, 13, 14] in the "limit" case.

(3.7) THEOREM. *Let $X$ be a complete metric space, $f : X \to \mathbb{R}$ a continuous function, $(D, S)$ a compact pair and $\psi : S \to X$ a continuous map. Let*

$$\Phi = \{\varphi \in C(D, X) : \varphi_{|S} = \psi\}.$$

*Let us assume that $\Phi \neq \emptyset$ and that there exists a closed subset $A$ of $X$ such that*

$$A \cap \psi(S) = \emptyset, \ \inf_A f \geq \max_S (f \circ \psi) \quad and \quad A \cap \varphi(D) \neq \emptyset \quad for\ all\ \varphi \in \Phi.$$

*If $f$ satisfies the Palais-Smale condition at level*

$$c = \inf_{\varphi \in \Phi} \max_D (f \circ \varphi),$$

*then* $K_c \neq \emptyset$. *Furthermore, if* $c = \inf_A f$, *then* $K_c \cap A \neq \emptyset$.

PROOF. It follows from the hypothesis that

$$c \geq \inf_A f.$$

If this inequality is strict and we suppose that $K_c = \emptyset$, we obtain a contradiction in a standard way (see e.g. [19]), using $(c)$ and $(d)$ of the Deformation Theorem.

If $c = \inf_A f$, we can proceed as in [4, Theorem 1]. Let us give the proof for the reader's convenience. Let us assume by contradiction that $K_c \cap A = \emptyset$ and let $\lambda > 0$ be such that

$$u \in K_c \Longrightarrow d(u, A) > \lambda,$$
$$u \in A, \ v \in \psi(S) \Longrightarrow d(u, v) \geq 2\lambda.$$

Let $\varepsilon > 0$ and $\eta : X \times [0,1] \to X$ continuous be as in the Deformation Theorem with

$$d(\eta(u,t), u) \leq \lambda t,$$
$$u \in f^{c+\varepsilon}, \ d(u, A) \leq \lambda \Longrightarrow f(\eta(u,1)) \leq c - \varepsilon.$$

Define, for $u \in X$, $t \in [0,1]$:

$$\rho(u) = \tfrac{1}{\lambda} \min\{d(u, \psi(S)), \lambda\}, \quad \tilde{\eta}(u,t) = \eta(u, \rho(u)t).$$

Let $\varphi \in \Phi$ be such that

$$\max_D (f \circ \varphi) \leq c + \varepsilon.$$

Then $\tilde{\varphi} \equiv \tilde{\eta}(\cdot, 1) \circ \varphi \in \Phi$. Let $x_0 \in D$ be such that $\tilde{\varphi}(x_0) \in A$, so that $f(\tilde{\varphi}(x_0)) \geq c$. On the other hand, $d(\varphi(x_0), A) \leq \lambda$, so that $\rho(\varphi(x_0)) = 1$ and $f(\tilde{\varphi}(x_0)) = f(\eta(\varphi(x_0), 1)) \leq c - \varepsilon$: a contradiction. □

(3.8) REMARK. Theorem (3.7) is similar to [4, Theorem 3] and, as in [4], the two parts of the proof can be deduced from a unique principle; indeed, in the case $c > \inf_A f$, the set $\{x \in X : \ f(x) \geq c\}$ has the same properties as the set $A$ with respect to $\Phi$.

## 4. Lower semi-continuous functionals

In the previous section we have proved some results of critical point theory for continuous functionals. In the lower semi-continuous case a further difficulty arises.

Let $X$ again denote a metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function. Because of Definition (2.6), $u \in \mathcal{D}(f)$ is critical for $f$ if and only if $(u, f(u))$ is critical for $\mathcal{G}_f$. Moreover $\mathcal{G}_f(u, f(u)) = f(u)$. Therefore, in order to get information about the critical points of $f$, we could study the function $\mathcal{G}_f$. The advantage is that $\mathcal{G}_f$ is continuous, so that the theory of the previous section applies to $\mathcal{G}_f$. Unfortunately, $\mathcal{G}_f$ has in general more critical points than $f$. Consider, for instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$, if $x < 0$, and $f(x) = x$, if $x \geq 0$. Then $(0,0)$ and $(0,1)$ are both critical points of $\mathcal{G}_f$. This is correct, because $\mathcal{G}_f$ satisfies the assumptions of the mountain pass theorem (which is, of course, a particular case of Theorem (3.7)). Actually, $(0,0)$ is a strict local minimum, while $(0,1)$ is the mountain pass point. On the other hand, 0 is the unique critical point of $f$ and $f(0) = 0$, in spite of the fact that $f$ also presents a behaviour of mountain pass type.

This difficulty cannot arise, if we assume

$$\inf\{|d\mathcal{G}_f|(u,\xi) : \ f(u) < \xi\} \ > \ 0. \tag{4.1}$$

For instance, in [10, Theorem 3.13] it has been proved that, if $X$ is a Banach space and $f = f_0 + f_1$ with $f_0 : X \to \mathbb{R} \cup \{+\infty\}$ convex and lower semi-continuous and $f_1 : X \to \mathbb{R}$ of class $C^1$, then

$$f(u) < \xi \Longrightarrow |d\mathcal{G}_f|(u,\xi) = 1.$$

In particular, (4.1) holds.

The same property is true for $(f + I_M)$, where $I_M$ is the indicator of a $C^1$–hypersurface $M$, provided that a suitable non-tangency condition is satisfied (see [2, Theorem 2.6]).

First of all, we want to show that the same result holds for another class of lower semi-continuous functionals. Let us recall some notions from [6, 7, 9].

(4.2) DEFINITION. Let $A$ be an open subset of a Banach space $E$ and $f : A \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function. For every $u \in \mathcal{D}(f)$ we denote by $\partial^- f(u)$ the (possibly empty) set of $\alpha$'s in $E'$ such that

$$\liminf_{v \to u} \frac{f(v) - f(u) - \langle \alpha, v - u \rangle}{\|v - u\|} \geq 0.$$

(4.3) DEFINITION. Let $A$ be an open subset of a Hilbert space $H$ and $p \geq 0$. We say that a lower semi-continuous function $f : A \to \mathbb{R} \cup \{+\infty\}$ has a *$\varphi$-monotone subdifferential of order $p$*, if there exists a continuous function

$$\chi : (\mathcal{D}(f))^2 \times \mathbb{R}^2 \to \mathbb{R}^+$$

such that

$$\langle \alpha - \beta, u - v \rangle \ \geq \ -\chi(u, v, f(u), f(v))(1 + \|\alpha\|^p + \|\beta\|^p)\|u - v\|^2$$

whenever $u, v \in \mathcal{D}(f)$, $\alpha \in \partial^- f(u)$ and $\beta \in \partial^- f(v)$.

(4.4) THEOREM. *Let $A$ be an open subset of a Hilbert space $H$ and $f : A \to \mathbb{R}\cup\{+\infty\}$ be a function. Assume that $f = f_0 + f_1$ with $f_0 : A \to \mathbb{R}\cup\{+\infty\}$ lower semi-continuous with $\varphi$-monotone subdifferential of order 2 and $f_1 : A \to \mathbb{R}$ of class $C^1$.*

*Then for every $(u, \xi) \in \mathrm{epi}(f)$*

$$f(u) < \xi \Longrightarrow |d\mathcal{G}_f|(u, \xi) = 1.$$

PROOF. Let $u \in \mathcal{D}(f)$ and $\xi > f(u)$. By [10, Proposition 2.7], it is sufficient to consider the case $f_1 = 0$.

For every $v \in \mathcal{D}(f)$ let $\mathcal{V} : [0, \mathcal{T}(v)[ \to \mathcal{D}(f)$ be the curve of maximal slope for $f$ such that $\mathcal{V}(0) = v$ defined on its maximal interval (see [9, section 3]). We recall that

$$\lim_{t \to \mathcal{T}(v)} \int_0^t \left(1 + \|\mathcal{V}'(\tau)\|^2 + d(\mathcal{V}(\tau), \partial A)^{-2}\right) d\tau = +\infty$$

with the convention $d(\mathcal{V}(\tau), \partial A)^{-2} = 0$ if $A = H$. Set

$$\Lambda = \{(v, t) : \ 0 \le t < \mathcal{T}(v)\}$$

and define $\Phi : \Lambda \to \mathcal{D}(f)$ by $\Phi(v, t) = \mathcal{V}(t)$. Then, if $((v_h, t_h))$ is a sequence in $\Lambda$ converging to $(v, t) \in \Lambda$ with $\sup_h f(v_h) < +\infty$, we have

$$\lim_h \Phi(v_h, t_h) = \Phi(v, t),$$
$$t > 0 \Longrightarrow \lim_h f(\Phi(v_h, t_h)) = f(\Phi(v, t)).$$

If, in addition, $f(v_h) \to f(v)$, then $f(\Phi(v_h, t_h)) \to f(\Phi(v, t))$ also when $t = 0$.

For every $(v, t) \in \Lambda$ let

$$\sigma_v(t) = \int_0^t \left(1 + \|\mathcal{V}'(\tau)\|^2 + d(\mathcal{V}(\tau), \partial A)^{-2}\right) d\tau.$$

Let us define $\Psi : \mathcal{D}(f) \times [0, +\infty[ \to \mathcal{D}(f)$ by $\Psi(v, s) = \Phi(v, \sigma_v^{-1}(s))$. We have

$$\sigma_v(t) = t + f(v) - f(\mathcal{V}(t)) + \int_0^t d(\mathcal{V}(\tau), \partial A)^{-2}\, d\tau.$$

Therefore, for every sequence $((v_h, s_h))$ in $\mathcal{D}(f) \times [0, +\infty[$ converging to $(v, s)$ with $f(v_h) \to f(v)$, we have

$$\lim_h \Psi(v_h, s_h) = \Psi(v, s),$$
$$\lim_h f(\Psi(v_h, s_h)) = f(\Psi(v, s)).$$

Let us define $\eta : \mathrm{epi}(f) \times [0, +\infty[ \to \mathrm{epi}(f)$ by

$$\eta((v, \mu), s) = \begin{cases} (v, \mu - s), & \text{if } 0 \le s \le \mu - f(v); \\ \left(\Psi(v, s - \mu + f(v)), f(\Psi(v, s - \mu + f(v)))\right), & \text{if } s \ge \mu - f(v). \end{cases}$$

Since $\eta$ is Lipschitz continuous of constant 1 in the last variable, it follows that

$$d\left(\eta((v,\mu),s),(v,\mu)\right) \leq s.$$

In order to show that $\eta$ is continuous, consider a sequence $((v_h,\mu_h),s_h)$ converging to $((v,\mu),s)$ with $f(v_h) \leq \mu_h$. Up to a subsequence, we can assume that $f(v_h) \to l$. Since $\eta$ is uniformly Lipschitz continuous in the last variable, we can suppose, without loss of generality, $s_h = s$ and treat only the cases $0 \leq s < \mu - l$, $\mu - l < s < \mu - f(v)$ and $s > \mu - f(v)$.

*Case* $[0 \leq s < \mu - l]$: It is clear that

$$\eta((v_h,\mu_h),s) = (v_h,\mu_h - s) \quad \longrightarrow \quad (v,\mu - s) = \eta((v,\mu),s).$$

*Case* $[\mu - l < s < \mu - f(v)]$: Suppose first $f(v_h) = \mu_h$ and set $t_h = \sigma_{v_h}^{-1}(s)$. Let us show that $t_h \to 0$. Since $t_h \leq s$, up to a subsequence $t_h \to t$. If, by contradiction, $t > 0$, it follows that

$$f(\Phi(v_h,t_h)) \to f(\Phi(v,t)) \leq f(v).$$

On the other hand we have

$$f(\Phi(v_h,t_h)) = f(\Psi(v_h,s)) \geq f(v_h) - s,$$

hence $f(v) \geq \mu - s$: a contradiction. Therefore

$$\Psi(v_h,s) = \Phi(v_h,t_h) \to v.$$

Moreover,

$$\begin{aligned} f(\Psi(v_h,s)) = f(\Phi(v_h,t_h)) &= f(v_h) - \int_0^{t_h} \|\mathcal{V}_h'(\tau)\|^2 \, d\tau \\ &= \mu_h - s + \int_0^{t_h} \left(1 + d(\mathcal{V}_h(\tau),\partial A)^{-2}\right) d\tau. \end{aligned}$$

It follows that

$$\begin{aligned} \eta((v_h,\mu_h),s) = \eta((v_h,f(v_h)),s) &= (\Psi(v_h,s), f(\Psi(v_h,s))) \\ &\longrightarrow (v,\mu - s) = \eta((v,\mu),s). \end{aligned}$$

If we remove the restriction $f(v_h) = \mu_h$, we have in any case

$$\begin{aligned} \eta((v_h,\mu_h),s) &= \eta((v_h,f(v_h)),s - \mu_h + f(v_h)), \\ \eta((v,\mu),s) &= \eta((v,l),s - \mu + l). \end{aligned}$$

Therefore we are reduced to the previous situation.

*Case* $[s > \mu - f(v)]$: Suppose first $l > f(v)$. Let

$$(w_h, f(w_h)) = \eta((v_h,\mu_h),\mu - f(v)).$$

By the previous step we have

$$(w_h, f(w_h)) \longrightarrow \eta((v,\mu), \mu - f(v)) = (v, f(v)),$$

hence,

$$\begin{aligned}\eta((v_h,\mu_h),s) &= (\Psi(w_h, s-\mu+f(v)), f(\Psi(w_h, s-\mu+f(v))))\\ &\longrightarrow (\Psi(v, s-\mu+f(v)), f(\Psi(v, s-\mu+f(v)))) = \eta((v,\mu),s).\end{aligned}$$

On the other hand, if $l = f(v)$, it follows that

$$\begin{aligned}\eta((v_h,\mu_h),s) &= (\Psi(v_h, s-\mu_h+f(v_h)), f(\Psi(v_h, s-\mu_h+f(v_h))))\\ &\longrightarrow (\Psi(v, s-\mu+f(v)), f(\Psi(v, s-\mu+f(v)))) = \eta((v,\mu),s).\end{aligned}$$

Therefore the map $\eta$ is continuous.

By [9, Theorem 1.18] for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(v, f(v)) \in B((u,\xi); 2\delta),\ \alpha \in \partial^- f(v) \Longrightarrow \varepsilon\|\alpha\|^2 \geq 1 + d(v, \partial A)^{-2}.$$

Let us consider

$$\eta : (B((u,\xi);\delta) \cap \mathrm{epi}(f)) \times [0,\delta] \to \mathrm{epi}(f).$$

If $s \leq \mu - f(v)$, of course we have

$$\mathcal{G}_f(\eta((v,\mu),s)) = \mu - s = \mathcal{G}_f(v,\mu) - s.$$

If $s > \mu - f(v)$, set $t = \sigma_v^{-1}(s - \mu + f(v))$, so that

$$s - \mu + f(v) = \int_0^t (1 + \|\mathcal{V}'(\tau)\|^2 + d(\mathcal{V}(\tau), \partial A)^{-2})\, d\tau.$$

Then

$$\begin{aligned}&\mathcal{G}_f(\eta((v,\mu),s))\\ &= f(\Psi(v, s-\mu+f(v))) = f(\Phi(v,t)) = f(v) - \int_0^t \|\mathcal{V}'(\tau)\|^2\, d\tau\\ &= f(v) - \int_0^t (1 + \|\mathcal{V}'(\tau)\|^2 + d(\mathcal{V}(\tau),\partial A)^{-2})\, d\tau + \int_0^t (1 + d(\mathcal{V}(\tau),\partial A)^{-2})\, d\tau\\ &= \mu - s + \int_0^t \frac{1 + d(\mathcal{V}(\tau),\partial A)^{-2}}{\|\mathcal{V}'(\tau)\|^2} \|\mathcal{V}'(\tau)\|^2\, d\tau\\ &\leq \mu - s + \varepsilon \int_0^t \|\mathcal{V}'(\tau)\|^2\, d\tau \ \leq\ \mathcal{G}_f(v,\mu) - (1-\varepsilon)s.\end{aligned}$$

It follows that $|d\mathcal{G}_f|(u,\xi) \geq 1 - \varepsilon$, hence the thesis by the arbitrariness of $\varepsilon$.

$\square$

Now let us show in more detail how the function $\mathcal{G}_f$ can be used to get informations on the critical points of $f$. As an example, let us prove an adaptation of Theorem (3.7).

(4.5) THEOREM. *Let $X$ be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous function satisfying (4.1). Let $D^n$, $S^{n-1}$ denote respectively the closed unit ball and sphere in $\mathbb{R}^n$ and let $\psi : S^{n-1} \to \mathcal{D}(f)$ be continuous. Consider*

$$\Phi = \{\varphi \in \mathcal{C}(D^n, \mathcal{D}(f)) : \varphi_{|S^{n-1}} = \psi\}.$$

*Assume that $\Phi \neq \emptyset$ and that there exists $A$ closed in $\mathcal{D}(f)$ such that*

$$A \cap \psi(S^{n-1}) = \emptyset, \inf_A f \geq \sup_{S^{n-1}} (f \circ \psi) \quad and \quad A \cap \varphi(D^n) \neq \emptyset \quad for\ all\ \varphi \in \Phi.$$

*Finally, suppose that*

$$c = \inf_{\varphi \in \Phi} \sup_{D^n} (f \circ \varphi)$$

*is a real number and that $f$ satisfies the Palais-Smale condition at level $c$. Then $K_c \neq \emptyset$; furthermore, if $c = \inf_A f$, then $K_c \cap A \neq \emptyset$.*

PROOF. Set $\alpha = \sup_{S^{n-1}} (f \circ \psi)$ and define:

$$\begin{aligned}
\tilde{\psi} &= (\psi, \alpha), \\
\widetilde{\Phi} &= \{\varphi \in \mathcal{C}(D^n, \mathrm{epi}(f)) : \varphi_{|S^{n-1}} = \tilde{\psi}\}, \\
\tilde{c} &= \inf_{\varphi \in \widetilde{\Phi}} \max_{D^n} (\mathcal{G}_f \circ \varphi).
\end{aligned}$$

We show that $\widetilde{\Phi} \neq \emptyset$ and that $\tilde{c} = c$. Let $\varphi \in \Phi$ with $\sup_{D^n} (f \circ \varphi) \equiv \beta < +\infty$; define, for $x \in D^n$:

$$\tilde{\varphi}_1(x) = \begin{cases} \varphi(2x), & \text{if } x \in \frac{1}{2}D^n; \\ \varphi\Big(\dfrac{x}{|x|}\Big), & \text{if } x \in \overline{D^n \setminus \frac{1}{2}D^n}, \end{cases}$$

$$\tilde{\varphi}_2(x) = \begin{cases} \beta, & \text{if } x \in \frac{1}{2}D^n; \\ (2|x| - 1)\alpha + 2(1 - |x|)\beta, & \text{if } x \in \overline{D^n \setminus \frac{1}{2}D^n}. \end{cases}$$

It is easy to verify that $\tilde{\varphi} \equiv (\tilde{\varphi}_1, \tilde{\varphi}_2) \in \widetilde{\Phi}$, with $\max_{D^n}(\mathcal{G}_f \circ \tilde{\varphi}) = \beta$. This shows that $\widetilde{\Phi} \neq \emptyset$ and that $\tilde{c} \leq c$. Conversely, if $\varphi = (\varphi_1, \varphi_2) \in \widetilde{\Phi}$, then $\varphi_1 \in \Phi$ and

$$\mathcal{G}_f(\varphi(x)) = \varphi_2(x) \geq f(\varphi_1(x)), \qquad \text{for all } x \in D^n,$$

whence $c \leq \tilde{c}$.

Now, let

$$\widetilde{A} = (A \times \mathbb{R}) \cap \mathrm{epi}(f).$$

The set $\widetilde{A}$ is closed in epi$(f)$,

$$\widetilde{A} \cap \tilde{\psi}(S^{n-1}) = \emptyset, \qquad \inf_{\widetilde{A}} \mathcal{G}_f = \inf_A f \geq \max_{S^{n-1}} (\mathcal{G}_f \circ \tilde{\psi}) = \alpha,$$

and

$$\widetilde{A} \cap \varphi(D^n) \neq \emptyset, \qquad \text{for all } \varphi \in \widetilde{\Phi}.$$

It follows from Definition (2.6) and (4.1) that $\mathcal{G}_f$ satisfies the Palais-Smale condition at level $c$ and that $(u, c)$ is a critical point for $\mathcal{G}_f$ if and only if $u$ is a critical point for $f$ at level $c$. The thesis thus follows by applying Theorem (3.7) to $\mathcal{G}_f$, making use of the set $\widetilde{A}$. □

For the Lusternik-Schnirelman theory, one has also to prove that epi$(f)$ is weakly locally contractible and to evaluate cat(epi$(f)$). The following result may help in this direction.

(4.6) THEOREM. *Let $A$ be an open subset of a Hilbert space $H$ and $f : A \to \mathbb{R} \cup \{+\infty\}$ be a function. Assume that $f = f_0 + f_1$ with $f_0 : A \to \mathbb{R} \cup \{+\infty\}$ lower semi-continuous with $\varphi$-monotone subdifferential of order 2 and $f_1 : A \to \mathbb{R}$ of class $C^1$. Let us define on $\mathcal{D}(f)$ the graph metric $d^*$ by*

$$d^*(u, v) = (\|u - v\|^2 + (f(u) - f(v))^2)^{1/2}.$$

*Then* epi$(f)$ *is weakly locally contractible and homotopically equivalent to $\mathcal{D}(f)$ endowed with the metric $d^*$.*

PROOF. First of all $\mathcal{D}(f_0) = \mathcal{D}(f)$ and the graph metric associated with $f_0$ is topologically equivalent to that associated with $f$. On the other hand, epi$(f_0)$ is homeomorphic to epi$(f)$. Therefore it is sufficient to treat the case $f_1 = 0$.

By [8, Theorem 3.14] $(\mathcal{D}(f), d^*)$ is an *ANR*, hence weakly locally contractible. Then it is enough to prove that epi$(f)$ is homotopically equivalent to $(\mathcal{D}(f), d^*)$.

The metric space $(\mathcal{D}(f), d^*)$ is isometric to the graph

$$G = \{(u, f(u)) : \ u \in \mathcal{D}(f)\} \subseteq \text{epi}(f).$$

Let us show that $G$ is a weak deformation retract of epi$(f)$.

Let $\eta : \text{epi}(f) \times [0, +\infty[ \to \text{epi}(f)$ be as in the proof of Theorem (4.4). Let $\vartheta : \text{epi}(f) \to \mathbb{R}$ be a continuous function such that $\vartheta(u, \xi) > \xi - f(u)$, and let $\mathcal{H} : \text{epi}(f) \times [0, 1] \to \text{epi}(f)$ be defined by

$$\mathcal{H}((u, \xi), s) = \eta((u, \xi), s\vartheta(u, \xi)).$$

Then we have

$$\mathcal{H}((u, \xi), 0) = (u, \xi),$$
$$\mathcal{H}((u, \xi), 1) = \eta((u, \xi), \vartheta(u, \xi)) \in G,$$
$$\mathcal{H}((u, f(u)), s) \in G.$$

□

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